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A SIMPLE RULE ON PARABOLIC REGGE TRAJECTORIES FOR PROTON-NEUTRON MULTIPLETS IN ODD-ODD NUCLEI

(I) EXCHANGE OF A QUADRUPOLE PHONON

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An analytical expression for the energies of the states of the proton-neutron multiplet $(j_p j_n) I = |j_p - j_n|, \dots, j_p + j_n$ owing to the exchange of the quadrupole phonon is derived. Detailed discussion of the resulting parabola and illustrative applications are presented.

1. *Introduction*

A systematic approach to even-even, odd-even and odd-odd nuclei based on the spherical quadrupole phonons$^{1)}$ offers an useful framework for the systematic study of nuclear structure of medium and heavy nuclei$^{2,3)}$.

For even-even nuclei far away from the closed shells a complete average over the shell-model fermion degrees of freedom is a reasonable approximation. In a consistent way such a nucleus can be described by the Truncated quadrupole phonon model (*TQM*), which is based on the approximate $SU(6)$ symmetry for the quadrupole phonons$^{4,5)}$. *TQM* corresponds to the Holstein — Primakoff representation of the $SU(6)$ group. *TQM* enables a smooth transition to the deformed limit ($SU(3)$ limiting symmetry), both for even-even$^{6,7)}$ and odd-even nuclei$^{7)}$.

However, in the situation not very far from the closed shells or with odd shell-model particles, important shell-model degrees of freedom appear explicitely. In this situation we describe the nucleus by coupling a few pronounced shell-model particles to the quadrupole vibration. The particles explicitely included in the quantum — mechanical basis are referred to as cluster, and the model of coupling selected clusters to the quadrupole vibration will be called the cluster — vibration model (*CVM*). For different system *CVM* was first introduced in Refs. 8—10. In the *CVM*, even-even, odd-even and odd-odd nuclei differ only in the composition of the cluster[8–12]. In the case of large cluster, we describe them by introducing a *BCS* superfluid approximation with particle-number projection and the model is referred to as the Quasicluster-vibration model (*QCVM*)[23,24].

2. *The present theoretical status of odd-odd nuclei*

The coupling of odd protons and neutrons in odd-odd nuclei has been studied extensively, both empirically and theoretically[25–41]. Complex calculations have been performed using different residual forces[26–31,33,34,36–38]. The basic assumption was that the wave function is a simple vector-coupled product of the wave functions of the two odd groups; the low-lying levels in odd-odd nuclei should result from combinations of the lowest configurations in the adjacent odd-proton and odd neutron nuclei.

Extensive calculations have been performed by using for residual proton-neutron interaction a delta-force with spin-dependent component [27,28]:

$$V = - V_0 \left[(1 - \alpha) + \alpha \vec{\sigma}_1 \cdot \vec{\sigma}_2\right] \delta(\vec{r}_1 - \vec{r}_2),$$

and also by using a general form[27]

$$V = V_0 (r_1, r_2, \cos \omega_{12}) + \vec{\sigma}_1 \cdot \vec{\sigma}_2 \, V_1 (r_1, r_2, \cos \omega_{12}).$$

Other forces and multipole decomposition have been also studied[26,31,33,34].

Inspecting the empirical evidence and from the numerical results of calculations, revised Nordheim's three rules for the ground-state spin have been formulated, one strong and two revised weak rules[25,30]:

Strong rule: $J = | j_1 - j_2 |$ for $j_1 = l_1 \pm \frac{1}{2}$, $j_2 = l_2 \mp \frac{1}{2}$ and $| j_1 >$ and $| j_2 >$ being both particle-like or both hole-like.

First weak rule: $J = j_1 + j_2$ or $| j_1 - j_2 |$ for $j_1 = l_1 \mp \frac{1}{2}$, $j_2 = l_2 \pm \frac{1}{2}$ and $| j_1 >$ and $| j_2 >$ being both particle-like or both hole-like.

Second weak rule: There is a tendency $J = j_1 + j_2 - 1$ for j_1 particle-like and j_2 hole-like, or vice versa. Here J, j_1 and j_2 are the angular momenta of the lowest state of the proton-neutron multiplet, of the odd proton and of the odd neutron, respectively. We note that the second weak rule is especially vague.

Similar rules appear for the lowest state of the multiplets in deformed odd-odd nuclei[40,41].

Less attention has been paid to the qualitative discussion of other states of the multiplet.

3. *Leading terms for the proton-neutron multiplet in CVM*

In the present paper we explore a proton-neutron multiplet pattern in odd-odd nuclei that arises in leading (second) order owing to the phonon exchange between the proton (quasiproton) and the neutron (quasineutron) of the multiplet. In other words, we consider the leading process of the *CVM* for the proton (quasiproton) — neutron (quasineutron) cluster. The resulting new parabolic rule has been introduced in Ref. 42. Here we present in more detailes derivation, discussion and illustrative examples for the application of the new rule.

Let us comment on the terms not included in the present parabolic rule.

In the second order we have additional proton- and neutron-self-energies (Figs. 1b_1, b_2); however, they lead only to an overall shift of the multiplet and do not influence the relative positions between the states of the multiplet.

