

CANONICAL FORMULATION OF A FREE VECTOR FIELD SATISFYING THE FIRST ORDER DIFFERENTIAL EQUATIONS*

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The canonical formalism of a free vector field satisfying the first order differential equations is considered and the correct canonical formulation is given. Some consequences of the new formulation are analysed. The connection with zero mass field of the standard theory is also presented.

1. Introduction

The standard field theory describes a vector field by the Lagrange's equations^{1,2)}

$$(\partial_\alpha \partial^\alpha + \kappa^2) F^\alpha = 0 \quad (1)$$

and the condition

$$\partial_\alpha F^\alpha = 0 \quad (2)$$

where κ is a constant. We use the notation

$$x^\mu = (t, \mathbf{x}), \quad x_\mu = (t, -\mathbf{x}), \quad \partial_\mu = \frac{\partial}{\partial x^\mu}$$

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with the metric tensor

$$g_{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad (3)$$

and units $c = 1$.

The Lagrangian density is usually taken in the form

$$\mathcal{L} = -\frac{1}{4} (\partial_\mu \Phi^\nu{}^* - \partial^\nu \Phi_\mu{}^*) (\partial^\mu \Phi_\nu - \partial_\nu \Phi^\mu) + \frac{\kappa^2}{2} \Phi_\mu{}^* \Phi^\mu \quad (4)$$

where we assumed a complex vector field. The corresponding Lagrange's equations are

$$\frac{\partial \mathcal{L}}{\partial \Phi_\nu^*} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi_\nu^*)} \right) = \kappa^2 \Phi_\nu + \partial_\mu (\partial^\mu \Phi_\nu - \partial_\nu \Phi^\mu) = 0. \quad (5)$$

From Eq. (5) follows the condition (2)

$$\partial^\nu \Phi_\nu = 0$$

and the Eq. (5) becomes

$$\partial_\mu \partial^\mu \Phi_\nu + \kappa^2 \Phi_\nu = 0.$$

The canonical conjugate momenta to Φ_ν and Φ_ν^* are

$$\Pi_{\Phi_\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_0 \Phi_\nu)} = -\frac{1}{2} (\partial_0 \Phi^{\nu*} - \partial^\nu \Phi_0^*),$$

$$\Pi_{\Phi_\nu^*} = -\frac{1}{2} (\partial_0 \Phi_\nu - \partial^\nu \Phi_0). \quad (6)$$

The Π_{Φ_ν} is (σ, ν) component of the second order tensor

$$-(\partial_\mu \Phi^{\sigma*} - \partial^\sigma \Phi_\mu^*) \quad (7)$$

and similarly $\Pi_{\Phi_\nu^*}$. Consequently, Φ_ν and Π_{Φ_ν} cannot be a canonically conjugate pair as well as Φ_ν^* and $\Pi_{\Phi_\nu^*}$.

The problem of canonical formulation of physical fields has been investigated in several papers. Recently this problem was successfully solved for the Dirac's field³⁾. The way how to treat this problem more generally was considered in Refs. 4 and 5. Following this line of thinking we consider in this paper the canonical formulation of a vector field satisfying the first order differential equations and give the solution.

In Section 2 we define the field equations. The canonical equations are given in Section 3. Solution of the canonical equations is given in Section 4. In Section 5 and 6 the energy-momentum and the angular momentum tensors are evaluated. Connection with a scalar field of the standard up to date theory is considered in Section 7. Conclusions are given in Section 8.

2. Vector field equations

Let F^e be a complex vector field satisfying the first order differential equations. According to Refs. 3, 4 and 5 we expect that F^e is the momentum of the field and that the corresponding equations are the canonical equations for this momentum. Therefore, we assume the Lagrangian

$$\mathcal{L} = k F_e^* F^e \quad (8)$$

and

$$F^e = \partial_a D^{ae\beta} \Phi_\beta \quad (9)$$

where Φ_β is canonical variable conjugate to F_β , $D^{ae\beta}$ a constant component tensor and k is a constant.

Eq. (9) is the most general connection of F^e and Φ_β for the first order canonical equations without a mass term. The case with a mass is considered at the end of the Section.

The canonical equations should satisfy the principle of special relativity. In order to simplify the situation at the beginning with respect to this principle, we write

$$D^{ae\beta} = D^{ae\beta\eta} \varphi_\eta \quad (10)$$

where φ_η is a constant component four-vector and $D^{ae\beta\eta}$ is a product of the metric tensors.

