YU ISSN 0015-3206
FZKAAA 11 (4) 171 (1979)

CANONICAL FORMULATIONS OF FIELDS SATISFYING THE D'ALAMBERT'S EQUATION

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Received 8 October 1979

UDC 530.19

Original scientific paper

The canonical formalism for fields satisfying the d'Alambert's equations is considered. It is found that there is no solution for a scalar field. The simplest fields for which such solutions exist are the Dirac's fields.

1. Introduction

Canonical formulation of physical fields suffers of several difficulties[1]. Among them there is one which is very fundamental. We illustrate this on a scalar field. Let Φ be a real scalar field which satisfies the d'Alambert's equation

$$\partial_a \partial^a \Phi = 0. \tag{1}$$

The Lagrangian of this equation is usually taken in the form

$$\mathscr{L} = k \, \partial_a \Phi \, \partial^a \Phi. \tag{2}$$

The canonical momentum is then defined by

$$\Pi_\Phi = \frac{\partial \mathscr{L}}{\partial \partial_0 \Phi} = 2k \, \partial_0 \Phi. \tag{3}$$

Φ is a scalar but the conjugate momentum Π_{Φ} is not a scalar, it is the zero-component of the four-vector $2k\,\partial_{\alpha}\Phi$. Consequently, Φ and Π_{Φ} cannot be a canonical pair. This situation is generally present in contemporary field theory[1].

The solution of this problem for the Dirac's field has been recently found[2]. Here we want to consider this problem for a more general point of view in order to see how it can generally be solved. First, we consider a scalar field satisfying the d'Alambert's equation. We find that such a solution doesn't exist. Then we consider more complex fields. We find that the simplest field for which such solution exists is the Dirac's field. From the obtained results we conclude what has to be done for other fields. We do not consider them here but we do it elsewhere.

In Section 2 we present the situation of real scalar fields and in Section 3 of complex scalar fields. An extension to a matrix scalar field which leads to the massless Dirac's field is given in Section 4. The solution for the Dirac's field with a mass is given in Section 5. Conclusions are given in Section 6.

2. A real scalar field

Let Φ be a real scalar field satisfying the Lagrange's equation

$$\partial_{\alpha}\,\partial^{\alpha}\,\Phi = 0. \tag{4}$$

The Lagrangian of this field is given by Eq. (2) and the conjugate momentum by Eq. (3). As we have mentioned, Π_{Φ} is not a scalar. If the assumed scalar field exists, the conjugate momentum should also be a scalar. The question is how to construct a scalar conjugate momentum to Φ from the four-vector ∂_{α} and the scalar Φ? There is only one way to do it and it is a contraction with a constant fourvector which we denote by φ^{α}:

$$\partial_{\alpha}\,\Phi\,\varphi^{\alpha}. \tag{5}$$

If it is possible, it has to be incorporated in the Lagrange's formalism.

Let us denote

$$F = \partial_{\alpha}\,\Phi\,\varphi^{\alpha} = \partial_{\alpha}\,\varphi^{\alpha}\,\Phi. \tag{6}$$

The Lagrangian is then

$$\mathscr{L} = k\,F^{2} = k\,(\partial_{\alpha}\,\varphi^{\alpha}\,\Phi)^{2} \tag{7}$$

where k is a constant. The corresponding Lagrange's equation is

$$\frac{\partial\mathscr{L}}{\partial\Phi} - \partial_{\mu}\left(\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\Phi)}\right) = 0 \rightarrow \partial_{\alpha}\,\partial_{\beta}\,\varphi^{\alpha}\,\varphi^{\beta}\,\Phi = 0$$

or

$$\frac{1}{2}\,(\varphi^{\alpha}\,\varphi^{\beta} + \varphi^{\beta}\,\varphi^{\alpha})\,\partial_{\alpha}\,\partial_{\beta}\,\Phi = 0. \tag{8}$$

The condition that this equation is the d'Alambert's equation requires

$$\varphi^{\alpha}\varphi^{\beta}+\varphi^{\beta}\varphi^{\alpha}=2a\,g^{\alpha\beta} \tag{9}$$

where a is a constant and $g^{\alpha\beta}$ the metric tensor

$$(g^{\alpha\beta})=\begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix} \tag{10}$$

Eq. (8) then becomes

$$\partial_{\alpha}\,\partial^{\alpha}\,\Phi=0. \tag{11}$$

Eq. (9) imposes conditions on φ^{α}. It is evident that they are fulfilled for matrices. From this we conclude that there is no solution of the problem of canonical formalism for a scalar field in the given framework.