In the fourth order there appear four vertex corrections (of the types in Figs. 1*c*, *d*) and the corresponding sixteen self-energies (of the types c_1 — c_4, d_1 — d_4), which involve one phonon exchange, are affected by the nuclear Ward identity. This relationship is analogous to the nuclear Ward identity introduced for the nuclear boson-fermion system in Refs. 43—45. It formally resembles the Ward identity in quantum electrodynamics[46] and to the Ward-Piaterskii identity in the solid-state physics[47]. In fact, the vertex corrections and the corresponding self energies asymptotically cancel. In the realistic case this systematic incoherence sizeably decreases the contribution from these diagrams. Nontrivial contribution in the fourth order is due to two-phonon exchange (diagrams e_1 — e_3); this contribution will be discussed in part II of this paper.

4. *Exchange of the quadrupole phonon*

Let us first consider the exchange of the quadrupole phonon between the proton and the neutron of the multiplet $|(j_p j_n) I>$ (Fig. 1*a*).

The particle-quadrupole vibration interaction is[1]

$$H_2 = \sqrt{20\pi}\, a_2\, [Y_2\, (b_2^\dagger + b_2)]_0. \tag{1}$$

Here a_2 is the interaction strength defined conveniently[28] as

$$a_2 = \frac{1}{\sqrt{4\pi}}\, k \left(\frac{\hbar\omega_2}{2c_2}\right)^{1/2}. \tag{1a}$$

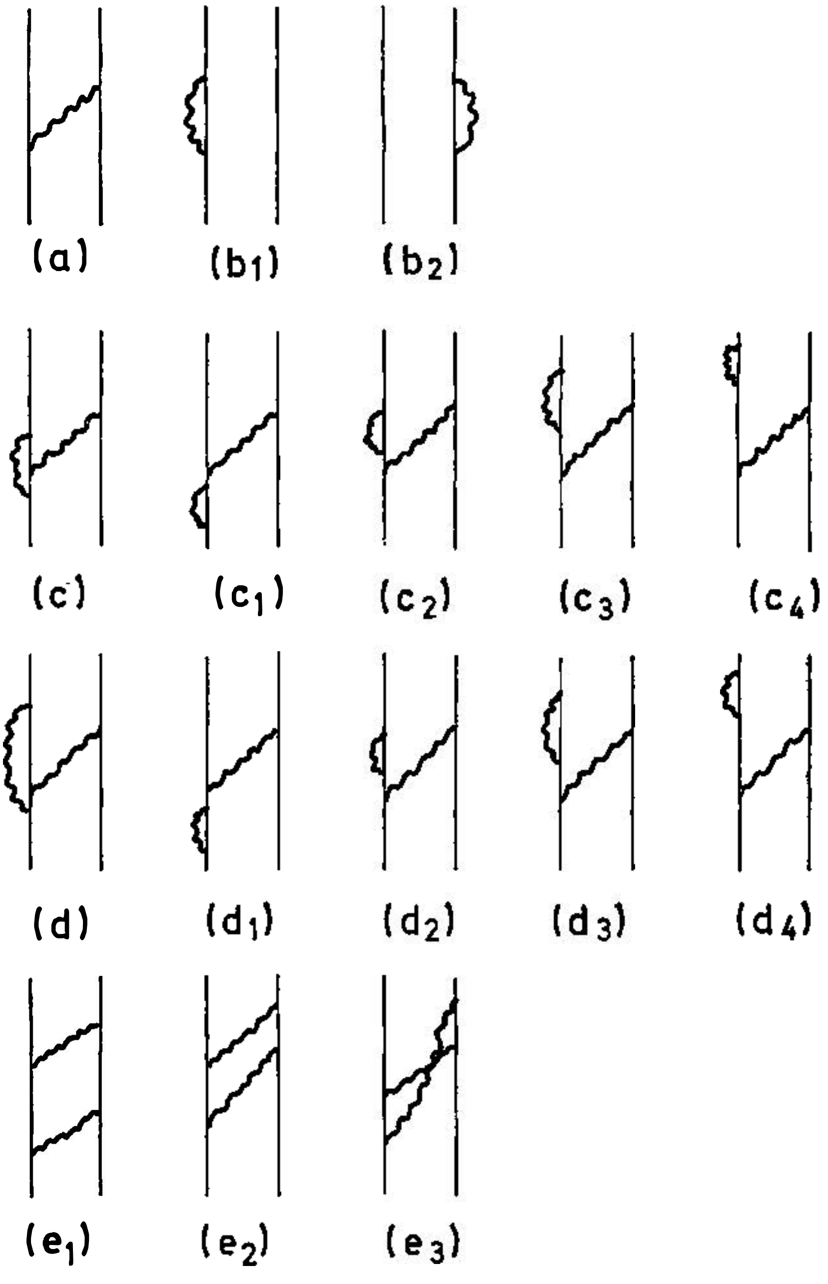


Fig. 1 Leading diagrams for the splitting of proton-neutron multiplet in *CVM*.
For description see Sec. 3.

For the quasiparticle, we also include the usual U, V-blocking factors in the interaction strength a_2. The symbol $b_2^\dagger$ denotes the creation operator of the quadrupole phonon.