We consider at first

$$D^{ae\beta\eta} = g^{a\beta} g^{e\eta} + g^{ae} g^{\beta\eta} - g^{\beta e} g^{a\eta}. \quad (11)$$

For this case

$$D^{ae\beta} = g^{a\beta} \varphi^e + g^{ae} \varphi^\beta - g^{\beta e} \varphi^a \quad (12)$$

and

$$F^e = \partial^\beta \varphi^e \Phi_\beta + \partial^e \varphi^\beta \Phi_\beta - \partial_\beta \varphi^\beta \Phi^e. \quad (13)$$

The Lagrange's equations for Φ^e and Φ^{e*} are then, respectively

$$\varphi^{e*} (\partial_\eta F^\eta) + \varphi^{e*} (\partial^\sigma F_e - \partial_e F^\sigma) = 0, \quad (14)$$

$$\varphi^e (\partial_\eta F^{\eta*}) + \varphi^e (\partial_\sigma F_e^* - \partial_e F^{\sigma*}) = 0$$

where F^e and F^{e*} are given by (9).

Eqs. (14) are the first order differential equations for F^e and F^{e*} and contain only these quantities. Consequently, they are equations of our interest.

In order that Eqs. (14) become physical equations we impose the principle of special relativity. It can be fulfilled in one of the two ways: (1) φ^a are constants equal in each reference system, (2) F^e do not depend of Φ^a at all. In the first case F^e cannot be a four-vector. Due to this we take the second case:

$$\begin{aligned} \partial^\eta F_\eta &= 0, \\ \partial_e F_e - \partial_0 F_0 &= 0, \end{aligned} \tag{15}$$

$$\begin{aligned} \partial_\eta F^{\eta*} &= 0, \\ \partial_e F_e^* - \partial_0 F_0^* &= 0. \end{aligned} \tag{15'}$$

Selecting Eqs. (15) φ^a becomes completely free with respect to F^e .

Eqs. (15) and (15') are the first set of dynamical equations of the vector field F^e . The second set is Eq. (13) and its conjugate complex. For a given F^e Eq. (13) gives Φ^a as a particular solution. This solution depends on φ^a . Because φ^a are free, there is a family of the particular solutions depending on φ^a . The principle of special relativity makes a selection. In order that Eq. (13) satisfies this principle φ^a has to be equal in each reference system. It means that at end of the calculation this selection has to be done. By this the vector character of φ^a and Φ^a is not changed. Only among possible vectors φ^a and Φ^a one makes a definite selection.

In Eq. (11) we have taken a defined combination of products of the metric tensor. There are three other combinations:

$$\begin{aligned} D^{\alpha e \beta \eta} &= g^{\alpha \beta} g^{e \eta} - g^{\alpha e} g^{\beta \eta} + g^{\beta e} g^{\alpha \eta}, \\ D^{\alpha e \beta \eta} &= g^{\alpha \beta} g^{e \eta} + g^{\alpha e} g^{\beta \eta} + g^{\beta e} g^{\alpha \eta}, \\ D^{\alpha e \beta \eta} &= g^{\alpha \beta} g^{e \eta} - g^{\alpha e} g^{\beta \eta} - g^{\beta e} g^{\alpha \eta}. \end{aligned}$$

One immediately sees that the first combinations gives the same equations as in (15). In Eq. (13) the last two terms change the sign. One can expect that this is equivalent to the selection (11). In subsequent sections it will turn out too be true. For the same reason the second and third combinations are equivalent. Thus, we need to look at the second combination only.

In accordance to (9) and (10), the field equations for this combination read

$$\partial_\eta F^\eta = 0,$$

$$\partial_e F_e + \partial_0 F_0 = 0$$

or

$$\partial_0 F^0 + \partial_i F^i = 0,$$

$$\partial_0 F_i + \partial_i F_0 = 0, \quad \partial_i F_j + \partial_j F_i = 0.$$

From here one has: $F^0 = 0$, $F^i = G^i(\mathbf{r})$, with $\text{rot } \mathbf{G}(\mathbf{r}) = 0$. Therefore, this field is of no physical interest.