3. A complex scalar field

Extension of previous analysis to a complex scalar field is trivial but it is instructive for the next step.

Let Φ be a complex field. Using the same notation as in Section 3, the Lagrangian is now

$$\mathscr{L}=k\,F^{*}F \tag{12}$$

and the Lagrange's equations for Φ and Φ^{*} are

$$\partial_{\alpha}\,\partial_{\beta}\,\varphi^{\alpha *}\varphi^{\beta}\,\Phi=0, \tag{13}$$

$$\partial_{\alpha}\,\partial_{\beta}\,\varphi^{\alpha}\varphi^{\beta *}\,\Phi^{*}=0,$$

or

$$\frac{1}{2}\,(\varphi^{\alpha *}\varphi^{\beta}+\varphi^{\beta *}\varphi^{\alpha})\,\partial_{\alpha}\,\partial_{\beta}\,\Phi=0,$$

$$\frac{1}{2}\,(\varphi^{\alpha *}\varphi^{\beta}+\varphi^{\beta *}\varphi^{\alpha})\,\partial_{\alpha}\,\partial_{\beta}\,\Phi^{*}=0.$$

The conditions that these equations become the d'Alambert's equations are

$$\varphi^{\alpha *}\varphi^{\beta}+\varphi^{\beta *}\varphi^{\alpha}=2ag^{\alpha\beta} \tag{14}$$

Taking $\alpha = \beta$, we have

$$\varphi^0\, {}^*\varphi^0 = a,$$

$$\varphi^{i*}\, \varphi^i = -a.$$

These equations show that there are no solutions of the conditions (14). What is the next object for which a solution might exist? We consider it in the next Section.

4. The Dirac's massless field

Eqs. (9) and (6) indicate that solution of the formulated problem might exist for a matrix field Φ. The simplest case is a one column matrix

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{pmatrix}. \tag{15}$$

Eq. (14) shows that the corresponding Lagrangian cannot be *const* $F^\dagger F$. One can start with a more general structure

$$\mathscr{L} = k\, F^\dagger \eta\, F \tag{16}$$

where η is a constant matrix. The reality condition of $\mathscr{L}$ requires

$$\eta^\dagger = \eta. \tag{17}$$

F is given again by

$$F = \partial_\alpha\, \varphi^\alpha\, \Phi, \tag{18}$$

where Φ, F, φ^α are now matrices.

The Lagrange's equations for the matrices Φ, Φ^* are

$$\begin{gathered} \frac{1}{2}\,(\varphi^{\alpha\dagger}\,\eta\,\varphi^\beta + \varphi^{\beta\dagger}\,\eta\,\varphi^\alpha)\,\partial_\alpha\,\partial_\beta\,\Phi = 0\cdot \\ \partial_\alpha\,\partial_\beta\,\Phi^\dagger\,\frac{1}{2}\,(\varphi^{\alpha\dagger}\,\eta\,\varphi^\beta + \varphi^{\beta\dagger}\,\eta\,\varphi^\alpha) = 0. \end{gathered} \tag{19}$$

The conditions that these equations become the d'Alambert's equations now read

$$\varphi^{\alpha\dagger}\,\eta\,\varphi^\beta + \varphi^{\beta\dagger}\,\eta\,\varphi^\alpha = 2\,g^{\alpha\beta}\,\varepsilon \tag{20}$$

where ε is a nonsingular constant matrix. Eqs. (19) become then

$$\partial_\alpha \partial^\alpha \varepsilon \Phi = 0, \qquad (21)$$

$$\partial_\alpha \partial^\alpha \Phi^\dagger \varepsilon = 0.$$

or after multiplications by ε^{-1}

$$\partial_\alpha \partial^\alpha \Phi = 0, \qquad (22)$$

$$\partial_\alpha \partial^\alpha \Phi^\dagger = 0.$$

The condition (20) can be also written in the form

$$\varepsilon^{-1} \varphi^{\alpha\dagger} \eta \varphi^\beta + \varepsilon^{-1} \varphi^{\beta\dagger} \eta \varphi^\alpha = 2 g^{\alpha\beta}. \qquad (23)$$

These conditions have the following solutions

$$\eta = \varepsilon = \gamma^0, \qquad (24)$$

$$\varphi^\alpha = \gamma^\alpha,$$

where γ^α are the Dirac's matrices. Indeed, substituting (24) in (23), we get

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2 g^{\alpha\beta}. \qquad (25)$$

