New results on system of variational inequalities and fixed point theorem

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Abstract. In (Croat. Oper. Res. Rev, 13(1), 131-135, (2022).) A. Benhadid showed by a counterexample that a number of publications in the research on the variational inequalities system contains inaccurate results about applying the Lipschitz continuity concept in relation to both the first and second variables during the proof of the main theorem on the part of authors. In this paper, We suggest to applying fixed point theorem to correct the main results of some publications.

Keywords: fixed point problem, relaxed (e_1, e_2)-cocoercive mappings, variational inequalities system's with nonlinear mappings.

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1. Introduction

Consider a real Hilbert space H, with its norm and inner product represented by the symbols $\|.\|$ and $\langle .,. \rangle$, respectively. Consider a closed convex set K in H. It is commonly known that a variety of problems with transportation, fluid mechanics, boundary value, and equilibrium issues (see [2, 6, 8, 9, 11]) reduce to variational inequality situation:

$$\text{look for a point } \alpha \in K : \langle G(\alpha), v - \alpha \rangle \geq 0, \forall v \in K, \tag{1}$$

where $G : H \to H$ is a nonlinear operator on H. Variational inequality also has the benefit of allowing for the investigation of a variety of topics in the domains of physics, industry, ecology, social sciences, finance, and economics.

In summary, systems of variational inequalities offer a unified and comprehensive approach to modeling and solving complex problems with multiple interacting components or agents. They provide several advantages over individual variational inequalities, including integrated modeling, efficient solution methods, and robustness to uncertainty, making them valuable tools in various fields such as engineering, economics, and multi-agent systems.

In recent years (see [3, 4, 10, 13]), several authors have been interested in the field of system of variational inequalities and they have also used it to create new iterative algorithms for handling other pertinent problems. The system of variational inequalities that follows was established by Verma [14] in 2004. It entails determining $(\alpha, \beta) \in K^2$ such that:

$$\begin{cases} \langle \rho G(\beta, \alpha) + \alpha - \beta, v - \alpha \rangle \geq 0, \forall v \in K, \rho > 0, \\ \langle \eta G(\alpha, \beta) + \beta - \alpha, v - \beta \rangle \geq 0, \forall v \in K, \eta > 0. \end{cases} \tag{2}$$

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Huang and Noor [7] then examined and researched the topic of determining $(\alpha, \beta) \in K^2$ such that :

$$\begin{cases} \langle \rho G_1(\beta, \alpha) + \alpha - \beta, v - \alpha \rangle \geq 0, \forall v \in K, \rho > 0, \\ \langle \eta G_2(\alpha, \beta) + \beta - \alpha, v - \beta \rangle \geq 0, \forall v \in K, \eta > 0, \end{cases} \tag{3}$$

where $G, G_1, G_2 : H \longrightarrow H$ are nonlinear operators. The goal of the current work is to investigate the existence and uniqueness of solutions for the system (3) and to correct the previous results of [7] and [14], by employing the definitions corrected by Benhadid in [1]. For this reason, we think revisiting the following well-known notion.

2. Preliminaries

Definition 1. *Let G be an operator from H to H, $\lambda > 0$ is a constant with:*

$$\forall (a_1, a_2) \in H^2 : \|G(a_1) - G(a_2)\| \leq \lambda \|a_1 - a_2\| \tag{4}$$

then G is called λ-Lipschitz.

Remark 1. *If $\lambda < 1$, then G is said to be a contraction.*

Definition 2. *Let G be an operator from $H \times H$ to H, $\lambda > 0$ is a constant with:*

$$\forall v \in H, \forall (a_1, a_2) \in H^2 : \|G(a_1, v) - G(a_2, v)\| \leq \lambda \|a_1 - a_2\| \tag{5}$$

then G is called λ-Lipschitz in the first variable.

Definition 3. *Let G be an operator from H to H, $r > 0$ is a constant with:*

$$\forall (a_1, a_2) \in H^2 : \langle G(a_1) - G(a_2), a_1 - a_2 \rangle \geq r \|a - b\|^2 \tag{6}$$

then G is called r-strongly monotone.