At the end of this Section we consider inclusion of mass in the field. If it is possible, it can be done in the following way:

$$F_e = \begin{cases} \partial_\alpha D^\alpha{}^\beta \Phi_\beta + m \Phi_e \\ \partial_\alpha D_\alpha{}^\beta \Phi_\beta + m \varphi^\alpha D_{\alpha\beta} \Phi^\beta. \end{cases}$$

The field equations are then

$$\begin{aligned} m F^\eta - \partial_\lambda F^\lambda \varphi^{\eta*} + (\partial_\lambda F^\eta - \partial^\eta F_\lambda) \varphi^{\lambda*} &= 0, \\ m \varphi^{\alpha*} (2\varphi^{\eta*} F_\alpha - \varphi_\alpha^* F^\eta) - \partial_\alpha \varphi^{\eta*} F^\alpha + \varphi^{\alpha*} (\partial_\alpha F^\eta - \partial^\eta F_\alpha) &= 0, \end{aligned}$$

and c.c.

The principle of special relativity and requirement that F^e is a vector field leads to $m = 0$ in both cases.

3. Canonical equations

The conjugate momenta to Φ'_β , Φ_β^* are defined by

$$\begin{aligned} \Pi_{\omega_\beta} &\equiv \Pi^\beta = \frac{\partial \mathcal{L}}{\partial (\partial_0 \Phi_\beta)} = k (g^{0\beta} F_e^* \varphi^e - F^{\beta*} \varphi^0 + F^{0*} \varphi^\beta), \\ \Pi_{\omega_\beta^*} &\equiv \Pi^{\beta*} = k (g^{0\beta} F_e \varphi^{e*} - F^\beta \varphi^{0*} + F^0 \varphi^{\beta*}). \end{aligned} \tag{16}$$

From Eqs. (16) it follows that Π^α is a linear combination of $F^{\beta*}$ and $\Pi^{\beta*}$ of F^β . On the other hand they are components of a second order tensor constructed from F^α , $F^{\alpha*}$, φ^α , $\varphi^{\alpha*}$. The selection of φ^α based on the principle of special relativity makes them four-vectors. Because the four-vector φ^α is free in a given reference system, we may take

$$\varphi^\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{17}$$

In this case we have

$$\begin{aligned} \Pi^\beta &= k F_{\beta^*}, \\ \Pi^{\beta*} &= k F_\beta \end{aligned} \tag{18}$$

i.e. direct proportionality of the momentum and field components. By this a simple connection of the field F^e and its momentum is established. The canonical equations

for the momenta are naturally Eqs. (15) and (15') after substitution of F^e by Π^e according to (18):

$$\begin{aligned} \partial_\eta \Pi_\eta^* &= 0, \\ \partial_\epsilon \Pi^{\epsilon*} - \partial_\sigma \Pi^{\sigma*} &= 0, \\ \partial_\eta \Pi_\eta &= 0, \\ \partial_\epsilon \Pi^\epsilon - \partial_\sigma \Pi^\sigma &= 0. \end{aligned} \tag{19}$$

For φ^a given by Eq. (17) F^a according to (13) reads

$$\begin{aligned} F^0 &= \partial^\beta \Phi_\beta, \\ F^i &= \partial^i \Phi_0 - \partial_0 \Phi^i \end{aligned} \tag{20}$$

or written in terms of the momentum components

$$\begin{aligned} \Pi_0^* &= k \partial^\beta \Phi_\beta, \\ \Pi_i^* &= k (\partial^i \Phi_0 - \partial_0 \Phi^i) \end{aligned} \tag{21}$$

and

$$\begin{aligned} \Pi_0 &= k \partial^\beta \Phi_\beta^*, \\ \Pi_i &= k (\partial^i \Phi_0^* - \partial_0 \Phi^{i*}). \end{aligned}$$

The second of Eqs. (19) leads to

$$\partial_0 (\partial_i \Phi_i - \partial_j \Phi_j) = 0 \tag{22}$$

and

$$\partial_i \Phi_j - \partial_j \Phi_i = \text{const in time.} \tag{23}$$

The only physically interesting solution is the one for which *const in time* = 0. Then, we have

$$\partial_i \Phi_j - \partial_j \Phi_i = 0$$

and

$$\partial_i \Phi_j^* - \partial_j \Phi_i^* = 0. \tag{24}$$

Eqs. (19) and (21) with the selection (24) are the canonical equations of the field. This can easily be checked. The Lagrangian is now

$$\mathcal{L} = k F_\epsilon^* F^\epsilon = k \{ \partial^\beta \Phi_\beta^* \partial^\beta \Phi_\beta + (\partial_i \Phi_0^* - \partial_0 \Phi_i^*) (\partial^i \Phi_0 - \partial_0 \Phi^i) \}. \tag{25}$$