With this solution, we have

$$F = \partial_\alpha \gamma^\alpha \Phi, \qquad (26)$$

$$\bar{F} = \partial_\alpha \bar{\Phi} \gamma^\alpha,$$

$$\mathscr{L} = k \bar{F} F \qquad (27)$$

where

$$\bar{F} = F^\dagger \gamma^0, \quad \bar{\Phi} = \Phi^\dagger \gamma^0. \qquad (28)$$

The conjugate momenta to Φ and $\bar{\Phi}$ are

$$\Pi_\Phi = \frac{\partial \mathscr{L}}{\partial (\partial_0 \Phi)} = k \bar{F} \gamma^0 = k (\partial_0 \bar{\Phi} + \partial_l \bar{\Phi} \gamma^l \gamma^0),$$

$$\Pi_{\bar{\Phi}} = \frac{\partial \mathscr{L}}{\partial (\partial_0 \bar{\Phi})} = k \gamma^0 F = k (\partial_0 \Phi + \gamma^0 \gamma^l \partial_l \Phi). \qquad (29)$$

The Hamiltonian density is then

$$\mathscr{H} = \Pi_{\Phi}\,\partial_0\,\Phi + \partial_0\,\bar{\Phi}\,\Pi_{\bar{\Phi}} - \mathscr{L} = \frac{1}{k}\,\Pi_{\Phi}\,\Pi_{\bar{\Phi}} - \Pi_{\Phi}\,\gamma^0\,\gamma^i\,\partial_i\,\Phi - \partial_i\,\bar{\Phi}\,\gamma^i\,\gamma^0\,\Pi_{\bar{\Phi}} \tag{30}$$

and the corresponding canonical equations

$$\begin{aligned} \partial_0\,\Pi_{\Phi} &= -\frac{\delta\,\mathscr{H}}{\delta\Phi} = -\,\partial_i\,\Pi_{\Phi}\,\gamma^0\,\gamma^i, \\ \partial_0\,\Pi_{\bar{\Phi}} &= -\frac{\delta\,\mathscr{H}}{\delta\,\bar{\Phi}} = -\,\partial_i\,\gamma^i\,\gamma^0\,\Pi_{\bar{\Phi}}, \end{aligned} \tag{31}$$

$$\begin{aligned} \partial_0\,\Phi &= \frac{\delta\,\mathscr{H}}{\delta\Pi_{\Phi}} = \frac{1}{k}\,\Pi_{\bar{\Phi}} - \gamma^0\,\gamma^i\,\partial_i\,\Phi, \\ \partial_0\,\bar{\Phi} &= \frac{\delta\,\mathscr{H}}{\delta\Pi_{\bar{\Phi}}} = \frac{1}{k}\,\Pi_{\Phi} - \partial_i\,\bar{\Phi}\,\gamma^i\,\gamma^0, \end{aligned} \tag{32}$$

or using (29)

$$\begin{aligned} \partial_\mu\,\bar{F}\,\gamma^\mu &= 0, \\ \partial_\mu\,\gamma^\mu\,F &= 0, \end{aligned} \tag{33}$$

$$\begin{aligned} F &= \partial_\alpha\,\gamma^\alpha\,\Phi, \\ \bar{F} &= \partial_\alpha\,\bar{\Phi}\,\gamma^\alpha. \end{aligned} \tag{34}$$

Up to this moment Φ, $\bar{\Phi}$, F,$\bar{F}$ have been scalar matrices and φ^a are four-vector matrices. The problem of canonical formalism has not been solved yet because Eqs. (33) and (34) are not physically acceptable. They do not satisfy the principle of special relativity.

The principle of special relativity requires that the field equations in a new coordinate sistem are

$$\begin{aligned} \tilde{\partial}_\mu\,\tilde{\bar{F}}\,\gamma^\mu &= 0, \\ \tilde{\partial}_\mu\,\gamma^\mu\,\tilde{F} &= 0, \end{aligned} \tag{35}$$

$$\begin{aligned} \tilde{F} &= \tilde{\partial}_\alpha\,\gamma^\alpha\,\tilde{\Phi}, \\ \tilde{\bar{F}} &= \tilde{\partial}_\alpha\,\tilde{\bar{\Phi}}\,\gamma^\alpha \end{aligned} \tag{36}$$

where $\sim$ denotes the quantities in the new coordinate system. Let the Lorentz transformations be denoted by

$$\tilde{x}^\mu = a^\mu_\nu\,x^\nu. \tag{37}$$

The conditions (35) and (36) cannot be fulfilled for scalar matrices $\bar{\Phi}$, Φ, F, $\bar{F}$. Thus, we assume

$$\tilde{\bar{F}} = S\,F,$$

$$\tilde{\bar{\Phi}} = S\,\Phi. \tag{38}$$

Substituting (38) into (35) and (36) and comparing it with (33) and (34) we find equation for S:

$$a_{\mu}{}^{\nu}\, S^{-1}\, \gamma^{\mu}\, S = \gamma^{\nu}. \tag{39}$$

As it is very well known from the Dirac's field theory there exists a solution of this equation and the quantities Φ, $\bar{\Phi}$, F, $\bar{F}$ are bispinors.