Definition 4. *Let G be an operator from H to H, $e_1 > 0$, $e_2 > 0$ are two constants with:*

$$\forall (a_1, a_2) \in H^2 : \langle G(a_1) - G(a_2), a_1 - a_2 \rangle \geq -e_1 \|G(a_1) - G(a_2)\|^2 + e_2 \|a_1 - a_2\|^2 \tag{7}$$

then G is called relaxed (e_1, e_2)-cocoercive.

Proposition 1. *[5]. Given an element $a \in H$, $b \in K$, where $K \subset H$ is a convex closed set. then the inequality*

$$\langle b - a, x - b \rangle \geq 0, \forall x \in K \tag{8}$$

is equivalent to

$$b = P_K(a), \tag{9}$$

where P_K is a projection of H into K and satisfies:

$$\|P_K(a) - P_K(b)\| \leq \|a - b\|, \forall a, b \in H.$$

Theorem 1. *[12] Let F be a contraction on H. Then F has a unique fixed point $a \in H$, i.e $F(a) = a$.*

Finding the solution $(\alpha, \beta) \in K^2$ of (3) is identical to finding $(\alpha, \beta) \in K^2$ such that:

$$\begin{cases} \alpha = \frac{1}{2}(\alpha + P_K[\beta - \rho G_1(\beta, \alpha)]) \\ \beta = \frac{1}{2}(\beta + P_K[\alpha - \eta G_2(\alpha, \beta)]) \end{cases} \tag{10}$$

as we can readily demonstrate using Proposition 1, which is similar to the operator's fixed point problem $F : H^2 \to H^2$ define by:

$$(u,v) \to \left(\frac{1}{2}\left(u + P_K\left[v - \rho G_1(v,u)\right]\right), \frac{1}{2}\left(v + P_K\left[u - \eta G_2(u,v)\right]\right)\right) \tag{11}$$

Proof. We have, $\forall v \in K, \rho > 0, \eta > 0$

$$\begin{cases} \langle \rho G_1(\beta,\alpha) + \alpha - \beta, v - \alpha \rangle \geq 0 \\ \langle \eta G_2(\alpha,\beta) + \beta - \alpha, v - \beta \rangle \geq 0 \end{cases} \Leftrightarrow \begin{cases} \langle \alpha - (\beta - \rho G_1(\beta,\alpha)), v - \alpha \rangle \geq 0 \\ \langle \beta - (\alpha - \eta G_2(\alpha,\beta)), v - \beta \rangle \geq 0 \end{cases}$$

By using Proposition 1, we get:

$$\begin{aligned} \begin{cases} \langle \alpha - (\beta - \rho G_1(\beta,\alpha)), v - \alpha \rangle \geq 0 \\ \langle \beta - (\alpha - \eta G_2(\alpha,\beta)), v - \beta \rangle \geq 0 \end{cases} &\Leftrightarrow \begin{cases} \alpha = P_K\left[\beta - \rho G_1(\beta,\alpha)\right] \\ \beta = P_K\left[\alpha - \eta G_2(\alpha,\beta)\right] \end{cases} \\ &\Leftrightarrow \begin{cases} \alpha = \frac{1}{2}\left(\alpha + P_K\left[\beta - \rho G_1(\beta,\alpha)\right]\right) \\ \beta = \frac{1}{2}\left(\beta + P_K\left[\alpha - \eta G_2(\alpha,\beta)\right]\right) \end{cases} \end{aligned} \tag{12}$$

which means that:

$$(\alpha,\beta) = F(\alpha,\beta) \quad \square$$

3. Main result

In this section, we'll show the existence and uniqueness of solution for the problem (3).

Theorem 2. *Let $G_1 : H^2 \to H$ be relaxed (γ_1, r_1)−cocoercive in the first variable, μ_1-Lipschitzian in the first variable and λ_1-Lipschitz in the second variable. Let $G_2 : H^2 \to H$ be relaxed (γ_2, r_2)−cocoercive in the first variable, μ_2 Lipschitzian in the first variable and λ_2-Lipschitz in the second variable. If*

$$\begin{cases} r_1 - \gamma_1\mu_1^2 > \mu_1, \\ \dfrac{r_1 - \gamma_1\mu_1^2 + \sqrt{\left(r_1 - \gamma_1\mu_1^2\right)^2 - \mu_1^2}}{\mu_1^2} < \rho < \dfrac{r_1 - \gamma_1\mu_1^2 + \sqrt{\left(r_1 - \gamma_1\mu_1^2\right)^2 - \frac{3}{4}\mu_1^2}}{\mu_1^2}, \rho < \dfrac{1}{2\lambda_1} \end{cases} \tag{13}$$

