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NUMERICAL SOLUTIONS OF A RELATIVISTIC SINGLE-PARTICLE BOUND-STATE EQUATION

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We give a short derivation of a relativistic single-particle two-body bound-state integral equation the kernel of which is symmetric, fully continuous and positive definite. We discuss a numerical solution of the eigenvalue problem based on the analytical structure of the kernel and different binding energies. We have estimated the error in the numerical calculation of the first 50 approximate eigenvalues and the corresponding eigenfunctions. We give a graphical representation of the first few eigenfunctions.

1. Introduction

Even though much progress has been made in the last thirty years in the understanding of nuclear forces, no one has yet been able to construct a simple relativistic two-body description of the deuteron. Following the work of Gross[1], several authors[2] have studied the equations of the Bethe-Salpeter type when one of the particles in the two-body bound state is restricted to stay on its mass shell. The result is a covariant three-dimensional single-particle equation. The dynamics of the second particle is greatly simplified by keeping it effectively on its mass shell.

The contributions omitted from the Bethe-Salpeter equation by restricting one particle to its mass shell can be reintroduced by enlarging the number of terms in the kernel of the corresponding integral equation. The actual kernel, which now plays the role of a potential, can be calculated to any desired accuracy from whatever quantum field theory that underlies the interaction. In this respect, the proposed equation is equal in rigour to that of Bethe-Salpeter and is simple to apply. It is usually applied when one of the two constituents in a bound state stays very near the mass shell, i.e. either when the ratio of constituent mass is large or when the binding is weak.

It is clearly possible to write down an infinite set of equations similar to those discussed above which differ only in their off-shell behaviour.

An important advantage of this approach is that there exists a Coulomb-like kernel for which an exact analytical solution is known; all other cases require a special numerical handling.

Following the work of Gross and others, we try in this paper to find a numerical solution of a bound-state equation when a deuteron-nucleon vertex is composed of spinless particles. We restrict one of the nucleons to its mass shell everywhere in the equation. This prescription proves to be essential for the reduction of the Bethe-Salpeter equation to a one-dimensional integral equation. The main purpose of our investigation is to discuss the eigenvalue problem and its numerical solution. The usual physical interpretation can be given only for the solution with the largest eigenvalue; it corresponds to the eigenfunction which is positive everywhere in the region of the definition of the integral equation. All other eigenvalues are also positive and decreasing towards zero in magnitude. They correspond to eigenfunctions which show oscillatory behaviour and have nodes.

The plan of the paper is as follows: In Sec. 2 we give a short derivation of a single-particle two-body bound-state integral equation the kernel of which we analyze in Sec. 3. In Sec. 4 we discuss the numerical solution of the eigenvalue problem. In Sec. 5 we present the main points of the paper and draw some conclusions.

2. *Integral equation*

Following the prescription of Gross[1], we can write an integral equation for a scalar deuteron-nucleon vertex when one of the nucleons is on its mass shell; the equation is of the form

$$\Gamma(t) = \frac{q^2}{(2\pi)^3} \int \frac{d^3 k}{2E_k} \frac{\Gamma(k^2)}{(k^2 - m^2)\,[(k-p)^2 - \mu^2]}, \tag{2.1}$$

where

$p^2 = t = (m - B)^2 - \zeta,$

B = binding energy $= 2m - M$ and

M = mass of the scalar deuteron.

Figure 1 shows a graphical representation of Eq. (2.1). In the rest frame of the deuteron, $p_D \equiv (M, \vec{0})$, we can perform the angular integration in Eq. (2.1).