The canonical momenta are

$$\begin{aligned} \Pi_{\Phi_\beta} &\equiv \Pi^\beta = \frac{\partial \mathcal{L}}{\partial (\partial_0 \Phi_\beta)} = k \{ \partial^\alpha \Phi_\alpha^* g^{0\beta} - g^{i\beta} (\partial_i \Phi_0^* - \partial_0 \Phi_i^*) \} \\ \Pi_{\Phi_\beta^*} &\equiv \Pi^{\beta*} = k \{ \partial^\alpha \Phi_\alpha g^{0\beta} - g^{i\beta} (\partial_i \Phi_0 - \partial_0 \Phi_i) \} \end{aligned} \tag{26}$$

or

$$\begin{aligned}
 \Pi^0 &= k \partial^\beta \Phi_\beta^*, \\
 \Pi^i &= k (\partial_i \Phi_0^* - \partial_0 \Phi_i^*), \\
 \Pi^{0*} &= k \partial^\beta \Phi_\beta, \\
 \Pi^{i*} &= k (\partial_i \Phi_0 - \partial_0 \Phi_i).
 \end{aligned}
 \tag{27}$$

The Hamiltonian density is

$$\begin{aligned}
 \mathcal{H} = \Pi^\beta \partial_0 \Phi_\beta + \Pi^{\beta*} \partial_0 \Phi_\beta^* - \mathcal{L} &= \frac{1}{k} \Pi_\alpha^* \Pi^\alpha + (-\Pi^0 \partial^i \Phi_i + \Pi^i \partial_i \Phi_0 - \\
 &- \Pi^{0*} \partial^i \Phi_i^* + \Pi^{i*} \partial_i \Phi_0^*).
 \end{aligned}
 \tag{28}$$

and the canonical equations

$$\begin{aligned}
 \partial_0 \Pi^0 &= \partial^i \Pi_i, \\
 \partial_0 \Pi^i &= -\partial^i \Pi^0, \\
 \partial_0 \Pi^{0*} &= \partial^i \Pi_i^*, \\
 \partial_0 \Pi^{i*} &= -\partial^i \Pi^{0*}, \\
 \partial_0 \Phi_0 &= \frac{1}{k} \Pi_0^* - \partial^i \Phi_i, \\
 \partial_0 \Phi_i &= \frac{1}{k} \Pi_i^* + \partial_i \Phi_0, \\
 \partial_0 \Phi_0^* &= \frac{1}{k} \Pi_0 - \partial^i \Phi_i^*, \\
 \partial_0 \Phi_i^* &= \frac{1}{k} \Pi_i + \partial^i \Phi_0^*
 \end{aligned}$$

or

$$\begin{aligned}
 \partial_\alpha \Pi_\alpha &= 0, \\
 \partial_0 \Pi^i - \partial_i \Pi^0 &= 0, \\
 \partial_\alpha \Pi_\alpha^* &= 0, \\
 \partial_0 \Pi^{i*} - \partial_i \Pi^{0*} &= 0, \\
 \Pi^0 &= k \partial^\beta \Phi_\beta^*, \\
 \Pi^i &= k (\partial_i \Phi_0^* - \partial_0 \Phi_i^*), \\
 \Pi^{0*} &= k \partial^\beta \Phi_\beta, \\
 \Pi^{i*} &= k (\partial_i \Phi_0 - \partial_0 \Phi_i).
 \end{aligned}
 \tag{29}$$

$$\tag{30}$$

The time derivative of $(\partial_t \Pi^j - \partial_j \Pi^t)$ taking into account the second of Eqs. (29) is

$$\partial_0 (\partial_t \Pi^j - \partial_j \Pi^t) = 0. \quad (31)$$

From here it follows

$$\partial_t \Pi^j - \partial_j \Pi^t = \text{const in time.} \quad (32)$$

Again the only physically interesting solution is

$$\partial_t \Pi^j - \partial_j \Pi^t = 0 \quad (33)$$

and

$$\partial_t \Pi^{j*} - \partial_j \Pi^{t*} = 0.$$

From Eqs. (33) and (30) it follows

$$\begin{aligned} \partial_0 (\partial_t \Phi_j^* - \partial_j \Phi_t^*) &= 0 \\ \partial_t \Phi_j^* - \partial_j \Phi_t^* &= 0, \\ \partial_t \Phi_j - \partial_j \Phi_t &= 0. \end{aligned} \quad (34)$$