According to (29) Π_{Φ}, $\Pi_{\bar{\Phi}}$ become also bispinors. By this we have found a solution of the problem of canonical formalism for the massless Dirac's field. This solution, as we have already mentioned, is known$^{2)}$.

5. *The Dirac' field with a mass*

Inclusion of masses in this analysis can be done by extension of (18) in the following way

$$F = \partial_{a}\, \varphi^{a}\, \Phi + b\, \Phi \tag{40}$$

where b is a constant. The Lagrangian density is then

$$\mathscr{L} = k\, F^{\dagger}\, \eta\, F = k\, (\partial_{a}\, \Phi^{\dagger}\, \varphi^{a\dagger} + b^{*}\, \Phi^{\dagger})\, \eta\, (\partial_{\beta} \varphi^{\beta}\, \Phi + b\, \Phi) \tag{41}$$

and the corresponding Lagrange's equations are

$$\frac{1}{2}\, (\varphi^{a\dagger}\, \eta\, \varphi^{\beta} + \varphi^{\beta\dagger}\, \eta\, \varphi^{a})\, \partial_{a}\, \partial_{\beta}\, \Phi + \partial_{a}\, (\varphi^{a\dagger}\, \eta\, b - \eta\, \varphi^{u}\, b^{*})\, \Phi - b^{*}\, b\, \eta\, \Phi = 0,$$

$$\partial_{a}\, \partial_{\beta}\, \Phi^{\dagger}\, \frac{1}{2}\, (\varphi^{a\dagger}\, \eta\, \varphi^{\beta} + \varphi^{\beta\dagger}\, \eta\, \varphi^{a}) + \partial_{a}\, \Phi^{\dagger}\, (b^{*}\, \eta\, \varphi^{a} - \varphi^{a\dagger}\, \eta\, b) - b^{*}\, b\, \Phi^{\dagger}\, \eta = 0. \tag{42}$$

The condition that these equations become the d'Alambert's equations we extend to

$$\partial_{a}\, \partial^{a}\, \Phi + const\, \Phi = 0,$$

$$\partial_{a}\, \partial^{a}\, \Phi^{\dagger} + const\, \Phi^{\dagger} = 0. \tag{43}$$

From (43) it follows

$$\begin{gathered}\varphi^{\alpha\dagger}\,\eta\,\varphi^{\beta}+\varphi^{\beta\dagger}\,\eta\,\varphi^{\alpha}=2g^{\alpha\beta}\,\varepsilon,\\ \varphi^{\alpha\dagger}\,\eta\,b-\eta\,\varphi^{\alpha}\,b^{*}=0,\\ \varepsilon=\pm\,\eta.\end{gathered}\tag{44}$$

The first conditions is the same as in (20). For the solution of the Dirac's matrices the second condition gives

$$b^{*}=b\tag{45}$$

and Eqs. (43) appear in the form

$$\begin{gathered}\partial_{\alpha}\,\partial^{\alpha}\,\Phi-b^{2}\,\Phi=0,\\ \partial_{\alpha}\,\partial^{\alpha}\,\Phi^{\dagger}-b^{2}\,\Phi=0.\end{gathered}\tag{46}$$

Writing the Lagrange's density in the form

$$\mathscr{L}=k\,\bar{F}F$$

the canonical momenta are

$$\begin{gathered}\Pi_{\Phi}=k\,\bar{F}\,\varphi^{0}=k\,(\partial_{0}\,\bar{\Phi}+\partial_{i}\,\bar{\Phi}\,\varphi^{i}\,\varphi^{0}+b\,\bar{\Phi}\,\varphi^{0}),\\ \Pi_{\bar{\Phi}}=k\,\varphi^{0}\,F=k\,(\partial_{0}\,\Phi+\varphi^{0}\,\varphi^{i}\,\partial_{i}\,\Phi+b\,\varphi^{0}\,\Phi)\end{gathered}\tag{47}$$

and the Hamiltonian density

$$\mathscr{H}=\frac{1}{k}\,\Pi_{\Phi}\,\Pi_{\bar{\Phi}}-\Pi_{\Phi}\,\varphi^{0}\,\varphi^{i}\,\partial_{i}\,\Phi-(\partial_{i}\,\bar{\Phi})\,\varphi^{i}\,\varphi^{0}\,\Pi_{\bar{\Phi}}-b\,(\Pi_{\Phi}\,\varphi^{0}\,\Phi+\bar{\Phi}\,\varphi^{0}\,\Pi_{\bar{\Phi}}).\tag{48}$$