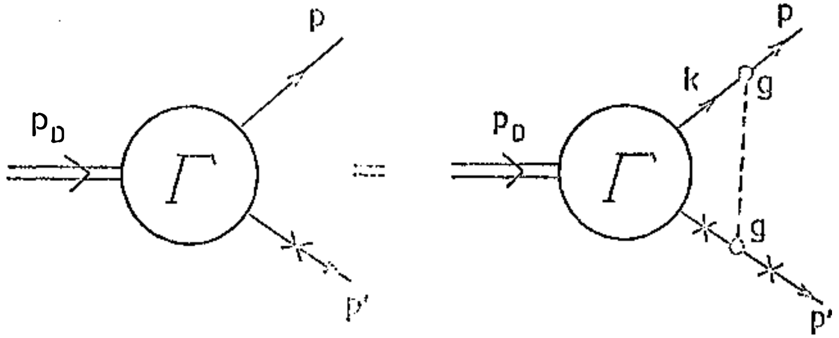

Fig. 1. Graphical representation of Eq. (2.1) for the bound-state vertex function $\Gamma(p)$. The symbol x indicates that the nucleon is on its mass shell.

After a suitable symmetrization of the kernel for $t < m^2$, we obtain the one-dimensional integral equation

$$\sigma \Phi(\zeta) = K\Phi \equiv \int_0^\infty K(\zeta, \zeta') \Phi(\zeta') \, \mathrm{d}\zeta', \tag{2.2}$$

where

$$\Phi(\zeta) = \left[\frac{\lambda^{1/2}(\zeta)}{2M(MB + \zeta)}\right]^{1/2} \Gamma(\zeta),$$

$$\lambda(\zeta) = \zeta(4Mm + \zeta), \tag{2.3}$$

$$\sigma = 4\pi/g^2,$$

$$K(\zeta, \zeta') = K(\zeta', \zeta) = \frac{1}{\pi} \frac{Q_0(\beta)}{[\lambda^{1/2}(\zeta) \lambda^{1/2}(\zeta') (MB + \zeta)(MB + \zeta')]^{1/2}}, \tag{2.4}$$

$$\beta = \frac{\zeta\zeta' + 2Mm(\zeta + \zeta') + 2M^2 \mu^2}{\lambda^{1/2}(\zeta) \lambda^{1/2}(\zeta')}, \tag{2.5}$$

and $Q_0(\beta)$ is the Legendre function of the second kind.

In the nonrelativistic limit, the bound-state wave function $\varphi_D(\zeta)$ is related to $\Gamma(\zeta)$ as

$$\varphi_D(\zeta) = \frac{M}{m} \frac{\Gamma(\zeta)}{MB + \zeta}. \tag{2.6}$$

3. Mathematical considerations

Equation (2.2) is known as a homogeneous integral equation with a symmetric kernel. The basis property of symmetric kernels is the identity[5)]

$$(K\Phi, \Psi) = (\Phi, K\Psi), \tag{3.1}$$

where $(K\Phi, \Psi)$ denotes the quadratic form

$$(K\Phi, \Psi) = \int_0^\infty \int_0^\infty K(\zeta, \zeta') \, \Phi(\zeta') \, \Psi(\zeta) \, \mathrm{d}\zeta \, \mathrm{d}\zeta' \tag{3.2}$$

if the kernel $K(\zeta, \zeta')$ is real as in our case.

In this section we investigate the eigenvalue problem for the linear operator $K\Phi = \sigma\Phi$, the action of which is given by Eq. (2.2).

The kernel $K(\zeta, \zeta')$ is a continuous symmetric function of ζ, ζ', defined on $[0, +\infty) \times [0, +\infty)$. To show this, we first examine the double integral

$$\| K \|^2 = \int_0^\infty \int_0^\infty K^2(\zeta, \zeta') \, \mathrm{d}\zeta \, \mathrm{d}\zeta'. \tag{3.3}$$

Let us split the integration region in (3.3) into three separate parts:

$$\| K \|^2 = \left(\int_0^A \int_0^A + 2 \int_A^\infty \int_0^A + \int_A^\infty \int_A^\infty \right) K^2(\zeta, \zeta') \, \mathrm{d}\zeta \, \mathrm{d}\zeta'.$$

In the region $0 \leq \zeta, \zeta' \leq A$, we find that

$$\lim_{\zeta \to 0} K(\zeta, \zeta') = 0 \tag{3.4}$$

$$\lim_{\zeta', \zeta \to 0} K(\zeta, \zeta') = 0.$$

We conclude that $K(\zeta, \zeta')$ is continuous on $[0, A] \times [0, A]$ and bounded, i.e.

$$\int_0^A \int_0^A K^2(\zeta, \zeta') \, \mathrm{d}\zeta \, \mathrm{d}\zeta' < \infty. \tag{3.5}$$

In the region $\underline{0 \leq} \zeta \leq A$, $A \leq \zeta' < \infty$, we find for A large enough the inequality

$$\int_0^A \int_A^\infty K^2(\zeta, \zeta') \, \mathrm{d}\zeta \, \mathrm{d}\zeta' < \mathrm{C} \frac{\ln^2 A}{A}, \tag{3.6}$$

where C is a finite constant.