The contribution to the splitting of the multiplet $|(j_p, j_n)\, I = |j_p - j_n|, \dots$ $\dots, j_p + j_n>$ coming from the exchange of quadrupole phonons (Fig. 1*a*) is

$$\delta E_2 = 2(-)^{j_p + j_n + I + 1} \begin{Bmatrix} j_p & j_p & 2 \\ j_n & j_n & I \end{Bmatrix} 4\pi \frac{a_2^2}{\hbar\omega_2} \cdot$$

$$\cdot <j_p \| Y_2 \| j_p> <j_n \| Y_2 \| j_n>. \tag{2}$$

Here $\hbar\omega_2$ is the energy of the quadrupole phonon. We assume the coupling strength a_2 to be equal for protons and neutrons. Inserting analytical expressions for the 6-j and 3-j symbols in Eq. (2), we obtain

$$\delta E_2 = -\alpha_2 \mathcal{O} \frac{[I(I+1) - j_n(j_n+1) - j_p(j_p+1)]^2 + [I(I+1) - j_n(j_n+1) - j_p(j_p+1)]}{2j_n(2j_n+2)\, 2j_p(2j_p+2)} +$$

$$+ \frac{\alpha_2 \mathcal{O}}{12}, \tag{3}$$

where

$$\alpha_2 = 15 \frac{a_2^2}{\hbar\omega_2}. \tag{4}$$

Keeping only the I-dependent terms in Eq. (3), we obtain

$$\delta E_2(I) = A\,[I(I+1)]^2 + BI(I+1), \tag{5}$$

with

$$A = -\alpha_2 \mathcal{O} \frac{1}{2j_p(2j_p+2)\, 2j_n(2j_n+2)}, \tag{6}$$

$$B = \alpha_2 \mathcal{O} \frac{2j_p(j_p+1) + 2j_n(j_n+1) - 1}{2j_p(2j_p+2)\, 2j_n(2j_n+2)}. \tag{7}$$

In Eqs. (3), (6) and (7), the quantity $\mathcal{O}$ is the occupation number defined in the following way:

$$\mathcal{O} = 1, \tag{8}$$

if $|j_p>$ and $|j_n>$ are both particle-like or both hole-like;

$$\mathcal{O} = -1, \tag{8a}$$

if $|j_p>$ is particle-like and $|j_n>$ is hole-like, or vice versa.

In addition to the I-dependent second-order contribution (5) coming from the exchange of quadrupole phonons, we obtain I-independent contributions in the same order coming from the phonon exchange itself (the last term in Eq. (3)) and from self-energy diagrams. Such contributions cause only an overall shift of the multiplet and do not give rise to the splitting of the multiplet.

Let us comment on the result (5) in the language of the wave function of the proton-neutron multiplet. In fact, we assume that the two-single-particle (or two-quasiparticle) configuration $|(j_p, j_n)\, I>$ is the largest component in the wave function of the corresponding state. In our approach, however, this leading component receives sizeable phonon admixtures coming from the particle-vibration coupling. In the second-order contribution (5), the only explicitly contributing admixtures are the one-phonon components $|(j_n j_p)\, I', 12; I>$. Here 12 denotes one phonon of angular momentum 2. However, first-order one-phonon admixtures of the type $|(j_n' j_p')\, I', 12, I>$, with $j_n' \neq j_n$ or $j_p' \neq j_p$, also contribute in second order as self-energy corrections to the energy of the multiplet; however, in this leading order, such admixtures produce only an overall shift of the multiplet and, therefore, they do not contribute to the energy splitting (5). Thus, the contribution (5) presents an I-dependent contribution that corresponds to a wave function which should be $|(j_n j_p)>$ in zeroth order and which should receive selected first-order admixtures:

$$\sum_{I'} C^{I}_{(j_n j_p) I', 12} |(j_n j_p)\, I', 12; I> + \sum_{I', j_n' \neq j_n} C^{I}_{(j_n' j_p) I', 12} |(j_n' j_p)\, I', 12; I> +$$
$$\sum_{I', j_p' \neq j_p} C^{I}_{(j_n j_p') I', 12} |(j_n j_p')\, I', 12; I>. \tag{9}$$

In the microscopic approach, this would correspond to introducing at least the selected coherent three-particle — one-hole (or four-quasiparticle) admixtures to two-particle zeroth-order components.

5. Parabola due to the exchange 2^+ phonons:

Discussion

The I-dependent energy shifts of the members of the proton-neutron multiplet states $|(j_p j_n)\, I>$ generate Eq. (5), i.e. a parabola in $I(I+1)$. The parabola is concave down for $\mathcal{O} = +1$, i.e. for particle-particle and hole-hole states, and is concave up for $\mathcal{O} = -1$, i.e. for particle-hole states. If the absolute value, of A is smaller, i.e. if the coupling strength a_2 is smaller, and the factor $j_p(j_p + 1) \cdot j_n(j_n + 1)$ larger, the focus will be closer to the vertex; so, the spreading of the parabola will be larger, i.e. the splitting of the proton-neutron multiplet will be smaller.