$$\begin{cases} r_2 - \gamma_2\mu_2^2 > \mu_2, \\ \dfrac{r_2 - \gamma_2\mu_2^2 + \sqrt{\left(r_2 - \gamma_2\mu_2^2\right)^2 - \mu_2^2}}{\mu_2^2} < \eta < \dfrac{r_2 - \gamma_2\mu_2^2 + \sqrt{\left(r_2 - \gamma_2\mu_2^2\right)^2 - \frac{3}{4}\mu_2^2}}{\mu_2^2}, \eta < \dfrac{1}{2\lambda_2} \end{cases} \tag{14}$$

then there existe a unique solution of the system (3).

Proof. To check the result, we need to assess $\|F(x,y) - F(z,t)\|$ where $(x,y), (z,t) \in H^2$ and $\|(x,y)\|_{H^2} = \|x\|_H + \|y\|_H$.

$$F(x,y) - F(z,t) = \left(\frac{1}{2}\left(x + P_K\left[y - \rho G_1(y,x)\right]\right), \frac{1}{2}\left(y + P_K\left[x - \eta G_2(x,y)\right]\right)\right) \tag{15}$$

$$- \left(\frac{1}{2}\left(z + P_K\left[t - \rho G_1(t,z)\right]\right), \frac{1}{2}\left(t + P_K\left[z - \eta G_2(z,t)\right]\right)\right) \tag{16}$$

$$= (A,B) \tag{17}$$

First we need to evaluate:

$$\|A\| = \frac{1}{2}\left\|x + P_K\left[y - \rho G_1(y,x)\right] - \left(z + P_K\left[t - \rho G_1(t,z)\right]\right)\right\| \tag{18}$$

So,

$$\begin{aligned}
2\|A\| &= \|(x-z)+[P_K[y-\rho G_1(y,x)]-P_K[t-\rho G_1(t,z)]]\| && (19)\\
&\le \|x-z\|+\|[P_K[y-\rho G_1(y,x)]-P_K[t-\rho G_1(t,z)]]\| && (20)\\
&\le \|x-z\|+\|[y-\rho G_1(y,x)]-[t-\rho G_1(t,z)]\| && (21)\\
&\le \|x-z\|+\|y-t-\rho[G_1(y,x)-G_1(t,z)]\| && (22)\\
&\le \|y-t-\rho[G_1(y,x)-G_1(t,x)+G_1(t,x)-G_1(t,z)]\| && (23)\\
&\quad +\|x-z\| && (24)\\
&\le \|x-z\|+\|y-t-\rho[G_1(y,x)-G_1(t,x)]\| && (25)\\
&\quad +\rho\|G_1(t,x)-G_1(t,z)\| && (26)
\end{aligned}$$

From the relaxed (γ_1, r_1) −cocoercive for the first variable on G_1, we have

$$\begin{aligned}
\|y-t-\rho[G_1(y,x)-G_1(t,x)]\|^2 &= \|y-t\|^2-2\rho\langle G_1(y,x)-G_1(t,x), y-t\rangle && (27)\\
&\quad +\rho^2\|G_1(y,x)-G_1(t,x)\|^2 && (28)\\
&\le -2\rho\left[-\gamma_1\|G_1(y,x)-G_1(t,x)\|^2+r_1\|y-t\|^2\right] && (29)\\
&\quad +\|y-t\|^2+\rho^2\|G_1(y,x)-G_1(t,x)\|^2 && (30)\\
&= 2\rho\gamma_1\|G_1(y,x)-G_1(t,x)\|^2-2\rho r_1\|y-t\|^2 && (31)\\
&\quad +\|y-t\|^2+\rho^2\|G_1(y,x)-G_1(t,x)\|^2 && (32)
\end{aligned}$$

From the μ_1-Lipschitzian definition for the first variable on G_1, we have:

$$\|y-t-\rho[G_1(y,x)-G_1(t,x)]\|^2 \le \left[1+2\rho\gamma_1\mu_1^2-2\rho r_1+\rho^2\mu_1^2\right]\|y-t\|^2 \qquad (33)$$