We conclude that

$$\lim_{A\to\infty} \int_0^A \int_A^\infty K^2(\zeta, \zeta')\, d\zeta\, d\zeta' = 0. \tag{3.7}$$

In the region $A \leq \zeta, \zeta' < \infty$, we find for A large enough the inequality

$$\int_A^\infty \int_A^\infty K^2(\zeta, \zeta')\, d\zeta\, d\zeta' < C' \frac{\ln^2 A}{A^2}, \tag{3.8}$$

where C' is also a finite constant.

We conclude that

$$\lim_{A\to\infty} \int_A^\infty \int_A^\infty K^2(\zeta, \zeta')\, d\zeta\, d\zeta' = 0. \tag{3.9}$$

From the fact that the double integral (3.3) is finite and, moreover, $(K\Phi, \Phi) \geq 0$, we conclude that the kernel $K(\zeta, \zeta')$ is fully continuous and positive definite. It has a countable set of eigenvalues in the region $[0, +\infty) \times [0, +\infty)$; the accumulation point in the spectrum of K is equal to zero. The largest eigenvalue, σ_{max}, is obtained as a maximum value of $(K\Phi, \Phi)$ under the condition that

$$\int_0^\infty d\zeta \Phi(\zeta)^2 = 1. \tag{3.10}$$

4. *Numerical solution*

In this section we analyze a method for obtaining an approximate solution of the integral equation (2.2).

We split the integral

$$K\Phi = \int_0^\infty K(\zeta, \zeta')\, \Phi(\zeta')\, d\zeta' \tag{4.1}$$

into two parts,

$$K\Phi = \int_0^{10} K(\zeta, \zeta')\, \Phi(\zeta')\, d\zeta' + \int_{10}^\infty K(\zeta, \zeta')\, \Phi(\zeta')\, d\zeta', \tag{4.2}$$

and replace it by a finite sum according to one of the approximate formulas of mechanical quadrature, i.e.

$$K\Phi = \sum_{k=1}^{34} \omega_k K(\zeta, \zeta_k)\Phi(\zeta_k) + \sum_{k=35}^{50} \omega_k K(\zeta, \zeta_k)\Phi(\zeta_k). \tag{4.3}$$

We have split the integral in Eq. (4.1) into two parts because the behaviour of the first derivative of the kernel $K(\zeta, \zeta')$ at $\zeta = 0$ is unsatisfactory.

We compute the first integral in (4.2) using Gauss' quadrature formula[6] with $n = 36$. The error is bounded by $10^{-117} B_{68}$.

We evaluate the second integral in (4.2) using the Gauss-Laguerre quadrature formula[6] with $n = 16$. The error is smaller than $6.4 \times 10^{-9} B_{30}$. Here B_l denotes the upper bound on the l-th derivative of the function under the integral sign. Note that B_{30} is small on the interval $[10, +\infty)$.

In the approximate equation

$$\sigma\Phi(\zeta) = \sum_{k=1}^{50} \omega_k K(\zeta, \zeta_k)\Phi(\zeta_k), \tag{4.4}$$

we replace ζ by $\zeta_1, \zeta_2, \dots, \zeta_{50}$, thus obtaining a system of linear algebraic equations for the unknowns $\Phi(\zeta_1), \dots, \Phi(\zeta_{50})$:

$$\sigma\Phi(\zeta_i) = \sum_{k=1}^{50} \omega_k K(\zeta_i, \zeta_k)\Phi(\zeta_k), \qquad i = 1, 2, \dots, 50. \tag{4.5}$$