and

Eqs. (29) and (30) with (33) and (34) can be written in the form

$$\begin{aligned} \partial_\alpha \Pi_\alpha &= 0, \\ \partial_\alpha \Pi^\beta - \partial_\beta \Pi^\alpha &= 0, \\ \Pi^0 &= k \partial^\beta \Phi_\beta^*, \\ \Pi^t &= k (\partial_t \Phi_0^* - \partial_0 \Phi_t^*), \\ \partial_t \Phi_j^* - \partial_j \Phi_t^* &= 0 \end{aligned} \quad (35)$$

and the complex conjugate equations. These are Eqs. (19), (21) and (24).

4. Solution of the canonical equations

It is useful to write Eqs. (15), (20) and (24) in the three-dimensional vector notation. For this purpose we write:

$$\begin{aligned} F^e &= \begin{pmatrix} F^0 \\ \mathbf{F} \end{pmatrix}, \quad \mathbf{F} \equiv (F^1, F^2, F^3) \equiv (F_x, F_y, F_z), \\ \Phi^e &= \begin{pmatrix} \Phi^0 \\ \Phi \end{pmatrix}, \quad \Phi \equiv (\Phi^1, \Phi^2, \Phi^3) \equiv (\Phi_x, \Phi_y, \Phi_z). \end{aligned} \quad (36)$$

Eqs. (15) and (20) then read

$$\begin{aligned}\partial_0 F^0 + \operatorname{div} \mathbf{F} &= 0, \\ \partial_0 \mathbf{F} + \operatorname{grad} F^0 &= 0, \\ \operatorname{rot} \mathbf{F} &= 0,\end{aligned}\tag{37}$$

$$\begin{aligned}F^0 &= \partial_0 \Phi^0 + \operatorname{div} \Phi, \\ \mathbf{F} &= -\operatorname{grad} \Phi^0 - \partial_0 \Phi, \\ \operatorname{rot} \Phi &= 0.\end{aligned}\tag{38}$$

A solution of Eqs. (37) can easily be found. Due to the third of these equations it follows

$$\mathbf{F} = -\operatorname{grad} f.\tag{39}$$

After substitution into (37) one gets

$$F^0 = \partial^0 f,\tag{40}$$

$$\square f = 0.$$

Thus, we have the solution of Eqs. (37):

$$F^a = \partial^a f\tag{41}$$

where f satisfies the equation

$$\partial_a \partial^a f = 0.\tag{42}$$

For this solution equations for Φ^a read

$$\begin{aligned}\partial^0 f &= \partial_0 \Phi^0 + \operatorname{div} \Phi, \\ -\operatorname{grad} f &= -\operatorname{grad} \Phi^0 - \partial_0 \Phi, \\ \operatorname{rot} \Phi &= 0.\end{aligned}\tag{43}$$

A particular solution of these equations is

$$\begin{aligned}\Phi^0 &= f, \\ \Phi^i &= 0.\end{aligned}\tag{44}$$

Therefore, a solution of the system (37) and (38) is

$$F^e = \begin{pmatrix} \partial^0 f \\ \partial^1 f \\ \partial^2 f \\ \partial^3 f \end{pmatrix}, \quad \Phi^e = \begin{pmatrix} f \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (45)$$

with $\square f = 0$.

This solution has a characteristic that the spatial momentum components have zero conjugate components. It causes some difficulties, particularly in the quantum theory.

There is another solution of the system (37) and (38) which is also of physical interest.

The first of Eqs. (37) suggests

$$\begin{aligned} F^0 &= \operatorname{div} \mathbf{g}, \\ \mathbf{F} &= -\partial_0 \mathbf{g}. \end{aligned} \quad (46)$$

For this F^e the first of Eqs. (37) is identically fulfilled. The rest of Eqs. (37) give

$$\begin{aligned} -\partial_0 \partial^0 \mathbf{g} + \operatorname{rot} \operatorname{rot} \mathbf{g} + \Delta \mathbf{g} &= 0, \\ \partial_0 \operatorname{rot} \mathbf{g} &= 0. \end{aligned}$$