From the λ_1-Lipschitzian definition for the second variable on G_1, we have:

$$\|G_1(t,x)-G_1(t,z)\| \le \lambda_1\|x-z\| \qquad (34)$$

As a result, we have:

$$\|A\| \le \frac{1}{2}(\|x-z\|+\theta_1\|y-t\|+\rho\lambda_1\|x-z\|) \qquad (35)$$

where,

$$\theta_1 = \left[1+2\rho\gamma_1\mu_1^2-2\rho r_1+\rho^2\mu_1^2\right]^{\frac{1}{2}} \qquad (36)$$

Similary we have:

$$\|B\| \le \frac{1}{2}(\|y-t\|+\theta_2\|x-z\|+\eta\lambda_2\|y-t\|), \qquad (37)$$

where,

$$\|B\| = \frac{1}{2}\|y+P_K[x-\eta G_1(x,y)]-(t+P_K[z-\eta G_1(z,t)])\| \qquad (38)$$

and

$$\theta_2 = \left[1+2\eta\gamma_2\mu_2^2-2\eta r_2+\eta^2\mu_2^2\right]^{\frac{1}{2}}. \qquad (39)$$

The conditions (13) and (14) make it evident that,

$$\begin{cases}\theta_1 < \frac{1}{2},\\ \eta\lambda_2 < \frac{1}{2},\end{cases} \text{and} \begin{cases}\theta_2 < \frac{1}{2},\\ \rho\lambda_1 < \frac{1}{2}.\end{cases} \qquad (40)$$

So,

$$\theta_1 + \eta\lambda_2 < 1 \quad \text{and} \quad \theta_2 + \rho\lambda_1 < 1. \tag{41}$$

Then from (35) and (37),

$$\begin{aligned}\|A\| + \|B\| \leq& \frac{1}{2}\left(\|x-z\| + \theta_1 \|y-t\| + \rho\lambda_1 \|x-z\|\right)\\ &+ \frac{1}{2}\left(\|y-t\| + \theta_2 \|x-z\| + \eta\lambda_2 \|y-t\|\right)\\ \leq& \frac{1}{2}\left[\|x-z\| + \|y-t\|\right] + \sigma\frac{1}{2}\left[\|x-z\| + \|y-t\|\right]\\ \leq& \frac{1+\sigma}{2}\left[\|x-z\| + \|y-t\|\right]\end{aligned} \tag{42}$$

where, $\sigma = \max\left(\theta_1 + \eta\lambda_2, \theta_2 + \rho\lambda_1\right) < 1$.
So,

$$\|F(x,y) - F(z,t)\|_{H\times H} = \|A\| + \|B\| \tag{43}$$

$$\leq k\,\|(x,y) - (z,t)\|_{H\times H} \tag{44}$$

where $k = \dfrac{\sigma+1}{2} < 1$,
which implies that the map F defined by (11) is a contraction, using Theorem 1 then F has a unique fixed point. ☐

Remark 2. *Through this note, we explain and illustrate more about the two conditions (13) and (14): We set,*

$$\begin{cases} P_1(\rho) = \mu_1^2\rho^2 + 2(\gamma_1\mu_1^2 - r_1)\rho + 1 \\ P_2(\rho) = \mu_1^2\rho^2 + 2(\gamma_1\mu_1^2 - r_1)\rho + \frac{3}{4} \end{cases} \tag{45}$$

Clearly, P_1 and P_2 are two polynomials of order 2 with

$$\begin{cases} \Delta'_{P_1} = (\gamma_1\mu_1^2 - r_1)^2 - \mu_1^2 \\ \Delta'_{P_2} = (\gamma_1\mu_1^2 - r_1)^2 - \dfrac{3}{4}\mu_1^2 \end{cases} \tag{46}$$

1. *About the condition $r_1 - \gamma_1\mu_1^2 > \mu_1$.*
   We have

$$\begin{aligned} r_1 - \gamma_1\mu_1^2 > \mu_1 > \frac{\sqrt{3}}{2}\mu_1 &\Rightarrow \begin{cases} (r_1 - \gamma_1\mu_1^2)^2 > \dfrac{3}{4}\mu_1^2, \\ (r_1 - \gamma_1\mu_1^2)^2 > \mu_1^2, \end{cases} \\ &\Rightarrow \begin{cases} \Delta'_{P_1} > 0, \\ \Delta'_{P_2} > 0. \end{cases}\end{aligned} \tag{47}$$