Since $\omega_k \geq 0$, we can easily transform Eq. (4.5) into an equation with a symmetric matrix. Multiplying both sides of Eq. (4.5) by $(\omega_i)^{1/2}$ and introducing new unknown functions $\Psi(\zeta_i) \equiv (\omega_i)^{1/2}\Phi(\zeta_i)$, we arrive at the eigenvalue problem for the system of linear equations

$$\sigma\Psi(\zeta_i) = \sum_{k=1}^{50} (\omega_i)^{1/2} K(\zeta_i, \zeta_k)(\omega_k)^{1/2}\Psi(\zeta_k) \tag{4.6}$$

the matrix of which is symmetric.

5. Conclusions

We have obtained a solution of the eigenvalue problem for the kernel (2.5) for particular values of the masses M, m and μ. We have chosen

$$M = 1.875,\ 1.870 \text{ and } 1.860 \text{ GeV}, \tag{5.1}$$

$$m = 0.939 \text{ GeV},$$

$$\mu = 0.139 \text{ GeV}.$$

Before drawing a conclusion on the eigenvalue problem, let us examine the behaviour of the wave function $\Phi(\zeta)$ in two extreme regions, $\zeta \to 0$ and $\zeta \to \infty$.

For $\zeta \to 0$, we find the following behaviour:

$$\Phi(\zeta) \to C \frac{\zeta^{1/4}}{(MB + \zeta)^{1/2}}, \quad \zeta \to 0. \tag{5.2}$$

The function $\zeta^{1/4}(MB + \zeta)^{-1/2}$ has its maximum at $\zeta = MB$. We may conclude that

$$\Phi(0) = 0. \tag{5.3}$$

If we take the limit $\zeta \to \infty$, careful examination of the kernel shows that $\Phi(\zeta)$ tends to zero, or more precisely

$$\Phi(\zeta) \to C_1 \frac{\ln \zeta}{\zeta} + C_2 \frac{1}{\zeta}; \quad \zeta \to \infty. \tag{5.4}$$

Since the kernel $K(\zeta, \zeta')$ is square-integrable and satisfies Eq. (3.5), the constants $C_{1,2}$ exist.

In this paper we have studied the eigenvalue problem and its numerical solutions for the kernel $K(\zeta, \zeta')$ given by Eq. (2.5). We may draw the following conclusions:

1. The kernel $K(\zeta, \zeta')$ is symmetric, fully continuous and positive definite in the region $[0, +\infty) \times [0, +\infty)$.

2. The spectrum of $K(\zeta, \zeta')$ is discrete and nondegenerate; it is of the type $\sigma_0 > \sigma_1 > \sigma_2 \dots; \quad \sigma_k \to 0$ as $k \to \infty$.

3. The largest eigenvalue σ_0 is obtained as a maximum value of $(K\Phi, \Phi$ under the condition $(\Phi, \Phi) = 1$, i.e.

$$Max\,(K\Phi, \Phi) = \| K \| = \sigma_0; \quad (\Phi, \Phi) = 1.$$

When we replace the integral equation (2.2) by a finite 50×50 matrix equation (4.6), we obtain the first 50 approximate eigenvalues.

The error is

$$|\sigma_0 - \sigma_0^{(50)}| = \varepsilon^{(50)}. \tag{5.5}$$

According to Eq. (4.3), the order of magnitude of the error is given by

$$\varepsilon^{(50)} \sim 10^{-9} (B_{30} + 10^{-108} B_{68}). \tag{5.6}$$

To this error we should also add the error introduced through the numerical diagonalization of the 50 × 50 matrix; the order of magnitude of the latter error is smaller than 10^{-3} in our case.

