

## CANONICAL FORMULATION OF THE CLASSICAL ELECTROMAGNETIC FIELD\*

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A canonical formulation of classical electromagnetic field is developed and some consequences are discussed.

### *1. Introduction*

The canonical formulation of the electromagnetic field is connected with serious difficulties<sup>1-9)</sup>. Experimental background of the field equations put these difficulties at the beginning somewhat aside. Mathematical properties of the field equations led to the electromagnetic field potentials and these potentials turned out later to be of fundamental importance for the Lagrangian and the Hamiltonian (canonical) formulation of the electromagnetic field. But it did not give correct formulation. Several modifications have been done so far but the situation still cannot be accepted as satisfactory. We consider this problem here and give a possible solution.

The canonical formulations of the Dirac's field and a vector field have been successfully developed recently<sup>10,11,12)</sup>. Analysis of the present work is based in great deal on the knowledge obtained in these papers.

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In Section 2 we summarize the standard Lagrangian and canonical procedure of the classical electromagnetic field and indicate difficulties. The photon method is presented in Section 3. The correct Lagrangian and canonical formulation of the free electromagnetic field is given in Section 4. Transformation properties are considered in Section 5. Some consequences of the new canonical formulation of the electromagnetic field are analysed in Section 6. The field with sources is considered in Section 7. Connection with the potential method is given in Section 8. Conclusions and some remarks are given in Section 9.

## 2. The standard Lagrangian and canonical formulation

The electromagnetic field equations are

$$\begin{aligned} \operatorname{rot} \mathbf{E} &= - \frac{\partial \mathbf{H}}{\partial t}, \\ \operatorname{rot} \mathbf{H} &= \frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{j}, \\ \operatorname{div} \mathbf{E} &= 4\pi \rho, \\ \operatorname{div} \mathbf{H} &= 0 \end{aligned} \quad (1)$$

or in the covariant form

$$\begin{aligned} \partial_\nu F^{\mu\nu} &= -4\pi j^\mu, \\ \partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} &= 0, \end{aligned} \quad (2)$$

where

$$x^\mu = (t, \mathbf{x}), \quad x_\mu = (t, -\mathbf{x}), \quad \partial_\mu = \frac{\partial}{\partial x^\mu} \quad \alpha = 0, 1, 2, 3,$$

$F_{\mu\nu}$  is the electromagnetic field tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix} \quad (3)$$

and  $j^\mu$  is the current four-vector satisfying the continuity equation

$$\partial_\nu j^\mu = 0. \quad (4)$$

We have chosen the velocity of light to be unity.

The electromagnetic field potentials are defined by

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi, \quad (5)$$

$$\mathbf{H} = \text{rot } \mathbf{A}$$

or

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (6)$$

where  $A_\mu = (\varphi, -\mathbf{A})$  is the potential four-vector. The potential  $A_\mu$  separates the Maxwell's equations (1). Taking it as a mathematical tool there is the gauge freedom

$$\tilde{A}_\mu = A_\mu - \partial_\mu f \quad (7)$$

where  $f$  is an arbitrary solution of the d'Alembert's equation. It allows an additional condition on  $A_\mu$ . The most frequently used is the Lorentz's condition

$$\text{div } \mathbf{A} + \frac{\partial \varphi}{\partial t} = 0. \quad (8)$$

or

$$\partial_\mu A^\mu = 0 \quad (9)$$

The Lagrange's procedure starts with the scalar  $F_{\mu\nu}F^{\mu\nu}$  and the interaction term  $A_\mu j^\mu$ . The standard Lagrangian density of the electromagnetic field is

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu. \quad (10)$$

The minus sign in front of the scalar  $F_{\mu\nu}F^{\mu\nu}$  secures the minimum of the action  $S = \int \mathcal{L} \, d\tau dt$ . The Lagrange's variables are  $A_\mu$  (not  $F_{\mu\nu}$ !).

Starting from (10) and the Lagrange's equations

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\alpha \left( \frac{\partial \mathcal{L}}{\partial (\partial_\alpha A_\mu)} \right) = 0 \quad (11)$$

follow the Lagrange's equations for  $A_\mu$ :

$$\Delta \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left( \text{div } \mathbf{A} + \frac{\partial \varphi}{\partial t} \right) = -4\pi \mathbf{j}, \quad (12)$$

$$\Delta \varphi + \text{div } \frac{\partial \mathbf{A}}{\partial t} = -4\pi \rho. \quad (13)$$

Eq. (13) is not a dynamical equation. It doesn't contain the second time derivative of  $\varphi$ . This is the first manifestation of incorrectness of the standard Lagrangian procedure of the electromagnetic field. Using the Lorentz condition (8), Eqs. (12) and (13) transform to

$$\Delta \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} = -4\pi \mathbf{j}, \quad (14)$$

$$\Delta \varphi - \frac{\partial^2 \varphi}{\partial t^2} = -4\pi \rho.$$

However, neither the Lorentz condition nor the way in which Eqs. (14) was derived are within the framework of the Lagrange's procedure.

The canonical momenta to  $A_\mu$  are defined by

$$\Pi_{A_\mu} = \frac{\partial \mathcal{L}}{\partial (\partial_t A_\mu)}$$

and they are

$$\Pi_{A_i} = -\frac{1}{4\pi} E_i, \quad (i = 1, 2, 3), \quad (15)$$

$$\Pi_\varphi = 0.$$

According to

$$\begin{aligned} \mathcal{H} &= \Pi_{A_\mu} \frac{\partial A_\mu}{\partial t} - \mathcal{L} \\ &= \frac{1}{4\pi} \mathbf{E} \nabla \varphi + \frac{1}{8\pi} \{ \mathbf{E}^2 + (\text{rot } \mathbf{A})^2 \} \end{aligned} \quad (16)$$

the canonical equations

$$\begin{aligned} \frac{\partial}{\partial t} \Pi_{A_\mu} &= -\frac{\delta \mathcal{H}}{\delta A_\mu} = -\frac{\partial \mathcal{H}}{\partial A_\mu} + \partial_j \left( \frac{\partial \mathcal{H}}{\partial (\partial_j A_\mu)} \right), \\ \frac{\partial}{\partial t} A_\nu &= \frac{\delta \mathcal{H}}{\delta \Pi_{A_\nu}} = \frac{\partial \mathcal{H}}{\partial \Pi_{A_\nu}} - \partial_j \left( \frac{\partial \mathcal{H}}{\partial (\partial_j \Pi_{A_\nu})} \right), \end{aligned}$$

for the Hamiltonian (16) are

$$\frac{\partial}{\partial t} \Pi_{A_i} = -\frac{\delta \mathcal{H}}{\delta A_i} \rightarrow \frac{\partial \mathbf{E}}{\partial t} = \text{rot rot } \mathbf{A},$$

$$\frac{\partial}{\partial t} \Pi_\varphi = -\frac{\delta \mathcal{H}}{\delta \varphi} \rightarrow \text{div } \mathbf{E} = 0,$$

$$\frac{\partial}{\partial t} A_i = \frac{\delta \mathcal{H}}{\delta \Pi_{A_i}} \rightarrow \frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E} - \nabla \varphi, \quad (17)$$

$$\frac{\partial}{\partial t} \varphi = \frac{\delta \mathcal{H}}{\delta \Pi_\varphi} \rightarrow ??.$$

The last equation is unknown due to zero momentum  $\Pi_\varphi$ . It is the second manifestation of the incorrectness of the standard Lagrangian and canonical formalism of the electromagnetic field.