The second equation leads to

$$\operatorname{rot} \mathbf{g} = \text{const in time}$$

and due to the physical reasons

$$\operatorname{rot} \mathbf{g} = 0. \quad (47)$$

The first equation then is

$$\square \mathbf{g} = 0. \quad (48)$$

F^e given by (46) is a solution of Eqs. (37) if \mathbf{g} satisfies Eqs. (47) and (48). For this solution equations for Φ^e are

$$\begin{aligned} \operatorname{div} \mathbf{g} &= \partial_0 \Phi^0 + \operatorname{div} \Phi, \\ -\partial_0 \mathbf{g} &= -\operatorname{grad} \Phi^0 - \partial_0 \Phi, \\ \operatorname{rot} \Phi &= 0. \end{aligned} \quad (49)$$

A particular solution of these equations is

$$\begin{aligned} \Phi^0 &= 0, \\ \Phi &= \mathbf{g}. \end{aligned} \quad (50)$$

Therefore, the solution of Eqs. (37) and (38) in this case reads

$$F^e = \begin{pmatrix} \text{div } \mathbf{g} \\ -\partial_0 g^1 \\ -\partial_0 g^2 \\ -\partial_0 g^3 \end{pmatrix} \quad \Phi^e = \begin{pmatrix} 0 \\ g^1 \\ g^2 \\ g^3 \end{pmatrix} \quad (51)$$

with $\square \mathbf{g} = 0$, $\text{rot } \mathbf{g} = 0$.

In contrast to the solution (45) this solution has the time component of Φ^e equal to zero.

Due to linearity of Eqs. (37) and (38) a solution is also a linear combination of (45) and (51). We take it in the form:

$$F^e = \begin{pmatrix} \partial^0 f + \text{div } \mathbf{g} \\ -\text{grad } f - \partial_0 \mathbf{g} \end{pmatrix}, \quad \Phi^e = \begin{pmatrix} f \\ \mathbf{g} \end{pmatrix} \quad (52)$$

with

$$\begin{aligned} \square f &= 0, \\ \square \mathbf{g} &= 0, \\ \text{rot } \mathbf{g} &= 0, \end{aligned}$$

This solution has all good properties.

It is worthwhile to look for the solution of Eqs. (37) and (38) on the Lagrange's scheme.

The Lagrange's equations for Φ_a^* read

$$\frac{\partial \mathcal{L}}{\partial \Phi_a^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi_a^*)} = 0$$

$$\square \Phi^0 = 0, \quad a = 0, \quad (53)$$

$$-\partial_0 \partial^0 \Phi + \Delta \Phi + \text{rotrot } \Phi = 0, \quad a = 1, 2, 3.$$

Taking $\text{rot } \Phi = 0$, in accordance to (37) and (38), it becomes

$$\begin{aligned} \square \Phi^0 &= 0, \\ \square \Phi &= 0, \\ \text{rot } \Phi &= 0. \end{aligned} \quad (54)$$

Φ^0 and Φ are independent. For the solution of (54) F^e is given by (38). These are just the solution (52) with $\Phi^0 = f$ and $\Phi = \mathbf{g}$.

The complex conjugate solutions are evident.

5. Energy momentum vector

The energy-momentum tensor

$$T_{\mu}^{\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_{\nu} \Phi_a)} \partial_{\mu} \Phi_a + \frac{\partial \mathcal{L}}{\partial (\partial_{\nu} \Phi_a^*)} \partial_{\mu} \Phi_a^* - \delta_{\mu}^{\nu} \mathcal{L} \quad (55)$$

for the Lagrangian (8) reads

$$\begin{aligned} T^{\mu\nu} = & k \{ F_a^* \varphi^a \partial^{\mu} \Phi^{\nu} - F_a^* \varphi^{\nu} \partial^{\mu} \Phi^a + F^{\nu*} \varphi^a \partial^{\mu} \Phi_a + \\ & + F_a \varphi^{a*} \partial^{\mu} \Phi^{\nu*} - F_a \varphi^{\nu*} \partial^{\mu} \Phi^{a*} + F^{\nu} \varphi^{a*} \partial^{\mu} \Phi_a^* \} - g^{\mu\nu} \mathcal{L}. \end{aligned} \quad (56)$$