We have also noted that the eigenvalues approach zero very rapidly. To illustrate this, we list the magnitudes of some of them for $M = 1.875$ GeV:

$$\sigma_0 = 1.519,$$

$$\sigma_1 = 0.453,$$

$$\sigma_2 = 0.198,$$

$$\sigma_3 = 0.101,$$

$$\sigma_4 = 0.349 \times 10^{-1},$$

$$\vdots$$

$$\sigma_8 = 0.979 \times 10^{-2},$$

$$\vdots$$

$$\sigma_{15} = 0.769 \times 10^{-3},$$

$$\vdots$$

$$\sigma_{21} = 0.780 \times 10^{-4},$$

$$\vdots$$

$$\sigma_{26} = 0.793 \times 10^{-5},$$

$$\vdots$$

$\sigma_{49} \cong 10^{-77}$. (This value is beyond the computer's accuracy.)

The number of nodes (zero values of the eigenfunction, excluding those which lie at the ends $\zeta = 0$ and $\zeta = \infty$) increases regularly with eigenvalue, increasing by unity in going from one eigenvalue to that lying immediately below in magnitude. The eigenfunction with the largest eigenvalue has no nodes; it is positive everywhere in the region $[0, +\infty)$ of the variable ζ.

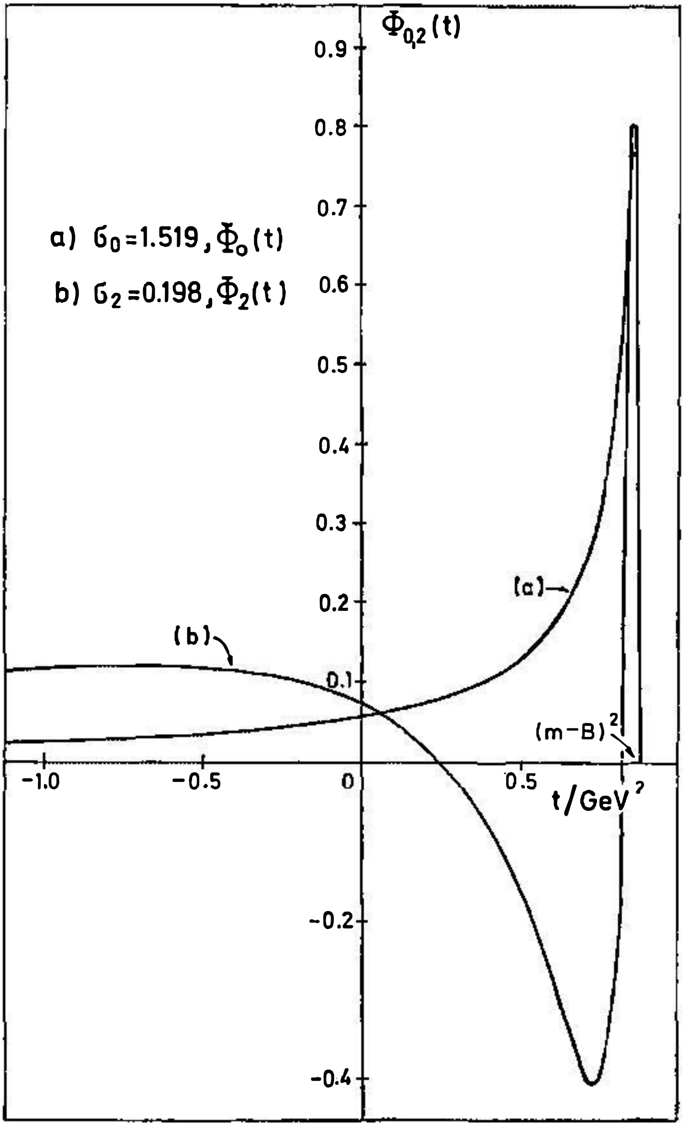

Fig. 2. The deuteron bound-state wave functions in momentum space for the eigenvalues (a) $\sigma_0 = 1.519$, (b) $\sigma_2 = 0.198$, and mass $M = 1.875$ GeV.

For the sake of the reader, we show in Fig. 2 the first two eigenfunctions when $M = 1.875$ GeV. As we increase the binding energy B (lower the mass M) ,we notice that the values of all eigenvalues are decreased. The general shape of the corresponding eigenfunctions remains, however, the same, except for a slight shift to the left. We show this in Fig. 3.