Considering the free field and taking  $\varphi = 0$  there is no problem with the zero momentum, but the canonical equations for the canonical pair  $(\mathbf{A}, -\frac{1}{4\pi} \mathbf{E})$  are not equivalent to the Maxwell's equations. With definition  $\mathbf{H} = \text{rot } \mathbf{A}$  Eqs. (17) for  $\partial \mathbf{E}/\partial t$  and  $\partial \mathbf{A}/\partial t$  give the Maxwell's equations for  $\partial \mathbf{E}/\partial t$  and  $\partial \mathbf{H}/\partial t$ . However, the Maxwell's equations for  $\partial \mathbf{E}/\partial t$  and  $\partial \mathbf{H}/\partial t$  do not give Eqs. (17) for  $\partial \mathbf{E}/\partial t$  and  $\frac{\partial \mathbf{A}}{\partial t}$ . The equation for  $\partial \mathbf{H}/\partial t$  gives

$$\text{rot} \left( \frac{\partial \mathbf{A}}{\partial t} + \mathbf{E} \right) = 0$$

what leads to

$$\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E} + \nabla f$$

where  $f$  is an arbitrary function.

Following this line of reasoning we would find some other similar inconsistencies. Besides these there is one which is very fundamental. The canonical momenta conjugate to the canonical variables  $A_\mu$  are components of the second order tensor  $F_{\mu\nu}$ . Therefore,  $A_\alpha$  and  $F_{\mu\nu}$  cannot be canonical pairs. Naturally, all these inconsistencies are closely connected.

## 3. Photon method

An introduction of photons in quantum electrodynamics uses the following expansions of  $\mathbf{E}$  and  $\mathbf{H}^{(1,3)}$ :

$$\begin{aligned}\mathbf{E} &= \sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}}(t) e^{i\mathbf{k}\mathbf{r}}, \quad \mathbf{E}_{\mathbf{k}} = \mathbf{E}_{-\mathbf{k}}^*, \\ \mathbf{H} &= \sum_{\mathbf{k}} \mathbf{n}_{\mathbf{k}} \times \mathbf{E}_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}}.\end{aligned}\tag{18}$$

In the second expression the orthogonality of  $\mathbf{k}, \mathbf{E}_{\mathbf{k}}, \mathbf{H}_{\mathbf{k}}$  is explicitly taken into account. The substitution of (18) into (1) gives

$$\begin{aligned}\dot{\mathbf{E}}_{\mathbf{k}} &= i\mathbf{k} \times (\mathbf{n}_{\mathbf{k}} \times \mathbf{E}_{\mathbf{k}}), \\ \mathbf{n}_{\mathbf{k}} \times \dot{\mathbf{E}}_{\mathbf{k}} &= -i\mathbf{k} \times \mathbf{E}_{\mathbf{k}}, \\ \mathbf{k} \cdot (\mathbf{n}_{\mathbf{k}} \times \mathbf{E}_{\mathbf{k}}) &= 0, \\ \mathbf{k} \cdot \mathbf{E}_{\mathbf{k}} &= 0.\end{aligned}\tag{19}$$

These equations are equivalent to

$$\begin{aligned}\dot{\mathbf{E}}_{\mathbf{k}} &= -ik\mathbf{E}_{\mathbf{k}}, \\ \mathbf{k} \cdot \mathbf{E}_{\mathbf{k}} &= 0.\end{aligned}\tag{20}$$

It is usually stated that Eqs. (20) are equivalent to the Maxwell's equations (1) for the free field.

The expansion of  $\mathbf{A}$  in plane waves with respect to  $\mathbf{r}$  gives

$$\mathbf{E}_{\mathbf{k}} = -\frac{\partial \mathbf{A}_{\mathbf{k}}}{\partial t}.$$

$\mathbf{A}_{\mathbf{k}}$  are now variables of the system: the Lagrange's variables. The corresponding Lagrangian is

$$L = \int \mathcal{L} d\tau = \frac{1}{8\pi} \int (\mathbf{E}^2 - \mathbf{H}^2) d\tau = \frac{1}{8\pi} \sum_{\mathbf{k}} (|\mathbf{E}_{\mathbf{k}}|^2 - |\mathbf{n}_{\mathbf{k}} \times \mathbf{E}_{\mathbf{k}}|^2) = 0.$$

We see that it is not the Lagrangian of the system (20). Consequently, Eqs. (20) are not dynamically equivalent to the Maxwell's equations (1). This is a consequence of the second of Eqs. (18) where some basic properties of solutions of the Maxwell's equations are already expressed.

We have presented the photon method in order: (1) to emphasize dynamical characteristics of the Maxwell's equations in total, and (2) to indicate that the tensor field  $F_{\mu\nu}$  can hardly be reduced to vector canonical pairs.

#### 4. Canonical formulation of the free electromagnetic field

In the canonical formulation of the Dirac's field<sup>11)</sup> it was found that the Dirac's field is the momentum of a canonical pair and the Dirac's equation is the canonical equation for this momentum. There are deep similarities between the Dirac's and the Maxwell's fields<sup>14, 15, 16)</sup>. Among them they satisfy the first order differential equations. Analogous situation appears with the vector field satisfying the first order differential equations where the field itself is also the momentum. Because of this we think that solution of the canonical formulation problem of the electromagnetic field has to be found on the same line. Consequently, we start with the Maxwell's equations as canonical equations for canonical momenta.

For the Lagrange's density we take, as usually, the simplest scalar  $F_{\mu\nu} F^{\mu\nu}$ :

$$\mathcal{L} = \text{const } F_{\mu\nu} F^{\mu\nu}. \quad (21)$$

We choose the constant to be  $-1/16\pi$ . The Lagrange's variables of these momenta we denote by  $\Phi_{\alpha\beta}$ . Let us emphasize that the Lagrange's variables are components of a second order tensor because the momenta are also a second order tensor. Then we look for

$$F_{\mu\nu} = F_{\mu\nu}(\Phi_{\alpha\beta}), \quad (22)$$

where  $\Phi_{\alpha\beta}$  satisfy the second order differential Lagrange's equations and in the corresponding canonical equations  $F_{\mu\nu}$  satisfy the Maxwell's equations.

Due to linearity of the Lagrange's equations the most general functional dependence  $F_{\mu\nu}$  of  $\Phi_{\alpha\beta}$  is

$$F_{\mu\nu} = \partial_\mu G_{\mu\nu}{}^{\gamma\alpha\beta} \Phi_{\alpha\beta}, \quad (23)$$

where  $G_{\mu\nu}{}^{\gamma\alpha\beta}$  is a constant tensor antisymmetric in the indices  $(\mu, \nu)$  and  $(\alpha, \beta)$ . We assume that the tensor  $\Phi_{\alpha\beta}$  is antisymmetric also because the tensor  $F_{\mu\nu}$  is antisymmetric.

The Lagrange's equations for  $\Phi_{\alpha\beta}$  are

$$\partial_\zeta \partial_\gamma G_{\zeta\gamma}{}^{\zeta\alpha\beta} G^{\mu\nu\gamma\eta\zeta} \Phi_{\eta\zeta} = 0. \quad (24)$$

The conjugate momenta to  $\Phi_{\alpha\beta}$  are

$$\Pi_{\Phi_{\alpha\beta}} = \Pi^{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_{\alpha\beta}} = -\frac{1}{4\pi} G_{\mu\nu}{}^{\alpha\beta} F^{\mu\nu}. \quad (25)$$

Without loosing generality we take

$$G_{\mu\nu}{}^{\alpha\beta} = \frac{1}{2} (\delta_{\mu}^{\alpha} \delta_{\nu}^{\beta} - \delta_{\mu}^{\beta} \delta_{\nu}^{\alpha}). \quad (26)$$