   Which means that P_1 and P_2 have two different roots

$$\begin{aligned} \rho_{p_1} &= \frac{r_1 - \gamma_1\mu_1^2 - \sqrt{\left(r_1 - \gamma_1\mu_1^2\right)^2 - \mu_1^2}}{\mu_1^2}, \\ \rho'_{p_1} &= \frac{r_1 - \gamma_1\mu_1^2 + \sqrt{\left(r_1 - \gamma_1\mu_1^2\right)^2 - \mu_1^2}}{\mu_1^2}, \\ \rho_{p_2} &= \frac{r_1 - \gamma_1\mu_1^2 - \sqrt{\left(r_1 - \gamma_1\mu_1^2\right)^2 - \frac{3}{4}\mu_1^2}}{\mu_1^2}, \\ \rho'_{p_2} &= \frac{r_1 - \gamma_1\mu_1^2 + \sqrt{\left(r_1 - \gamma_1\mu_1^2\right)^2 - \frac{3}{4}\mu_1^2}}{\mu_1^2}. \end{aligned} \tag{48}$$

With:

$$0 < \rho_{p_1} < \rho_{p_2} < \rho'_{p_1} < \rho'_{p_2} \tag{49}$$

2. *About the condition:* $\dfrac{r_1 - \gamma_1\mu_1^2 + \sqrt{(r_1 - \gamma_1\mu_1^2)^2 - \mu_1^2}}{\mu_1^2} < \rho < \dfrac{r_1 - \gamma_1\mu_1^2 + \sqrt{(r_1 - \gamma_1\mu_1^2)^2 - \frac{3}{4}\mu_1^2}}{\mu_1^2}$.

   We have :

$$\begin{aligned}
\rho'_{p_1} < \rho < \rho'_{p_2} &\Rightarrow \begin{cases} \rho_{p_2} < \rho < \rho'_{p_2} \\ 0 < \rho_{p_1} < \rho'_{p_1} < \rho \end{cases} \\
&\Rightarrow \begin{cases} P_2(\rho) < 0 \\ P_1(\rho) > 0 \end{cases} \\
&\Rightarrow \begin{cases} \mu_1^2\rho^2 + 2(\gamma_1\mu_1^2 - r_1)\rho + \frac{3}{4} < 0 \\ \mu_1^2\rho^2 + 2(\gamma_1\mu_1^2 - r_1)\rho + 1 > 0 \end{cases} \\
&\Rightarrow 0 < \mu_1^2\rho^2 + 2(\gamma_1\mu_1^2 - r_1)\rho + 1 < \tfrac{1}{4} \\
&\Rightarrow \theta_1^2 < \frac{1}{4} \\
&\Rightarrow \theta_1 < \frac{1}{2}
\end{aligned} \tag{50}$$

   The same explanation with θ_2 and η.

Remark 3. *From the preceding proof, we can replace conditions (13) and (14) by a weaker condition* $\begin{cases} \mu_1^2\rho^2 + 2(\gamma_1\mu_1^2 - r_1)\rho + 1 > 0 \\ \theta_1 + \eta\lambda_2 < 1 \end{cases}$ *and* $\begin{cases} \mu_2^2\eta^2 + 2(\gamma_2\mu_1^2 - r_2)\eta + 1 > 0 \\ \theta_2 + \rho\lambda_1 < 1 \end{cases}$.

Remark 4. *We obtain an approximation of (α, β) via the fixed point Algorithm $X_{n+1} = F(X_n)$ i.e:*

$$\begin{cases} x_{n+1} = \dfrac{1}{2}\left(x_n + P_K\left[y_n - \rho G_1(y_n, x_n)\right]\right) \\ y_{n+1} = \dfrac{1}{2}\left(y_n + P_K\left[x_n - \eta G_2(x_n, y_n)\, .\right]\right) \end{cases} \tag{51}$$

For any given initial points $x_0, y_0 \in H$.

4. Conclusion

In this work, we introduced and described a novel system of variational inequalities using two different operators. We suggested and presented the correct Lipschitz definition concerning the first and/or second variable in order to solve this system of variational inequalities (see [1]) via the fixed point technique. This result corrected the main result of [7], and since this new system includes the system of variational inequalities using the single operator as a special case, it also corrected the main result of [14].

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