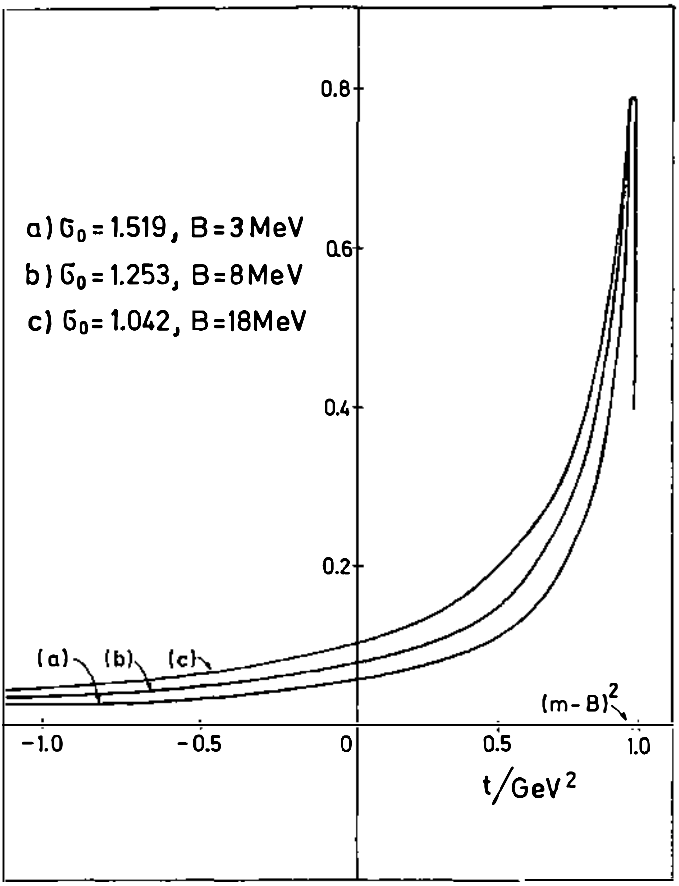

Fig. 3. Bound-state wave functions for increasing binding energies:

(a) $\sigma_0 = 1.519$, $M = 1.875$ GeV,

(b) $\sigma_0 = 1.253$, $M = 1.870$ GeV,

(c) $\sigma_0 = 1.042$, $M = 1.860$ GeV.

References

1) F. Gross, Phys. Rev. **186** (1969) 1448;

2) M. H. Partovi and E. L. Lomon, Phys. Rev. **D2** (1970) 1999; M. Chemtob, J. W. Durso and D. O. Riska, Nucl. Phys. **B 38** (1972) 141; R. M. Woloshyn and A. D. Jackson, Nucl. Phys **B 64** (1973) 269; F. Gross, Phys. Rev. **D 10** (1974) 223; W. W. Buck and F. Gross, A Family of Relativistic Deuteron Wave Functions, 1979 (preprint);

3) C. Fronsdal and R. W. Huff, Phys. Rev. **D 3** (1971) 933;

4) A. A. Logunov and A. N. Tavkhelidze, Nuovo Cimento **29** (1963) 380; R. Blankenbecler and R. Sugar, Phys. Rev. **142** (1966) 1051; R. Blankenbecler and R. Sugar, Phys. Rev. **183** (1969) 1387; H. Grotch and D. R. Yennie, Rev. Mod. Phys. **41** (1969) 350; C. Itzykson, V. G. Ka-

dyshevsky and I. T. Todorov, Phys. Rev. **D 1** (1970) 3247; M. Martinis, Lett. Nuovo Cimento **1** (1971) 843; G. P. Lepage, Phys. Rev. **A 16** (1977) 863;
5) S. G. Mikhlin, Integral Equations (Pergamon Press, London, 1957);
6) M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, New York 1965).

NUMERIČKA RJEŠENJA RELATIVISTIČKE JEDNADŽBE ZA DVOČESTIČNO VEZANO STANJE

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U radu je prikazan izvod relativističke integralne jednadžbe za dvočestično vezano stanje, čija je jezgra simetrična, kontinuirana i pozitivno definitna. Diskutirano je numeričko rješenje problema vlastitih vrijednosti za razne vrijednosti energije veze.