The vertex of the parabola (5) lies at the position

$$I(I+1) = j_n(j_n + 1) + j_p(j_p + 1) - \frac{1}{2}. \tag{10}$$

We denote the solution of Eq. (10) by I_v; from (10) we have:

$$I_v = \left((j_n (j_n + 1) + j_p (j_p + 1) - \frac{1}{4} \right)^{1/2} - \frac{1}{2}. \tag{11}$$

Now, we evaluate the relative position of the multiplet states with maximum and minimum angular momenta:

$$\varDelta E_2 \equiv \delta E_2 (I = I_{min} \equiv |j_p - j_n|) - \delta E_2 (I = I_{max} \equiv j_p + j_n). \tag{12}$$

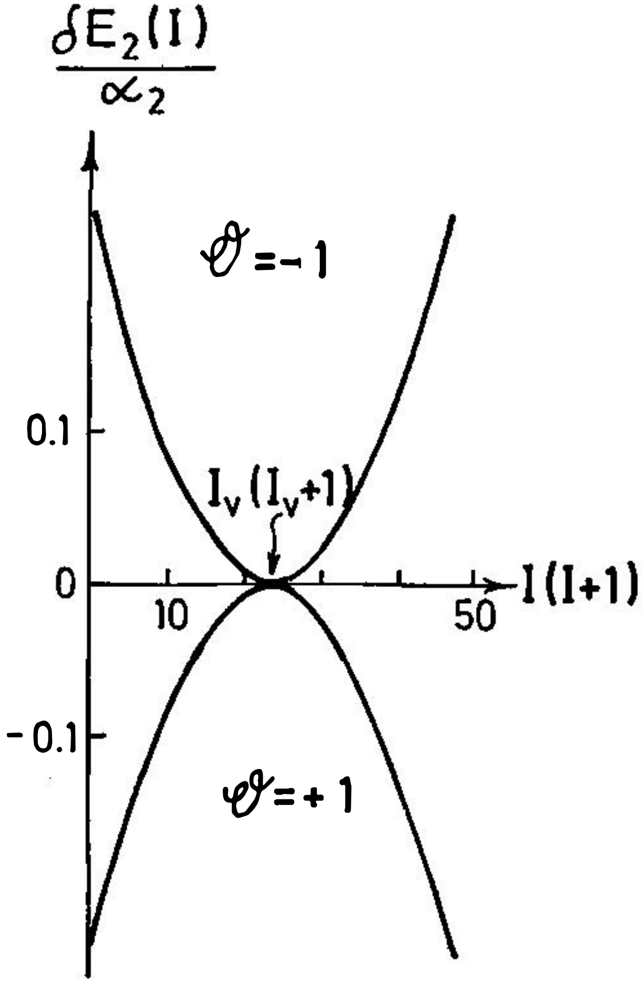


Fig. 2 Illustration of energy contributions coming from the exchange of the quadrupole phonon for the multiplet

$$\left(j_p = \frac{5}{2}, \; j_n = \frac{7}{2} \right) \text{ for } \vartheta = \pm 1.$$

Using Eq. (3), we obtain the following expression for $j_p > j_n$:

$$\Delta E_2 = - a_2 \mathcal{O} \frac{(2j_p + 1)(2j_n - 1)}{(2j_n + 2)\, 2j_p (2j_p + 2)}. \tag{13}$$

The sign of this expression is opposite to that of $\mathcal{O}$. Therefore, the state $|(j_p j_n) I = I_{min} = j_p - j_n >$ lies below the state $|(j_p j_n) I = I_{max} = j_p + j_n >$, if $\mathcal{O} = 1$. If $|(j_p j_n) I >$ is the lowest multiplet, this means that I_{min} is the angular momentum of the ground state. This result coincides with the prediction of Nordheim's strong rule[25,30]. For $\mathcal{O} = -1$, ΔE_2 is larger than zero; the state of angular momentum I_{min} lies above the state of angular momentum I_{max}. In this case the lowest state of the multiplet is $I = \{I_v\}$, where $\{I_v\}$ denotes the integer number that is closest to I_v from Eq. (11).

We may draw the same conclusions for $j_n > j_p$.

Fig. 2*a* illustrates both situations, $\mathcal{O} = +1$ and $\mathcal{O} = -1$.

For large $j_p \approx j_n \equiv j$, Eq. (13) becomes

$$\Delta E_2 \approx - \frac{a_2 \mathcal{O}}{2j}. \tag{14}$$

In the asymptotic limit $j \gg 1$, $\Delta E_2 \to 0$, i.e. the states $| I_{min} >$ and $| I_{max} >$ are degenerate in energy. Because of Eq. (11), the position of the vertex I_v in the same limit is

$$I_v \approx \sqrt{2}\, j. \tag{15}$$

6. Parabola due to the exchange of 2^+ phonons: Illustrative applications

In this section we apply the parabolic rule to a few multiplets determined experimentally in odd-odd nuclei. In each case the parabola is fitted to one experimental point adopting the following simplified procedure:

(*i*) Of the experimental states of the multiplet $|(j_p j_n) I >$, we choose as a reference state that which lies highest for $\mathcal{O} = 1$ and lowest for $\mathcal{O} = -1$. We denote the angular momentum of this reference state by $| I_R >$. We present each experimental state $| I >$ graphically relative to the reference state, i.e. $y_{exp} \equiv E_{exp}(I) - E_{exp}(I_R)$, as a function of the variable $x = I(I + 1)$.