In this case we have

$$\Pi^{\alpha\beta} = -\frac{1}{4\pi} F^{\alpha\beta} = -\frac{1}{4\pi} \partial^{\nu} G_{\gamma\eta\delta}{}^{\alpha\beta} \Phi^{\gamma\delta} \quad (27)$$

and

$$F_{\mu\nu} = \partial_0 \Phi_{\mu\nu} + \partial_i G_{\mu\nu}{}^{\lambda\alpha\beta} \Phi_{\alpha\beta}. \quad (28)$$

The Hamiltonian density is then

$$\mathcal{H} = \frac{1}{2} \Pi_{\phi_{\lambda\eta}} \dot{\Phi}_{\lambda\eta} - \mathcal{L} = -\pi \Pi_{\lambda\eta} \Pi^{\lambda\eta} - \frac{1}{2} \Pi^{\lambda\eta} \partial_i G_{\lambda\eta}{}^{\nu\gamma\delta} \Phi_{\gamma\delta}, \quad (29)$$

and the canonical equations

$$\dot{\Pi}^{\alpha\beta} = -\partial_k \Pi^{\lambda\eta} G_{-\lambda\eta}{}^{k\alpha\beta}, \quad (30)$$

$$\dot{\Phi}^{\alpha\beta} = -4\pi \Pi^{\alpha\beta} - \partial_i G^{\alpha\beta i\gamma\delta} \Phi_{\gamma\delta}. \quad (31)$$

We see that Eqs. (30) contain only the canonical momenta. Now, we require that these equations are the Maxwell's equations for time derivatives of the field components (!). According to (27), we first write Eq. (30) in the form

$$\dot{F}^{\alpha\beta} = -G_{\lambda\eta}{}^{k\alpha\beta} \partial_k F^{\lambda\eta}. \quad (32)$$

The Maxwell's equations (2) of the free field for time derivatives of the field components are

$$\partial_0 F^{i0} = -\partial_j F^{ij}, \quad (33)$$

$$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0.$$

Comparison of (32) and (33) gives

$$G_{\lambda\eta}{}^{kio} = \frac{1}{2} (\delta_{\lambda}^i \delta_{\eta}^k - \delta_{\lambda}^k \delta_{\eta}^i), \quad (34)$$

$$G_{\lambda\eta}{}^{kij} = \frac{1}{2} g_{\lambda 0} (\delta_{\eta}^i g^{kj} - \delta_{\eta}^j g^{ki}) - \frac{1}{2} g_{\eta 0} (\delta_{\lambda}^i g^{kj} - \delta_{\lambda}^j g^{ki}).$$

Eqs. (34) and (26) completely determine the tensor  $G_{\mu\nu}{}^{\alpha\beta}$ .

With this tensor we have

$$\begin{aligned} F_{0i} &= \partial_0 \Phi_{0i} - \partial_k \Phi_{ik}, \\ F_{ni} &= \partial_0 \Phi_{ni} - \partial_i \Phi_{on} + \partial_n \Phi_{oi}. \end{aligned} \tag{35}$$

It is convenient to introduce the notation

$$\Phi_{\mu\nu} = \begin{pmatrix} 0 & A_1 & A_2 & A_3 \\ -A_1 & 0 & -B_3 & B_2 \\ -A_2 & B_3 & 0 & -B_1 \\ -A_3 & -B_2 & B_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -A_x & -A_y & -A_z \\ A_x & 0 & B_z & -B_y \\ A_y & -B_z & 0 & B_x \\ A_z & B_y & -B_x & 0 \end{pmatrix} \tag{36}$$

with  $\mathbf{A} = (A_x, A_y, A_z)$  and  $\mathbf{B} = (B_x, B_y, B_z)$ .

Eqs. (35) then become

$$\begin{aligned} F_{0i} &= -\partial_0 (\mathbf{A})_i - (\text{rot } \mathbf{B})_i, \quad i = x, y, z, \\ F_{12} &= \partial_0 B_z - (\text{rot } \mathbf{A})_z, \\ F_{13} &= -\partial_0 B_y + (\text{rot } \mathbf{A})_y, \\ F_{23} &= \partial_0 B_x - (\text{rot } \mathbf{A})_x, \end{aligned} \tag{37}$$

or in three-dimensional notation

$$\begin{aligned} \mathbf{E} &= -\partial_t \mathbf{A} - \text{rot } \mathbf{B}, \\ \mathbf{H} &= -\partial_t \mathbf{B} + \text{rot } \mathbf{A}. \end{aligned} \tag{38}$$

Having the solution (32) and the notation (37) we may now check this solution.

According to (21) and (38) the Lagrangian density is

$$\mathcal{L} = \frac{1}{8\pi} (\mathbf{E}^2 - \mathbf{H}^2) = \frac{1}{8\pi} [(\partial_t \mathbf{A} + \text{rot } \mathbf{B})^2 - (-\partial_t \mathbf{B} + \text{rot } \mathbf{A})^2] \tag{39}$$

The Lagrange's variables are  $A_i, B_i$ . The corresponding Lagrange's equations are

$$\mathbf{A}: \quad \partial_t^2 \mathbf{A} + \text{rot rot } \mathbf{A} = 0, \tag{40}$$

$$\mathbf{B}: \quad \partial_t^2 \mathbf{B} + \text{rot rot } \mathbf{B} = 0. \tag{41}$$

The canonical momenta are

$$\Pi_{A_i} = \frac{1}{4\pi} (\partial_t \mathbf{A} + \text{rot } \mathbf{B})_i = -\frac{1}{4\pi} E_i, \quad i = x, y, z, \quad (42)$$

$$\Pi_{B_i} = \frac{1}{4\pi} (-\partial_t \mathbf{B} + \text{rot } \mathbf{A})_i = \frac{1}{4\pi} H_i \quad (43)$$

and the Hamiltonian density

$$\mathcal{H} = \Pi_{A_i} \dot{A}_i + \Pi_{B_i} \dot{B}_i - \mathcal{L} = \frac{1}{8\pi} (\mathbf{E}^2 - \mathbf{H}^2 + 2\mathbf{E} \text{ rot } \mathbf{B} + 2\mathbf{H} \text{ rot } \mathbf{A}). \quad (44)$$

The canonical equations are then

$$\dot{\Pi}_{A_i} = -\frac{1}{4\pi} (\text{rot } \mathbf{H})_i, \quad \rightarrow \quad \partial_\nu F^{1\nu} = 0,$$

$$\dot{\Pi}_{B_i} = -\frac{1}{4\pi} (\text{rot } \mathbf{E})_i, \quad \rightarrow \quad \partial_0 F_{\mu\nu} + \partial_\mu F_{\nu 0} + \partial_\nu F_{0\mu} = 0, \quad (45)$$

$$\dot{A}_i = -E_i - (\text{rot } \mathbf{B})_i$$

$$\dot{B}_i = -H_i + (\text{rot } \mathbf{A})_i$$

or written in a compact form using the vectors  $\mathbf{E}$  and  $\mathbf{H}$

$$\dot{\mathbf{E}} = \text{rot } \mathbf{H},$$

$$\dot{\mathbf{H}} = -\text{rot } \mathbf{E},$$

$$\dot{\mathbf{A}} = -\mathbf{E} - \text{rot } \mathbf{B},$$

$$\dot{\mathbf{B}} = -\mathbf{H} + \text{rot } \mathbf{A}. \quad (46)$$

The first two of Eqs. (46) are the Maxwell's equations for  $\mathbf{E}$  and  $\mathbf{H}$ . Therefore, the canonical equations for the canonical momenta are the Maxwell's equations for the time derivatives of  $\mathbf{E}$  and  $\mathbf{H}$ . In Eqs. (46), however, there are not

$$\text{div } \mathbf{E} = 0, \quad (\partial_t F^{0i} = 0),$$

$$\text{div } \mathbf{H} = 0, \quad (\partial_t F_{jk} + \partial_k F_{ij} + \partial_j F_{ki} = 0).$$

One may ask how one can get these equations? First of all, it is evident that they are not canonical equations because they do not contain any time derivatives. We can get them in the following way. Let us take divergence of the canonical equations for the canonical momenta:

$$\operatorname{div} \dot{\mathbf{E}} = 0,$$

$$\operatorname{div} \dot{\mathbf{H}} = 0.$$

From these equations follow

$$\partial_t (\operatorname{div} \mathbf{E}) = 0,$$

$$\partial_t (\operatorname{div} \mathbf{H}) = 0$$

and

$$\operatorname{div} \mathbf{E} = f(\mathbf{r}),$$

$$\operatorname{div} \mathbf{H} = g(\mathbf{r}).$$

In the case of the free field all space points are equivalent and consequently only physically meaningful solutions are  $f(\mathbf{r}) = g(\mathbf{r}) = 0$ . One can select these solutions by initial conditions. So, we have

$$\operatorname{div} \mathbf{E} = 0, \tag{47}$$

$$\operatorname{div} \mathbf{H} = 0.$$

These Maxwell's equations are, therefore, a consequence of the canonical equations and appropriate initial conditions.