From here the constant of motion is

$$\begin{aligned} P^{\mu} = & \text{const} \int \{ \varphi^a F_a^* \partial^{\mu} \Phi^0 - \varphi^0 F_a^* \partial^{\mu} \Phi_a + F^{0*} \varphi^a \partial^{\mu} \Phi_a + \\ & + \varphi^{a*} F_a \partial^{\mu} \Phi^{0*} - F^a \varphi^{0*} \partial^{\mu} \Phi_a^* + F^0 \varphi^{a*} \partial^{\mu} \Phi_a^* - g^{\mu\nu} F_a^* F^a \} d^3 x. \end{aligned} \quad (57)$$

For φ^a given by (17) it is

$$P^{\mu} = \text{const} \int F_a^* \partial^{\mu} \Phi_a + F_a \partial^{\mu} \Phi_a^* - g^{\mu 0} F_a^* F^a d^3 x \quad (58)$$

and

$$\begin{aligned} P^0 = & - \text{const} \int (F_a^* F^a - F^{0*} \partial^l \Phi_l - F_l^* \partial^l F^0 - F^0 \partial^l \Phi_l^* - F_l \partial^l \Phi^{0*}) d^3 x, \\ P^l = & \text{const} \int (F_a^* \partial^l \Phi_a + F_a \partial^l \Phi_a^*) d^3 x, \end{aligned} \quad (59)$$

where we have used Eq. (20).

We evaluate this four-vector for the solution (53):

$$\begin{aligned} P^0 = & \int (F_1^* F_1^a - F_2^* F_2^a) d^3 x, \\ P^l = & \int \{ (F_1^{0*} F_1^l + F_1^0 F_1^{l*}) - (F_2^{0*} F_2^l + F_2^0 F_2^{l*}) \} d^3 x, \end{aligned} \quad (60)$$

where

$$F_1^a = \partial^a f, \quad F_2^a = \begin{pmatrix} \text{div } \mathbf{g} \\ -\partial_0 \mathbf{g} \end{pmatrix}. \quad (61)$$

Let us notice that for $\mathbf{g} = 0$ this four-vector becomes

$$\begin{aligned} P^0 = & \int (F_0^* F^0 - F_l^* F^l) d^3 x, \quad (F_1^a = F^a), \\ P^l = & \int (F^{0*} F^l + F^0 F^{l*}) d^3 x. \end{aligned}$$

6. Angular momentum tensor

The angular momentum density tensor

$$M_a^\gamma{}_\beta = x_\beta T_a^\gamma - x_a T_\beta^\gamma + S_a^\gamma{}_\beta, \tag{62}$$

with the spin part

$$S_a^\gamma{}_\beta = - \frac{\partial \mathcal{L}}{\partial (\partial_\gamma \Phi_\eta)} A_{\eta\alpha\beta} \zeta \Phi_\zeta - \frac{\partial \mathcal{L}}{\partial (\partial_\gamma \Phi_\eta^*)} A_{\eta\alpha\beta} \zeta \Phi_\zeta^* \tag{63}$$

and

$$A_{\eta\alpha\beta}{}^\gamma = g_{\eta\alpha} \delta_\beta^\zeta - g_{\eta\beta} \delta_\alpha^\zeta, \quad \alpha \leq \beta, \tag{64}$$

for the Lagrangian (8) reads

$$\begin{aligned} M_a^\gamma{}_\beta &= x_\beta T_a^\gamma - x_a T_\beta^\gamma - \\ &- k \{ \delta_a^\gamma (F^{\alpha*} \varphi_\alpha \Phi_\beta + F^\alpha \varphi_\alpha^* \Phi_\beta^*) - \delta_\beta^\gamma (F^{\alpha*} \varphi_\alpha \Phi_a + F^\alpha \varphi_\alpha^* \Phi_a^*) + \\ &+ (\varphi_\alpha^* \Phi_\beta^* - \varphi_\beta^* \Phi_\alpha^*) F^\gamma + (F_\beta^* \varphi_\alpha - F_\alpha^* \varphi_\beta) \Phi^\gamma + \\ &+ (\varphi_\alpha \Phi_\beta - \varphi_\beta \Phi_\alpha) F^{\gamma*} + (F_\beta \varphi_\alpha^* - F_\alpha \varphi_\beta^*) \Phi^{\gamma*} \}. \end{aligned} \tag{65}$$