(*ii*) In the same diagram we draw a parabola:

$$y(x) \equiv A(x - x_v)^2, \tag{16}$$

where

$$x_v = I_v (I_v + 1) \tag{16a}$$

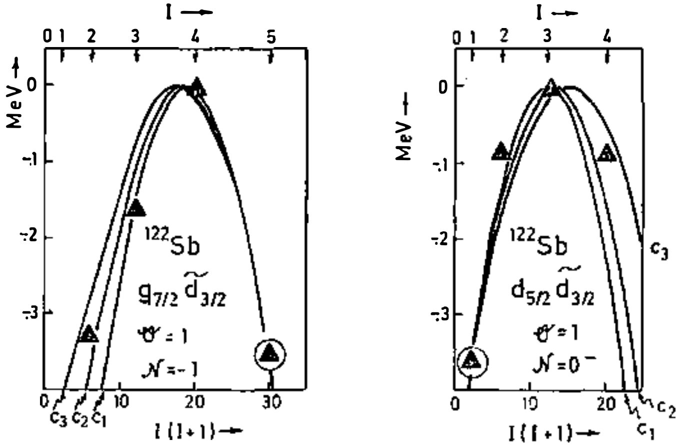
is given by Eq. (10). The parameter A is determined so that one experimental point (preferably that which is farthest away from the vertex) lies on the parabola (16). Let us denote the angular momentum of this fitting state by I_0. Denoting

$$x_0 = I_0 (I_0 + 1), \qquad y_0 = E_{exp}(I_0) - E_{exp}(I_R), \tag{16b}$$

we obtain

$$A = \frac{y_0}{(x_0 - x_v)^2}. \tag{16c}$$

We should note that because of our definition (16b), we have sign $(y_0) = -\mathcal{O}$, i.e. $y_0 < 0$ for $\mathcal{O} = 1$ and $y_0 > 0$ for $\mathcal{O} = -1$.



(a) (b)

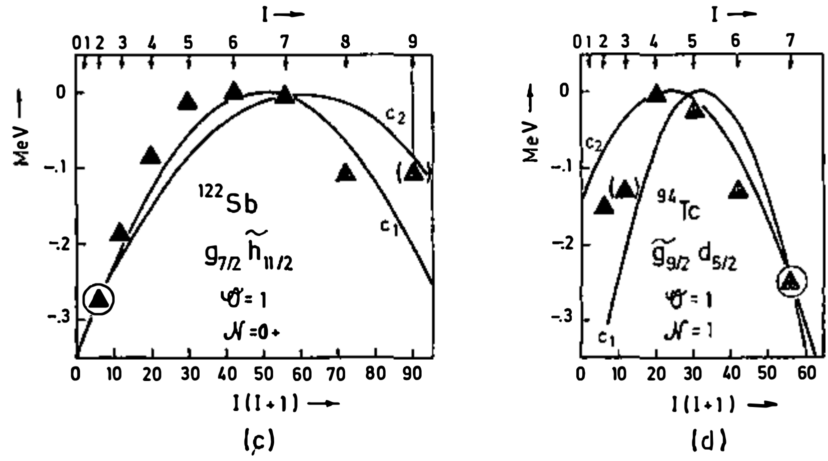
Figs. 3*a*—*h*. Application of the parabolic rule to several proton-neutron multiplets in odd-odd nuclei.

The relative energies of the multipole states are presented versus $x = I(I + 1)$. The available experimental states are marked with triangles. The circle around the triangle denotes the experimental state used to fit the spreading-out of the parabola. The corresponding parabolas δE_2 resulting from the exchange of the 2^+ phonon (prescriptions *(i)* and *(ii)*) are presented by curves c_i.

Figs. 3*a*-3*h* illustrate a few experimental cases. Triangles mark the experimental multiplet states drawn using procedure (*i*) and C_i indicates the parabola (16) drawn following procedure (*ii*). The encircled triangle denotes the fitting state.

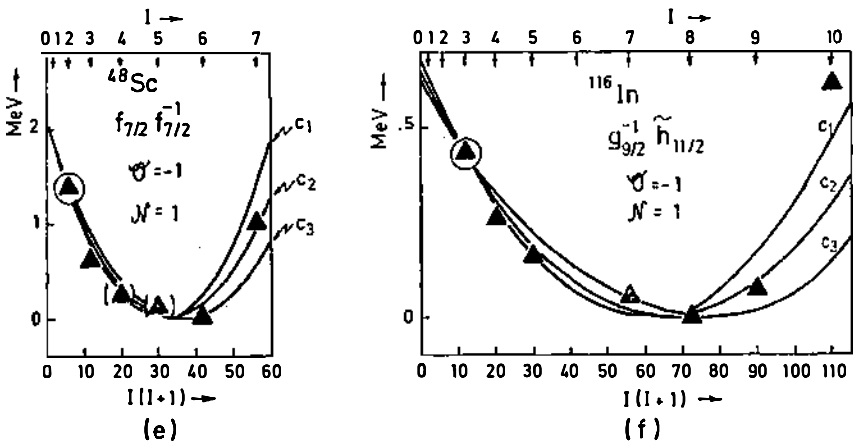
Figs. 3*a*-3*c* represent three multiplets in ^{122}Sb, ($\pi g_{7/2}\,\nu\tilde{d}_{3/2}$, $\pi d_{5/2}\,\nu\tilde{d}_{3/2}$ and $\pi g_{7/2}\,\nu\tilde{h}_{11/2}$, respectively), with the experimental data taken from Ref. 38. Fig. 3*d* shows the multiplet $\pi\tilde{g}_{9/2}\,\nu d_{5/2}$ in $^{94}_{43}Tc_{51}$. Fig. 3*e* represents the multiplet $\pi f_{7/2}\,\nu\tilde{f}_{7/2}^{-1}$ in ^{48}Sc; Figs. 3*f* and 3*g* show the multiplets $\pi g_{9/2}^{-1}\,\nu\tilde{h}_{11/2}$ and $\pi g_{9/2}^{-1}\,\nu\tilde{d}_{3/2}$ in ^{116}In (Ref. 37), Fig. 3*h* represents the multiplet $\pi h_{9/2}\,\nu\tilde{i}_{13/2}$ in ^{196}Tl (Ref. 35). In the last case, only four out of ten states of the multiplet are available experimentally.