Similarly from the last two of Eqs. (46) follow

$$\partial_t (\operatorname{div} \mathbf{A}) = 0,$$

$$\partial_t (\operatorname{div} \mathbf{B}) = 0$$

and by initial conditions

$$\operatorname{div} \mathbf{A} = 0, \tag{48}$$

$$\operatorname{div} \mathbf{B} = 0.$$

Let us mention that Eqs. (48) lead to (47). It can be easily checked.

With Eqs. (48) and the notation (36) the Lagrange's equations (24) become

$$\Delta \mathbf{A} - \partial_t^2 \mathbf{A} = 0, \quad (49)$$

$$\Delta \mathbf{B} - \partial_t^2 \mathbf{B} = 0$$

and the field equations (45)

$$\partial_\nu F^{\mu\nu} = 0,$$

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\lambda\nu} = 0.$$

### 5. Transformation properties of the canonical formulation

In this Section we consider the transformation properties of the developed canonical formulation of the electromagnetic field. We follow the procedure adopted at the canonical formulation of the vector field satisfying the first order differential equations<sup>1 2)</sup>.

The tensor  $G^{\mu\nu\alpha\beta}$  we write by means of the six order tensor and a four-vector:

$$G^{\mu\nu\alpha\beta} = G^{\mu\nu\gamma\alpha\beta\zeta} \Phi_\zeta \quad (50)$$

where the tensor  $G^{\mu\nu\gamma\alpha\beta\zeta}$  is constructed from the products of the metric tensors  $g^{\alpha\beta}$  and  $\varphi^\alpha$  is a four-vector with constant components.

Due to symmetry properties of the tensor  $G^{\mu\nu\alpha\beta}$  the tensor  $G^{\mu\nu\gamma\alpha\beta\zeta}$  can be written in the form

$$\begin{aligned} G^{\mu\nu\gamma\alpha\beta\zeta} = & \frac{1}{2} \{ (g^{\mu\alpha} g^{\nu\beta} - g^{\nu\alpha} g^{\mu\beta}) g^{\gamma\zeta} + (g^{\mu\alpha} g^{\nu\gamma} - g^{\nu\alpha} g^{\mu\gamma}) g^{\beta\zeta} + \\ & + (g^{\nu\beta} g^{\mu\gamma} - g^{\mu\beta} g^{\nu\gamma}) g^{\alpha\zeta} + (g^{\mu\beta} g^{\nu\zeta} - g^{\nu\beta} g^{\mu\zeta}) g^{\alpha\gamma} + \\ & + (g^{\nu\alpha} g^{\mu\zeta} - g^{\mu\alpha} g^{\nu\zeta}) g^{\beta\gamma} \}. \end{aligned} \quad (51)$$

The tensor  $G^{\mu\nu\alpha\beta}$  is then

$$\begin{aligned} G^{\mu\nu\alpha\beta} = & \frac{1}{2} \{ (g^{\mu\alpha} g^{\nu\beta} - g^{\nu\alpha} g^{\mu\beta}) \varphi^\gamma + (g^{\mu\alpha} g^{\nu\gamma} - g^{\nu\alpha} g^{\mu\gamma}) \varphi^\beta + \\ & + (g^{\nu\beta} g^{\mu\gamma} - g^{\mu\beta} g^{\nu\gamma}) \varphi^\alpha + (g^{\mu\beta} g^{\alpha\gamma} - g^{\mu\alpha} g^{\beta\gamma}) \varphi^\nu + \\ & + (g^{\nu\alpha} g^{\beta\gamma} - g^{\nu\beta} g^{\alpha\gamma}) \varphi^\mu \}. \end{aligned} \quad (52)$$

After substitution of (52) into (23) one gets

$$F_{\mu\nu} = \partial_\alpha \varphi^\alpha \Phi_{\mu\nu} + \partial_\mu \varphi^\alpha \Phi_{\alpha\nu} - \partial_\nu \varphi^\alpha \Phi_{\alpha\mu} + \partial^\alpha \varphi_\mu \Phi_{\nu\alpha} - \partial^\alpha \varphi_\nu \Phi_{\mu\alpha}. \quad (53)$$

The Lagrange's equations for  $\Phi_{\eta\zeta}$ , according to (21) and (53), are now

$$\varphi^\eta \partial_\lambda F^{\lambda\zeta} + \varphi^\zeta \partial_\lambda F^{\eta\lambda} + \varphi_\lambda (\partial^\lambda F^{\eta\zeta} + \partial^\zeta F^{\lambda\eta} + \partial^\eta F^{\zeta\lambda}) = 0 \quad (54)$$

where  $F_{\mu\nu}$  is given by (53).

As we have said in the previous Section, we expect that  $F_{\mu\nu}$  are components of the momentum of the electromagnetic field and that the canonical equations of the field determine  $F_{\mu\nu}$  independently of  $\Phi_{\alpha\beta}$ . Eqs. (54) are just equations for  $F_{\mu\nu}$ . In order that these equations become physical equations one has to impose the principle of special relativity. Similarly to the vector case<sup>1 2)</sup>, the only interesting solution is

$$\begin{aligned} \partial_\lambda F^{\lambda\zeta} &= 0, \\ \partial^\lambda F^{\eta\zeta} + \partial^\zeta F^{\lambda\eta} + \partial^\eta F^{\zeta\lambda} &= 0. \end{aligned} \quad (55)$$

Selecting these equations for  $F_{\mu\nu}$  the four-vector  $\varphi^\alpha$  becomes completely free with respect to  $F_{\mu\nu}$ .

Eqs. (55) are the first set of dynamical equations of the tensor field  $F_{\mu\nu}$ . The second set are Eqs. (53). For a given  $F_{\mu\nu}$  Eqs. (53) give  $\Phi_{\alpha\beta}$  as a particular solution. This solution depends on  $\varphi^\alpha$ . Because  $\varphi^\alpha$  are free, there is a family of particular solutions depending on  $\varphi^\alpha$ . The principle of special relativity makes a selection. In order that Eqs. (53) satisfy this principle  $\varphi^\alpha$  has to be equal in each reference system. It means that at the end of the calculation this selection has to be done. By this tensor characters of  $\varphi^\alpha$  and  $\Phi_{\alpha\beta}$  have not been changed. Only among possible tensors  $\varphi^\alpha$  (four—vectors) and  $\Phi_{\alpha\beta}$  one makes a definite selection.