The canonical equations are

$$\begin{gathered}\partial_{0}\,\Pi_{\Phi}=b\,\Pi_{\Phi}\,\varphi^{0}-\partial_{i}\,\Pi_{\Phi}\,\varphi^{0}\,\varphi^{i},\\ \partial_{0}\,\Pi_{\bar{\Phi}}=b\,\varphi^{0}\,\Pi_{\bar{\Phi}}-\partial_{i}\,\varphi^{i}\,\varphi^{0}\,\Pi_{\bar{\Phi}},\end{gathered}\tag{49}$$

$$\begin{gathered}-\,\partial_{0}\,\Phi=-\,\frac{1}{k}\,\Pi_{\bar{\Phi}}+b\,\varphi^{0}\,\Phi+\varphi^{0}\,\varphi^{i}\,\partial_{i}\,\Phi,\\ -\,\partial_{0}\,\Phi=-\,\frac{1}{k}\,\Pi_{\Phi}+b\,\bar{\Phi}\,\varphi^{0}+\partial_{i}\,\bar{\Phi}\,\varphi^{i}\,\varphi^{0}\end{gathered}\tag{50}$$

or using (47)

$$\begin{gathered}\partial_{\mu}\,\bar{F}\,\varphi^{\mu}-b\,\bar{F}=0,\\ \partial_{\mu}\,\varphi^{\mu}\,F-b\,F=0,\end{gathered}\tag{51}$$

$$F = \partial_\mu \varphi^\mu \Phi + b \Phi,$$
$$\overline{F} = \partial_\mu \overline{\Phi} \varphi^\mu + b \overline{\Phi}. \tag{52}$$

The principle of special relativity leads to the same solution as in the case of the massless field. This can be easily checked. Therefore, we have the solution of the problem of canonical formalism also in this case.

Let us mention that Eqs. (51) and (52) do not describe the Dirac's electrons and positrons. They become the Dirac's equations for the electron-positron field for $b = i\varkappa$, where $\varkappa$ is a real constant. Having in mind the energy dispersion relation

$$E_p = \pm \sqrt{\varkappa^2 + p^2}$$

we conclude that Eqs. (51), (52) described tachyons[3] and we call this field the Dirac's tachyon field.

For $b = i\varkappa$ the second equation in (44) is no more zero and the Lagrange's equations are

$$\partial_a \partial^a \Phi + 2i\varkappa\, \partial_a \gamma^a \Phi - \varkappa^2 \Phi = 0,$$
$$\partial_a \partial^a \Phi^\dagger - 2i\varkappa\, \partial_a \Phi^\dagger \gamma^{a\dagger} - \varkappa^2 \Phi^\dagger = 0.$$

This case is, therefore, out of the scope of this paper. Let us only mention that the presented canonical formalism is valid here also[2].

Existence of other solutions of Eq. (23) besides the Dirac's matrices as well as for $\varepsilon = -\eta$ and the physical meaning of such solutions will be considered elsewhere.

6. *Conclusion*

Analysis of canonical formalism in Sections 2—5 shows how the canonical formulation of physical fields can or has to be constructed. We have started with the simplest possible case, a real scalar field, and finished with the Dirac's field. It turns out that the constant four-vector φ^a plays an important role. In the case of the Dirac's field the contraction $\partial_a \varphi^a$ was found to be essential. For the quantities ∂_a, φ^a, Φ^a some other contractions are possible and must be taken into account. Then a more complex field should be analysed and so on.

The main goal of this paper was to show the way of construction of canonical formalism of physical fields. Due to this reason we didn't consider many other interesting questions related to this problem such as constants of motion, interacting fields, quantum rules and so on. These problems as well as other fields will be considered in subsequent papers.

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KANONSKA FORMULACIJA ZA POLJA KOJA ZADOVOLJAVAJU D'ALAMBERTOVU JEDNADŽBU

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Originalni znanstveni rad

Razmotren je kanonski formalizam za polja koja zadovoljavaju d'Alambertovu jednadžbu. Nađeno je da ne postoji rješenje za skalarno polje. Najjednostavnije polje za koje postoji rješenje su Diracova polja.