Figs. 3*a*-3*d* represent concave-down parabolas ($A < 0$, i.e. $\mathcal{O} = 1$) and Figs. 3*e*-3*h* represent concave-up parabolas ($A > 0$, i.e. $\mathcal{O} = -1$). Let us discuss how in the present approach this arises as a simple shell effect.



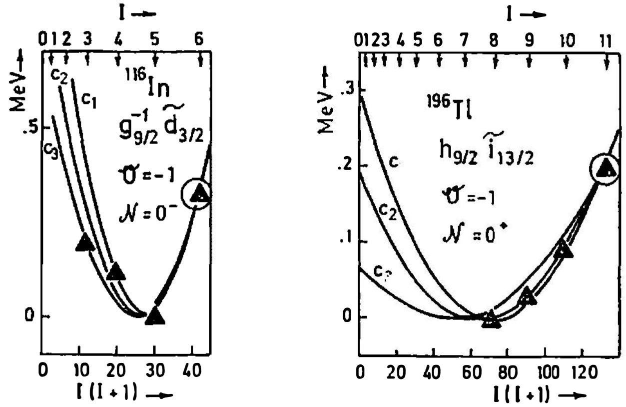
(c) (d)

In the cases shown in Figs. 3*a*-3*c* the respective multiplets are of the proton-particle — neutron-quasiparticle type, while in the cases shown in Figs. 3*f* and 3*g* they are of the proton-hole — neutron-quasiparticle type. In all these cases



(e) (f)

the quasineutron is in the $N = 50$—82 shell. The highest single-particle configurations in the $N = 50$—82 shell seem to be $h_{11/2}$ and $d_{3/2}$ because the ground state in the lighter $N = 81$ nuclei is $\frac{3^+}{2}$ (predominantly $d_{3/2}^{-1}$), followed by $\frac{11^-}{2}$ (predominantly $h_{11/2}^{-1}$). Thus, in the absence of the residual force, the $h_{11/2}$ level is less than half-filled in both ^{122}Sb and ^{116}In (five particles for ^{122}Sb and one partice for ^{116}In), while the $d_{3/2}$ configuration, lying above $h_{11/2}$ is empty. In this way, in both $^{122}_{51}$Sb$_{71}$ and $^{116}_{49}In_{67}$, the $\tilde{h}_{11/2}$ and $\tilde{d}_{3/2}$ neutron quasiparticles lie above the Fermi surface and therefore are expected to be particle-like ($U_j > V_j$). Thus, we have $\mathcal{O}\,(g_{7/2}\,\tilde{d}_{3/2}) = \mathcal{O}\,(g_{7/2}\,d_{3/2}) = 1$ (case a), $\mathcal{O}\,(d_{5/2}\,\tilde{d}_{3/2}) = \mathcal{O}\,(d_{5/2}\,d_{3/2}) = 1$ (case b), $\mathcal{O}\,(g_{7/2}\,h_{11/2}) = \mathcal{O}\,(g_{7/2}\,\tilde{h}_{11/2}) = 1$ (case c), $\mathcal{O}\,(g_{9/2}^{-1}\,h_{11/2} = \mathcal{O}(g_{9/2}^{-1}\,\tilde{h}_{11/2}) = -1$ (case f), $\mathcal{O}\,(g_{9/2}^{-1}\,\tilde{d}_{3/2}) = \mathcal{O}\,(g_{9/2}^{-1}\,d_{3/2}) = -1$ (case g). On the other hand, since in ^{122}Sb the $\tilde{h}_{11/2}$ quasiparticle is closer to the Fermi surface than the $\tilde{d}_{3/2}$ quasiparticle, it is obvious that $V\tilde{h}_{11/2} > V\tilde{d}_{3/2}$. Thus, the blocking effect on A (via the coupling strength a), which is in leading order proportional to $(U_j^2 - V_j^2)$, is more pronounced for multiplets with the quasineutron configuration $\tilde{h}_{11/2}$ than for those with the configuration $\tilde{d}_{3/2}$. This is clearly shown in Fig. 3. The spreading of the parabola is larger in case (c), involving $\tilde{h}_{11/2}$, than in cases (a) and (b), involving $\tilde{d}_{3/2}$; the larger spreading of the parabola indicates that A is smaller. On the other hand, since the occupation number of the $\tilde{h}_{11/2}$ configuration in ^{122}Sb is larger (five particles in the absence of the residual interaction) than in ^{116}In (one particle in the absence of the residual interaction), $V\tilde{h}_{11/2}$ (^{122}Sb) is larger than $V\tilde{h}_{11/2}$ (^{116}In). Thus, the blocking effect is stronger in the $g_{7/2}\,\tilde{h}_{11/2}$ multiplet in ^{122}Sb than in the $g_{9/2}^{-1}\,h_{11/2}$ multiplet in ^{116}In, i.e. the parameter A for ^{122}Sb is smaller than the parameter A for ^{116}In. Indeed, the parabola for ^{122}Sb in Fig. 3c is more spread out than the parabola for ^{116}In in Fig. 3f, i.e. A for ^{122}Sb is smaller.