The conjugate momenta to  $\Phi_{\alpha\beta}$  are defined by

$$\begin{aligned} \Pi_{\Phi_{\alpha\beta}} \equiv \Pi^{\alpha\beta} &= \frac{\partial \mathcal{L}}{\partial (\partial_0 \Phi_{\alpha\beta})} = -\frac{1}{8\pi} (\varphi^0 F^{\alpha\beta} + \varphi^\alpha F^{0\beta} + \varphi^\beta F^{\alpha 0} + g^{0\beta} \varphi_\mu F^{\mu\alpha} + \\ &+ g^{\alpha 0} \varphi_\mu F^{\beta\mu}). \end{aligned} \quad (56)$$

$\Pi^{\alpha\beta}$  is a linear combination of  $F_{\mu\nu}$ . On the other hand  $\Pi^{\alpha\beta}$  are components of the third order tensor constructed from  $F_{\mu\nu}$ ,  $\varphi^\alpha$  and  $g_{\mu\nu}$ . The selection of  $\varphi^\alpha$  based on the principle of relativity makes them the second order tensor. Due to the free selection of the four-vector  $\varphi^\alpha$  in a given reference system, we may take

$$\varphi^\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (57)$$

In this case  $\Pi^{a\beta}$  becomes

$$\Pi^{a\beta} = -\frac{1}{4\pi} F^{a\beta} \quad (58)$$

i. e. the momentum components are directly proportional to the field components. By this a simple connection of the field  $F_{\mu\nu}$  and its momentum is established. The canonical equations are naturally Eqs. (55) after substitution of  $F_{\mu}$  by  $\Pi_{\mu\nu}$  according to (58):

$$\partial_\lambda \Pi^{\lambda\zeta} = 0, \quad (59)$$

$$\partial^\lambda \Pi^{\eta\zeta} + \partial^\zeta \Pi^{\lambda\eta} + \partial^\eta \Pi^{\zeta\lambda} = 0.$$

For  $\varphi^a$  given by (57)  $F_{\mu\nu}$ , according to (53), reads

$$F_{\mu\nu} = \partial_0 \Phi_{\mu\nu} + \partial_\mu \Phi_{0\nu} - \partial_\nu \Phi_{0\mu} + \partial^\alpha \Phi_{\nu\alpha} \delta_{\mu,0} - \partial^\alpha \Phi_{\mu\alpha} \delta_{\nu,0}$$

or

$$F_{0l} = \partial_0 \Phi_{0l} - \partial_k \Phi_{lk}, \quad (60)$$

$$F_{nl} = \partial_0 \Phi_{nl} - \partial_l \Phi_{0n} + \partial_n \Phi_{0l}.$$

Eqs. (60) are equal to Eqs. (35). Therefore, the canonical formulation follows in the form as it is developed and expressed in the previous Section.

By this the character of the developed canonical formulation of the electromagnetic field with respect to the Lorentz transformations or the principle of special relativity is established. At the end let us mention that the whole analysis can be completely based on  $\Phi_{a\beta}$ .

### 6. Energy-momentum vector

In order to see some consequences of the developed formulation of the electromagnetic field we evaluate the energy-momentum vector of the free field.

The energy-momentum density tensor

$$T_\mu^\nu = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial (\partial_\nu \Phi_{a\beta})} \partial_\mu \Phi_{a\beta} - \delta_\mu^\nu \mathcal{L} \quad (61)$$

for the Lagrangian (21) with  $F_{\mu\nu}$  given by (53) is

$$T_\mu^\nu = -\frac{1}{8\pi} (\varphi^\nu F^{a\beta} + \varphi^a F^{\nu\beta} - \varphi^\beta F^{ra} + g^{\nu\beta} \varphi_\gamma F^{r\alpha} - g^{a\nu} \varphi_\gamma F^{\gamma\beta}) \partial_\mu \Phi_{a\beta} - \delta_\mu^\nu \mathcal{L}. \quad (62)$$

From

$$\partial_\nu T^\nu_\mu = 0 \quad (63)$$

follows conservation of the four-vector

$$P^\mu = \text{const} \int \{ 2(\varphi^0 F^{\alpha\beta} \partial^\mu \Phi_{\alpha\beta} + 2\varphi^\alpha F^{0\beta} \partial^\mu \Phi_{\alpha\beta} + 2\varphi_\gamma F^{\gamma\alpha} \partial^\mu \Phi_{\alpha^0}) - \\ - g^{\mu 0} F_{\alpha\beta} F^{\alpha\beta} \} d^3x. \quad (64)$$

For  $\varphi^a$  given by (57) it is

$$P^\mu = \text{const} \int (2 F^{\alpha\beta} \partial^\mu \Phi_{\alpha\beta} - g^{\mu 0} F_{\alpha\beta} F^{\alpha\beta}) d^3x. \quad (65)$$

The energy component is

$$P^0 = \text{const} 2 \int (F^{\alpha\beta} \partial^0 \Phi_{\alpha\beta} - \frac{1}{2} F_{\alpha\beta} F^{\alpha\beta}) d^3x \quad (66)$$

and the momentum components are

$$P^i = \text{const} 2 \int F^{\alpha\beta} \partial^i \Phi_{\alpha\beta} d^3x. \quad (67)$$

Introducing the notation

$$\mathbf{E}_1 = -\partial_t \mathbf{A}, \quad \mathbf{E}_2 = -\text{rot } \mathbf{B}, \quad (68)$$

$$\mathbf{H}_2 = -\partial_t \mathbf{B}, \quad \mathbf{H}_1 = \text{rot } \mathbf{A} \quad (69)$$

with, according to (38),

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2, \quad \mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 \quad (70)$$

the energy becomes

$$P^0 = -\text{const} 2 \int \{ (\mathbf{E}_1^2 + \mathbf{H}_1^2) - (\mathbf{E}_2^2 + \mathbf{H}_2^2) \} d^3x \quad (71)$$

and the momentum components

$$P^i = -\text{const} \int (\mathbf{E}_1 \times \mathbf{H}_1 - \mathbf{E}_2 \times \mathbf{H}_2) d^3x. \quad (72)$$

Expressions (71) and (72) are very similar to the standard energy-momentum vector but also very different. First, the energy is no more positive definite. This is a consequence of the term  $-(\partial_0 \mathbf{B})^2$  in the Lagrangian (39). Second, the standard

expressions one obtains for  $\mathbf{B} = 0$ . An inspection shows that this selection of the solutions gives the completely standard free electromagnetic field. However, in the quantum theory the solution  $\mathbf{B} = 0$  cannot be accepted because of the quantum rules. Third, due to (71) and (72) and the quantum rules the ground state energy of the field is zero as well as the momentum.

### 7. Field with sources

In the standard theory interaction of the electromagnetic field with charges is described by

$$\mathcal{L}_{\text{int}} = - A_\mu j^\mu \quad (73)$$

where  $A_\mu$  is the potential four-vector and  $j^\mu$  is the charge density current satisfying the continuity equation

$$\partial_\mu j^\mu = 0. \quad (74)$$

We now have the tensor  $\Phi_{\alpha\beta}$  instead of  $A_\mu$ . This tensor has to be connected with the vector  $j^\mu$ . It is evident that it cannot be done directly but with certain contractions. In the new canonical formulation for the free field appears the tensor  $G^{\mu\nu\gamma\alpha\beta}$ . This tensor can be used for this purpose.

The following contractions can be considered:

$$\begin{aligned} G^\mu_{\ \gamma}{}^{\gamma\alpha\beta} \Phi_{\alpha\beta} &\rightarrow C^\nu = 3 \varphi_\alpha \Phi^{\mu\alpha}, \\ G^\mu_{\ \alpha}{}^{\gamma\alpha\beta} \Phi_{\gamma\beta} &\rightarrow C^\mu = -\frac{1}{2} \varphi_\alpha \Phi^{\alpha\mu}, \\ G^{\mu\nu\gamma}{}_{\ \mu}{}^\beta \Phi_{\gamma\nu} &\rightarrow C^\beta = \frac{5}{2} \varphi_\alpha \Phi^{\alpha\beta}, \\ G^{\mu\nu}{}_{\ \gamma}{}^{\gamma\beta} \Phi_{\mu\nu} &\rightarrow C^\beta = \varphi_\alpha \Phi^{\beta\alpha}. \end{aligned} \quad (75)$$

Consequently, the interaction can be taken in the form

$$\mathcal{L} = a \varphi^\beta \Phi^\mu{}_\beta j_\mu \quad (76)$$

where  $a$  is a constant.