The spin momentum is then

$$\begin{aligned} S_{\alpha\beta} &= \int S_{\alpha\beta}{}^0 d^3x = k \int \{ g_{0\beta} (F^{\alpha*} \varphi_\alpha \Phi_0 + F^\alpha \varphi_\alpha^* \Phi_0^*) - g_{0\alpha} (F^{\alpha*} \varphi_\alpha \Phi_\beta + \\ &+ F^\alpha \varphi_\alpha^* \Phi_\beta^*) + (F_\alpha^* \varphi_\beta - F_\beta^* \varphi_\alpha) \Phi_\alpha + (F_\alpha^* \varphi_0 - F_0^* \varphi_\alpha) \Phi_\beta + \\ &+ (F_0 \varphi_\beta^* - F_\beta \varphi_0^*) \Phi_\alpha^* + (F_\alpha \varphi_0^* - F_0 \varphi_\alpha^*) \Phi_\beta^* \} d^3x. \end{aligned} \tag{66}$$

The three dimensional spin pseudo-vector is defined by

$$S_l = \varepsilon_{ljk} S_{jk}. \tag{67}$$

For φ^α given by (17) it is

$$\begin{aligned} S_1 &= 2k \int (-F_3^* \Phi_2 + F_2^* \Phi_3 - F_3 \Phi_2^* + F_2 \Phi_3^*) d^3x, \\ S_2 &= 2k \int (-F_1^* \Phi_3 + F_3^* \Phi_1 - F_1 \Phi_3^* + F_3 \Phi_1^*) d^3x, \\ S_3 &= 2k \int (-F_2^* \Phi_1 + F_1^* \Phi_2 - F_2 \Phi_1^* + F_1 \Phi_2^*) d^3x \end{aligned} \tag{68}$$

or

$$\mathbf{S} = 2k \int (\mathbf{F}^* \times \Phi + \mathbf{F} \times \Phi^*) d^3x \tag{69}$$

and for the solution (52)

$$\mathbf{S} = -2k \int (\partial_0 \mathbf{g}^* \times \mathbf{g} + \partial_0 \mathbf{g} \times \mathbf{g}^*) d^3x \quad (70)$$

due to $\text{rot } \mathbf{g} = 0$, $\mathbf{S} = 0$.

7. Connection with standard scalar field

In the standard field theory a scalar field is described by a scalar function satisfying d'Alembert's or Klein-Gordon's equation. Recently it has been shown⁵⁾ that there is no correct canonical formulation of such a field with zero mass.

In Section 4 it has been found that the field equation of the vector field F^e among others has the following solution:

$$F^e = \partial^e f, \quad \Phi^e = \begin{pmatrix} f \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (71)$$

where f satisfies the equation

$$\partial_a \partial^a f = 0.$$

This solution contains only one scalar function f . Therefore, this is the case which corresponds to the standard massless scalar field.

The solution (71), as it was mentioned in Section 4, has zero spatial components of the canonically conjugate variables to the field components F^e . Due to this reason the standard quantization is not possible. It indicates that such a field is not a physical object.

8. Conclusion

The canonical formulation of the massless vector field satisfying the first order differential equations is constructed. For the field given by Eqs. (15) the conjugate variables are given as particular solutions of Eqs. (20) for the selection (17).

The field is described by two independent functions.

The standard scalar massless field corresponds to the vector field with $\mathbf{g} = 0$. The quantum theory doesn't allow $\mathbf{g} = 0$.

Other aspects of this field will be considered in subsequent papers.

References

- 1) Bresteckij V. B., Lifšic E. M., Pitaevskij L. P., *Relativistskaja kvantovaja teorija I*, Moskva 1968;
- 2) K. Nishijima, *Field and particles*, W. A. Benjamin, Inc., New York 1969;
- 3) J. Brana and K. Ljolje, FIZIKA 10 (1978) 85;
- 4) K. Ljolje, The canonical formalism for particles and fields, prepared for press;
- 5) K. Ljolje and S. Vobornik, FIZIKA 11 (1979) 171.

KANONSKA FORMULACIJA ZA VEKTORSKA POLJA KOJA ZADOVOLJAVAJU DIFERENCIJALNE JEDNADŽBE PRVOG REDA

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Izgrađena je ispravna kanonska formulacija vektorskog polja koje zadovoljava diferencijalne jednačbe prvog reda. Uz prikladan izbor konstantnog četverovektora φ^a određene su kanonske jednačbe za impulse i za varijable. Polje je općenito opisano sa dvije proizvoljne funkcije. Skalarno polje standardne teorije odgovara slučaju $\mathbf{g} = 0$. Izračunat je četverovektor energije-impulsa i analizirane su neke posljedice novog formalizma.