(g) (h)

In case (h), the multiplet $\pi h_{9/2}\,\nu\tilde{i}_{13/2}$ is built from a proton-particle and a neutron-quasiparticle. In this case the $N = 82$—126 valence shell contains two neutron holes. It is obvious that the $\nu\tilde{i}_{13/2}$ quasiparticle is hole-like. Thus we have $\mathcal{O}\,(\pi h_{9/2}\,\nu\tilde{i}_{13/2}) = \mathcal{O}\,(h_{9/2}\,i_{13/2}^{-1}) = -1$.

In cases (*d*) and (*e*) the occupation numbers are as follows: $\mathcal{O}\,(\pi \tilde{g}_{9/2}\, \nu d_{5/2\,2}) = \mathcal{O}\,(\pi g^{3}_{9/2}\, \nu d_{5/2}) = \mathcal{O}\,(\pi g_{9/2}\, \nu d_{5/2}) = 1$ and $\mathcal{O}\,(\pi f_{7/2}\, \nu \bar{f}^{-1}_{7/2}) = -1$, respectively.

Now, we consider in detail the application of procedures (*i*) and (*ii*) to the multiplet $|\,(\pi g_{7/2}\ \nu \tilde{d}_{3/2})\, I = 2^+, 3^+, 4^+, 5^+ >$ in ^{122}Sb. The position of the vertex is given by Eq. (10) : $x_v \equiv I_v\,(I_v + 1) = \frac{7}{2} \cdot \frac{9}{2} + \frac{3}{2} \cdot \frac{5}{2} - \frac{1}{2} = 19.$

Because of $\mathcal{O} = 1$, the reference state is the highest state of the multiplet $I = 4 = I_R$. As the fitting state we choose the state $I = 5 = I_0$, which lies farthest away from the reference state. Thus we obtain:

$x_0 \equiv I_0\,(I_0 + 1) = 30,\ \ y_0 \equiv E_{exp}\,(I_R) - E_{exp}\,(I_0) = E_{exp}(4) - E_{exp}\,(5) = -0.35$ MeV.

Now, because of Eq. (16*c*), the value of the parameter A is $A = -0.35/11 \cdot 11 = -0.00289$. From this value, because of Eq. (6), we obtain $a_2 = |\,A\,| \cdot 7 \cdot 9 \cdot 3 \cdot 5 = 2.73$.

The parabola (16) reads

$$y = -0.00289\,(x - 19)^2.$$

This parabola is presented by the curve c_1 in Fig. 3*a*.

Similarly, we illustrate the application of procedures (*i*) and (*ii*) to the multiplet $|\,(\pi f_{7/2}\, \nu \bar{f}^{-1}_{7/2})\, I = 0^+, 1^+, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+ >$ in ^{48}Sc (Fig. 3*e*). In this case, $\mathcal{O} = -1$, so the reference state is the lowest state of the multiplet $I = 6 = I_R$. Because of Eq. (10), the vertex of the parabola is $x_v = \frac{7}{2} \cdot \frac{9}{2} + \frac{7}{2} \cdot \frac{9}{2} - \frac{1}{2} = 31$. As the fitting state we choose the multiplet $I = 2 = I_0$ (the states $I = 1$ and $I = 0$ should lie still farther away from the reference state but are not yet known experimentally). In this case we have $x_0 = 2 \cdot 3 = 6, y_0 = E_{exp}(I_0) - E_{exp}\,(I_R) = E_{exp}\,(2) - E_{exp}\,(6) = 1.4$ MeV. According to Eq. (16*c*), the value of the parameter A is 0.00224. Because of Eq. (6), this value corresponds to $\alpha_2 = 8.89$.

The parabola (16) reads

$$y = 0.00224\,(x - 31)^2.$$

7. *A new rule instead of Nordheim's weak rule*

For $\mathcal{O} = -1$, i.e. for the proton-neutron state of the particle-hole type, the parabola resulting from the exchange of a quadrupole phonon is concave up, with a minimum given by Eq. (11). We rewrite this expression in the form

$$I_v = j_n + j_p - \{\beta\}, \tag{17}$$

with

$$\beta = (j_n + j_p)\left[1 - \left(1 - \frac{2j_n j_p - j_n - j_p + \frac{1}{4}}{(j_n + j_p)^2}\right)^{1/2}\right] + \frac{1}{2}. \tag{18}$$

In the asymptotic limit of large angular momenta, Eq. (18) gives

$$\beta \approx \frac{1}{\dfrac{1}{j_p} + \dfrac{1}{j_n}}. \tag{19}$$

The symbol $\{\beta\}$ in Eq. (17) denotes an integer that is closest to β. For example, $\{3.2\} = 3$, $\{4.7\} = 5$.