The total Lagrangian density is then

$$\mathcal{L} = \mathcal{L}_{\text{free field}} + \mathcal{L}_{\text{int}} + \mathcal{L}_j$$

where  $\mathcal{L}_j$  is the Lagrangian density of the free current.

The Lagrange's equation for  $\Phi_{\eta\zeta}$  is now (instead of (53))

$$\begin{aligned} \varphi^\eta \left\{ \partial_\lambda F^{\lambda\zeta} + \frac{a}{4k} g^{\lambda\zeta} j_\lambda \right\} - \varphi^\zeta \left\{ \partial_\lambda F^{\lambda\eta} + \frac{a}{4k} g^{\lambda\eta} j_\lambda \right\} + \varphi_\lambda \left\{ \partial^\zeta F^{\lambda\eta} + \partial^\lambda F^{\eta\zeta} + \right. \\ \left. + \partial^\eta F^{\zeta\lambda} \right\} = 0 \end{aligned} \quad (77)$$

where  $F_{\mu\nu}$  is given by (53) and  $k = -1/16\pi$ .

The principle of special relativity now leads to

$$\begin{aligned} \partial_\lambda F^{\zeta\lambda} &= \frac{a}{4k} j^\zeta, \\ \partial^\zeta F^{\lambda\eta} + \partial^\lambda F^{\eta\zeta} + \partial^\eta F^{\zeta\lambda} &= 0. \end{aligned} \quad (78)$$

Selection of these equations for  $F_{\mu\nu}$  leaves again free the four-vector  $\varphi^\alpha$  with respect to  $F_{\mu\nu}$ . Comparison with Eq. (2) gives for  $a = 1$ .

The second set of dynamical equations for the field  $F_{\mu\nu}$  remains the same as in the case of the free field, i. e. they are given by Eq. (53). Therefore, the application of the principle of special relativity to Eq. (53) gives the same results.

The conjugate momenta to  $\Phi_{\alpha\beta}$  are given by (56) or selecting  $\Phi_{\alpha\beta}$  according to (57) by (58). After substitution of  $F_{\alpha\beta}$  from (58) into (78) one gets the canonical equations for the momenta:

$$\begin{aligned} \partial_\lambda \Pi^{\zeta\lambda} &= j^\zeta, \\ \partial^\lambda \Pi^{\eta\zeta} + \partial^\eta \Pi^{\zeta\lambda} + \partial^\zeta \Pi^{\lambda\eta} &= 0. \end{aligned} \quad (79)$$

After substitution of  $\varphi^\alpha$  from (57) into (53) one gets the corresponding equations for the Lagrange's variables of the field:

$$\begin{aligned} \partial_0 \Phi_{0l} &= -\frac{1}{4\pi} \Pi_{0l} + \partial_k \Phi_{lk}, \\ \partial_0 \Phi_{nl} &= -\frac{1}{4\pi} \Pi_{nl} + \partial_l \Phi_{on} - \partial_n \Phi_{ol}. \end{aligned} \quad (80)$$

The selection of  $\varphi^\alpha$  can be incorporated in the Lagrangian in a given reference system at the beginning, as it was done for the free field in Section 4. Here we present it in a short form for the field with sources. We use the notation (3) and (36).

Starting from the Lagrangian (39) and

$$\mathcal{L}_{int} = \Phi^\mu j_\mu = \mathbf{A} \cdot \mathbf{j}$$

the Lagrange's equations for  $\mathbf{A}$  and  $\mathbf{B}$  are, respectively,

$$\begin{aligned}\partial_t^2 \mathbf{A} + \text{rot rot } \mathbf{A} &= 4\pi \mathbf{j}, \\ \partial_t^2 \mathbf{B} + \text{rot rot } \mathbf{B} &= 0.\end{aligned}\tag{81}$$

The Hamiltonian density is

$$\mathcal{H} = \mathcal{H}_{free\ field} - \mathbf{A} \cdot \mathbf{j}.\tag{82}$$

The corresponding canonical equations are

$$\dot{H}_{A_i} = -\frac{1}{4\pi} (\text{rot } \mathbf{H})_i + j_i,\tag{83}$$

$$\dot{H}_{B_i} = -\frac{1}{4\pi} (\text{rot } \mathbf{E})_i,$$

$$\dot{A}_i = -E_i - (\text{rot } \mathbf{B})_i,\tag{84}$$

$$\dot{B}_i = -H_i + (\text{rot } \mathbf{A})_i$$

or

$$\dot{\mathbf{E}} = \text{rot } \mathbf{H} - 4\pi \mathbf{j},\tag{85}$$

$$\dot{\mathbf{H}} = -\text{rot } \mathbf{E},$$

$$\dot{\mathbf{A}} = -\mathbf{E} - \text{rot } \mathbf{B},\tag{86}$$

$$\dot{\mathbf{B}} = -\mathbf{H} + \text{rot } \mathbf{A}.$$

From Eqs. (85) and (74) follow

$$\text{div } \dot{\mathbf{E}} = 4\pi \dot{\rho},$$

$$\text{div } \dot{\mathbf{H}} = 0$$

or

$$\partial_t (\text{div } \mathbf{E} - 4\pi \rho) = 0,\tag{87}$$

$$\partial_t (\text{div } \mathbf{H}) = 0.$$

From here we have

$$\begin{aligned}\operatorname{div} \mathbf{E} &= 4\pi\rho + f(\mathbf{r}), \\ \operatorname{div} \mathbf{H} &= F(\mathbf{r}).\end{aligned}\tag{88}$$

Similar arguments as in the case of Eqs. (47) lead to

$$\begin{aligned}\operatorname{div} \mathbf{E} &= 4\pi\rho, \\ \operatorname{div} \mathbf{H} &= 0.\end{aligned}\tag{89}$$

The corresponding equations for  $\mathbf{A}$  and  $\mathbf{B}$  read (equation for  $\mathbf{B}$  remains in the same form)

$$\begin{aligned}\operatorname{div} \dot{\mathbf{A}} &= -4\pi\rho, \\ \operatorname{div} \dot{\mathbf{B}} &= 0.\end{aligned}\tag{90}$$

Using Eqs. (89) the field equations become

$$\begin{aligned}\dot{\mathbf{E}} &= \operatorname{rot} \mathbf{H} - 4\pi \mathbf{j}, \\ \dot{\mathbf{H}} &= -\operatorname{rot} \mathbf{E}, \\ \operatorname{div} \mathbf{E} &= 4\pi\rho, \\ \operatorname{div} \mathbf{H} &= 0, \\ \dot{\mathbf{A}} &= -\mathbf{E} - \operatorname{rot} \mathbf{B}, \\ \dot{\mathbf{B}} &= -\mathbf{H} + \operatorname{rot} \mathbf{A}.\end{aligned}\tag{91}$$

Besides the current  $j^\mu$  one can introduce also the pseudo-vector current  $j_p^\mu$  satisfying the continuity equation

$$\partial_\mu j_p^\mu = 0\tag{92}$$

making contractions of the pseudo-tensor  $\tilde{\Phi}_{\alpha\beta}$  with  $\varphi^\alpha$ , where  $\Phi_{\alpha\beta}$  is dual tensor to  $\tilde{\Phi}_{\alpha\beta}$ , i. e.