Thus, Eq (17) presents a new rule for the spin of the lowest state of the multiplet with $\mathcal{O} = -1$. This rule is more predictive than Nordheim's second weak rule, which for the lowest multiplet with $\mathcal{O} = -1$ states that the ground state shows a tendency to be $I_{g.s.} = j_p + j_n - 1$ (Ref. 30). The prediction (17) applies to the lowest multiplet for which $| I_v >$ is the ground state, as well as to higher multiplets. It applies also to the multiplets with $\mathcal{O} = 1$, but in such a case it predicts the spin of the highest state of the multiplet.

For the multiplets presented in Fig. 3, the predictions of Eq. (17) with Eq. (18) (or with Eq. (19)) are in agreement with experiment for cases (*a*) with $\{\beta\} = 1$, (*b*) with $\{\beta\} = 1$, (*g*) with $\{\beta\} = 1$ and (*h*) with $\{\beta\} = 3$. In cases (*c*) and (*d*), the predicted values for $\{\beta\}$ are too small by one unit; the values for $\{\beta\}$ are 2 and 2 instead of 3 and 3, respectively. In cases (*e*) and (*f*), the predicted values for $\{\beta\}$ are too large by one unit: they are 2 and 3 instead of 1 and 2, respectively. We will account for these deviations by including the spin-vibrational 1^+ phonon, in addition to the quadrupole phonon. This will slightly modify expression (17):

$$I_v = j_p + j_n - \{\beta'\}, \tag{20}$$

with

$$\beta' = \beta - | \delta | \mathcal{O} [2 \, | j_p - l_p + j_n - l_n | - 1], \tag{20a}$$

where $| \delta | < 1$. Because of expression (20a), the vertex is shifted to the left (toward lower I) or to the right (toward higher I), depending on whether $\mathcal{O} \, [2 | j_p - l_p + j_n - l_n | - 1] = +1$ or $\mathcal{O} = -1$, respectively. This may change the position of the vertex by about one unit of angular momentum. These situations are illustrated by cases (*d*) and (*f*) in Fig. 3. ($\mathcal{O} = 1$, $\mathcal{N} \equiv | j_p - l_p + j_n - l_n | = 1$ and $\mathcal{O} = -1$, $\mathcal{N} \equiv | j_p - l_p + j_n - l_n | = 1$, respectively). In these cases, the values of $\{\beta\}$ and $\{\beta'\}$ are $\{\beta\} = 2$, $\{\beta'\} = 3$ and $\{\beta\} = 3$, $\{\beta'\} = 2$, respectively.

The correction (20a) will be discussed in part II.

8. *Conclusion*

We have derived an analytical expression for the relative energies of the proton-neutron multiplet $| (j_p, j_n) \, I = | j_p - j_n |, \dots, j_p + j_n >$ owing to the exchange of the quadrupole phonon. The expression has the form of a quadratic function of $x = I (I + 1)$:

$$E_2 (I) = A (x - x_v)^2, \tag{21}$$

with x_v depending only on j_p and j_n (Eq. (10)), and A given by Eq. (16c) if its magnitude is fitted to one experimental state. A simple approximate expression for I_v at the vertex of the parabola (16) is

$$I_v \approx j_p + j_n - \left(\frac{1}{\frac{1}{j_p} + \frac{1}{j_n}} \right). \tag{22}$$

This parabola is concave down for $\mathcal{O} = 1$ (i.e. for the proton and neutron configuration being both particle-like or both hole-like) and concave up for $\mathcal{O} = -1$ (i.e. for the proton configuration being particle-like and the neutron configuration being hole-like or-vice versa).

It should be emphasized that the sensitivity of the splitting of the multiplet to the other admixtures in the wave function that are not included in the parabolic rule is sizeably reduced. This is because these admixtures give rise to an overall shift (self-energies) or, in higher orders, they introduce terms that are partly mutually incoherent (Ward-like identities).

The parabolic rule is more effective in nuclei with a pronounced collective quadrupole mode. More deviations, therefore, are expected in near-magic nuclei. The parabolic rule, therefore, should not be interpreted too rigidly. It could generally provide a simple classification of multiplet states, and larger deviations from this rule could indicate proton-neutron multiplets that are particularly strongly mixed and/or the importance of other modes and correlations. In particular, by its simple prescriptions (Sects. 5,6 and 7), the parabolic rule could sevre as a guideline in the experimental search and identification of the states of proton-neutron multiplets in odd-odd nuclei.

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JEDNOSTAVNO PRAVILO O PARABOLIČNIM REGGEOVIM TRAJEKTORIJAMA ZA PROTONSKO-NEUTRONSKE MULTIPLETE U NEPARNO-NEPARNIM JEZGRAMA

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Izveden je analitički izraz za energije stanja protonsko-neutronskog multipleta $(j_p j_n)\, I = |j_p - j_n|, \ldots, j_p + j_n$ kao posljedica izmjene jednog fonona. Na osnovi toga uvedena su nova izborna pravila. Izvršena je detaljna diskusija rezultirajuće parabole i ilustrirana je primjena pravila na niz konkretnih situacija.