$$\tilde{\Phi}_{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} \Phi^{\mu\nu}\tag{93}$$

and  $\varepsilon_{\alpha\beta\mu\nu}$  is a unit totally antisymmetric tensor. Similarly to (75), in this case one gets

$$\begin{aligned} G^{\mu}{}_{\gamma}{}^{\nu\alpha\beta} \tilde{\Phi}_{\alpha\beta} &\rightarrow D^{\mu} = 3 \varphi_{\alpha} \tilde{\Phi}^{\mu\alpha}, \\ G^{\mu}{}_{\gamma}{}^{\nu\alpha\beta} \tilde{\Phi}_{\gamma\beta} &\rightarrow D^{\mu} = -\frac{1}{2} \varphi_{\alpha} \tilde{\Phi}^{\alpha\mu}, \\ G^{\mu\nu\gamma}{}_{\mu}{}^{\beta} \tilde{\Phi}_{\gamma\nu} &\rightarrow D^{\beta} = \frac{5}{2} \varphi_{\alpha} \tilde{\Phi}^{\alpha\beta}, \\ G^{\mu\nu}{}_{\gamma}{}^{\nu\beta} \tilde{\Phi}_{\mu\nu} &\rightarrow D^{\beta} = \varphi_{\alpha} \tilde{\Phi}^{\beta\gamma}. \end{aligned} \tag{94}$$

and consequently

$$\mathcal{L}_{int,p} = b \varphi^{\beta} \tilde{\Phi}^{\mu}{}_{\beta} (j_p)_{\mu} \tag{95}$$

where  $b$  is a constant. The total interaction is

$$\mathcal{L}_{int} = a \varphi^{\beta} \Phi_{\beta}{}^{\mu} j_{\mu} + b \varphi^{\beta} \tilde{\Phi}_{\beta}{}^{\mu} (j_p)_{\mu}. \tag{96}$$

The Lagrange's equation for  $\Phi_{\eta\zeta}$  is now

$$\begin{aligned} \varphi^{\zeta} \left( \frac{a}{4k} j^{\eta} - \partial_{\lambda} F^{\eta\lambda} \right) - \varphi^{\eta} \left( \frac{a}{4k} j^{\zeta} + \partial_{\lambda} F^{\lambda\zeta} \right) + \\ + \varphi_{\lambda} \left\{ \frac{b}{4k} \varepsilon^{\mu\lambda\eta\zeta} (j_p)_{\mu} - (\partial^{\lambda} F^{\eta\zeta} + \partial^{\zeta} F^{\lambda\eta} + \partial^{\eta} F^{\zeta\lambda}) \right\} = 0. \end{aligned} \tag{97}$$

The principle of special relativity leads to

$$\partial_{\lambda} F^{\zeta\lambda} = \frac{a}{4k} j^{\zeta}, \tag{98}$$

$$\partial^{\lambda} F^{\eta\zeta} + \partial^{\zeta} F^{\lambda\eta} + \partial^{\eta} F^{\zeta\lambda} = \frac{a}{4k} \varepsilon^{\mu\lambda\eta\zeta} (j_p)_{\mu}$$

or

$$\dot{\mathbf{E}} = \text{rot } \mathbf{H} - 4\pi \mathbf{j},$$

$$\dot{\mathbf{H}} = -\text{rot } \mathbf{E} + 4\pi \mathbf{j}_p, \tag{99}$$

$$\operatorname{div} \mathbf{E} = 4\pi \rho,$$

$$\operatorname{div} \mathbf{H} = -4\pi \rho_p.$$

Selection of Eqs. (98) leaves again free the four-vector  $\varphi^a$  and follow the conclusions as in the previous cases. We give it in a short form.

Using the notation (3) and (36), the total  $\mathcal{L}_{int}$  is

$$\mathcal{L} = \mathbf{A} \cdot \mathbf{j} + \mathbf{B} \cdot \mathbf{j}_p \quad (100)$$

where  $a$  and  $b$  are selected to be 1. The Lagrange's equations, the Hamiltonian density and the canonical equations are then, respectively,

$$\partial_t^2 \mathbf{A} + \operatorname{rot} \operatorname{rot} \mathbf{A} = 4\pi \mathbf{j}, \quad (101)$$

$$\partial_t^2 \mathbf{B} + \operatorname{rot} \operatorname{rot} \mathbf{B} = -4\pi \mathbf{j}_p,$$

$$\mathcal{H} = \mathcal{H}_{free\ field} - \mathbf{A} \cdot \mathbf{j} - \mathbf{B} \cdot \mathbf{j}_p, \quad (102)$$

$$\dot{H}_{A_i} = -\frac{1}{4\pi} (\operatorname{rot} \mathbf{H})_i + (\mathbf{j})_i, \quad (103)$$

$$\dot{H}_{B_i} = -\frac{1}{4\pi} (\operatorname{rot} \mathbf{E})_i + (\mathbf{j}_p)_i,$$

$$\dot{A}_i = +E_i - (\operatorname{rot} \mathbf{B})_i, \quad (104)$$

$$\dot{B}_i = -H_i + (\operatorname{rot} \mathbf{A})_i$$

or

$$\dot{\mathbf{E}} = \operatorname{rot} \mathbf{H} - 4\pi \mathbf{j}, \quad (105)$$

$$\dot{\mathbf{H}} = -\operatorname{rot} \mathbf{E} + 4\pi \mathbf{j}_p,$$

$$\dot{\mathbf{A}} = -\mathbf{E} - \operatorname{rot} \mathbf{B}, \quad (106)$$

$$\dot{\mathbf{B}} = -\mathbf{H} + \operatorname{rot} \mathbf{A}.$$

From Eqs. (105), (74) and (92) now follow

$$\begin{aligned}\partial_t (\operatorname{div} \mathbf{E} - 4\pi\rho) &= 0, \\ \partial_t (\operatorname{div} \mathbf{H} + 4\pi\rho_p) &= 0\end{aligned}\tag{107}$$

and

$$\begin{aligned}\operatorname{div} \mathbf{E} &= 4\pi\rho, \\ \operatorname{div} \mathbf{H} &= -4\pi\rho_p.\end{aligned}\tag{108}$$

With this selection the final form of the canonical equations becomes

$$\begin{aligned}\dot{\mathbf{E}} &= \operatorname{rot} \mathbf{H} - 4\pi \mathbf{j}, \\ \dot{\mathbf{H}} &= -\operatorname{rot} \mathbf{E} + 4\pi \mathbf{j}_p, \\ \operatorname{div} \mathbf{E} &= 4\pi\rho,\end{aligned}\tag{109}$$

$$\operatorname{div} \mathbf{H} = -4\pi\rho_p,$$

$$\dot{\mathbf{A}} = -\mathbf{E} - \operatorname{rot} \mathbf{B},\tag{110}$$

$$\dot{\mathbf{B}} = -\mathbf{H} + \operatorname{rot} \mathbf{A}.$$

One gets again the standard Maxwell's field by taking  $\mathbf{B} = 0$ .

### 8. Standard potential method

One can get the connection of the developed canonical formulation of the electromagnetic field and the standard potential method by decomposition of the vectors  $\mathbf{A}$  and  $\mathbf{B}$  in two parts:

$$\mathbf{A} = \mathbf{a} + \boldsymbol{\alpha},\tag{111}$$

$$\mathbf{B} = \mathbf{b} + \boldsymbol{\beta}.$$

Eqs. (101) then read

$$\ddot{\mathbf{a}} - \Delta \mathbf{a} + \{\operatorname{grad} \operatorname{div} \mathbf{a} + \ddot{\boldsymbol{\alpha}} + \operatorname{rot} \boldsymbol{\alpha}\} = 4\pi \mathbf{j},\tag{112}$$

$$\ddot{\mathbf{b}} - \Delta \mathbf{b} + \{\operatorname{grad} \operatorname{div} \mathbf{b} + \ddot{\boldsymbol{\beta}} + \operatorname{rot} \operatorname{rot} \boldsymbol{\beta}\} = -4\pi \mathbf{j}_p.$$

The freedom introduced by the decomposition (111) we use now to eliminate the curly brackets in these equations:

$$\begin{aligned}\text{grad div } \mathbf{a} + \ddot{\mathbf{a}} + \text{rot rot } \mathbf{a} &= 0, \\ \text{grad div } \mathbf{b} + \ddot{\mathbf{b}} + \text{rot rot } \mathbf{b} &= 0.\end{aligned}\tag{113}$$

These are equations for  $\mathbf{a}$  and  $\mathbf{b}$ . In the next step we introduce new functions  $\varphi$  and  $\Phi$  instead of  $\mathbf{a}$  and  $\mathbf{b}$  according to

$$\begin{aligned}\dot{\mathbf{a}} &= \nabla \varphi, \text{rot } \mathbf{a} = 0, \\ \dot{\mathbf{b}} &= \nabla \Phi, \text{rot } \mathbf{b} = 0.\end{aligned}\tag{114}$$

Eqs. (113) then become

$$\begin{aligned}\text{grad} (\text{div } \mathbf{a} + \dot{\varphi}) &= 0, \\ \text{grad} (\text{div } \mathbf{b} + \dot{\Phi}) &= 0.\end{aligned}\tag{115}$$

The rest of the introduced freedom, now present in the function  $\varphi$  and  $\Phi$ , we eliminate by

$$\begin{aligned}\text{div } \mathbf{a} + \dot{\varphi} &= 0, \\ \text{div } \mathbf{b} + \dot{\Phi} &= 0.\end{aligned}\tag{116}$$

After this step Eqs. (111) and (90) with extension to the pseudo-vector source term become

$$\begin{aligned}\ddot{\mathbf{a}} - \Delta \mathbf{a} &= 4\pi \mathbf{j}, \\ \ddot{\varphi} - \Delta \varphi &= 4\pi \rho,\end{aligned}\tag{117}$$

$$\begin{aligned}\ddot{\mathbf{b}} - \Delta \mathbf{b} &= -4\pi \mathbf{j}_p, \\ \ddot{\Phi} - \Delta \Phi &= -4\pi \rho_p.\end{aligned}\tag{118}$$

The substitution of  $\mathbf{A}$  and  $\mathbf{B}$  from (111) into last two equations of the canonical equations (110) gives

$$\begin{aligned}\dot{\mathbf{a}} + \nabla \varphi &= -\mathbf{E} - \text{rot } \mathbf{B}, \\ \dot{\mathbf{b}} + \nabla \Phi &= -\mathbf{H} + \text{rot } \mathbf{A},\end{aligned}\tag{119}$$

or

$$\begin{aligned}\mathbf{E} &= -\dot{\mathbf{a}} - \nabla\varphi - \text{rot } \mathbf{b}, \\ \mathbf{H} &= -\dot{\mathbf{b}} - \nabla\Phi + \text{rot } \mathbf{a},\end{aligned}\tag{120}$$

where Eqs. (114) are also taken into account.

One gets the standard theory by taking  $\mathbf{B} = 0$ , i. e.  $\mathbf{b} = \boldsymbol{\beta} = 0$ :

$$\begin{aligned}\mathbf{E} &= -\dot{\mathbf{a}} - \nabla\varphi, \\ \mathbf{H} &= \text{rot } \mathbf{a}, \\ \dot{\mathbf{a}} - \Delta \mathbf{a} &= 4\pi \mathbf{j}, \\ \ddot{\varphi} - \Delta \varphi &= 4\pi \rho, \\ \text{div } \mathbf{a} + \dot{\varphi} &= 0.\end{aligned}$$

## 9. Conclusions

Correct canonical formulation of the electromagnetic field is developed. In this formalism two tensors, (3) and (39), are canonical pair. The standard potentials are not canonical pairs to the field components. The standard potential can be connected with the canonical variables by suitable decomposition of the tensor components (111) and suitable conditions on the new quantities, (114) and (115).

The canonical equations for the momenta can always be taken in the form of the Maxwell's equations or the Maxwell's equations with two kind of sources ((78), (79), (98)). Their conjugate variables are particular solution of last two of Eqs. (91) (or (80)) or (106).

The new canonical formulation has serious consequences. Some of them are manifested in the energy-momentum vector. The energy is no more positive definite quantity. Selecting solution  $\mathbf{B}=0$  it becomes positive definite. However, in the quantum theory this solution is not allowed because the quantum rules in this case cannot be satisfied. Another remark can be made with respect to the quantum theory. The ground state energy as well as the momentum are zero, according to (71) and (72).

The obtained results illustrate also the problem of magnetic monopoles. We do not consider particularly this problem here<sup>17, 18)</sup>.

At the end, one has to say that the tensor  $\Phi_{ab}$  allows for an interaction with some other sources besides the standard charges.

Application of the new canonical formulation of the electromagnetic field to various problems and investigation of related consequences will be considered in subsequent papers.

## References

- 1) W. Heisenberg and W. Pauli, *Z. Physik* **56** (1926) 1; **59** (1930) 168;
- 2) E. Fermi, *Rev. Mod. Phys.* **4** (1932) 125;
- 3) E. C. G. Stueckelberg, *Helv. Phys. Acta* **11** (1938) 225;
- 4) S. Gupta, *Proc. Phys. Soc. London* **A 63** (1950) 681;
- 5) K. Nishijima, *Field and particles*, W. A. Benjamin, Inc., New York, (1969);
- 6) P. A. M. Dirac, *Lectures on Quantum Field Theory*, Academic Press, Inc., New York, (1966);
- 7) R. V. Kamat, *Nuovo Cimento* **43 B** (1978) 141;
- 8) L. I. Schiff, *Quantum Mechanics*, Mc Graw-Hill, Inc., New York, (1968);
- 9) K. Ljolje, The canonical formalism for particles and fields, prepared for press;
- 10) J. Brana and K. Ljolje, *FIZIKA* **10** (1978) 85;
- 11) K. Ljolje and S. Vobornik, *FIZIKA* **11** (1979) 171;
- 12) K. Ljolje and N. Marjanović-Gabela, *FIZIKA* **11** (1979) 181;
- 13) A. I. Ahiezer, V. B. Beresteckii, *Kvantovaja elektrodinamika*, Nauka, Moskva, (1969);
- 14) J. Brana and K. Ljolje, *FIZIKA* **6** (1974) 117;
- 15) J. Brana and K. Ljolje, *FIZIKA* **7** (1975) 1;
- 16) J. Brana and K. Ljolje, *RADOVI — LIX* **16** Akademija nauka i umjetnosti Bosne i Hercegovine, (1976);
- 17) F. Rohrlich, *Phys. Rev.* **150** (1966) 1104;
- 18) V. I. Stražev, L. M. Tomiljčik, *Elektrodinamika s magnitnim zarjadom*, »Nauka i tehnika«, Moskva, (1975).

## KANONSKA FORMULACIJA KLASIČNOG ELEKTROMAGNETSKOG POLJA

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Razmatrane su poteškoće Lagrangeove i kanonske formulacije elektromagnetskog polja i data je nova formulacija u kojoj su kanonski parovi tenzori, od kojih je jedan klasični tenzor elektromagnetskog polja. Utvrđena je i veza sa standardnom metodom potencijala. U okviru izučavanja posljedica nove kanonske formulacije izračunat je četverovektor energije-impulsa i ustanovljene su neke posljedice u tom smislu u kvantnoj teoriji ovog polja. Razmatranja uključuju i magnetske monopole.