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DIFFRACTION DUE TO A CROSS-SHAPED APERTURE AND CROSSED SLITS

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Theoretical solution of Fresnel and Fraunhofer diffraction due to cross-shaped apertures of finite, symmetrical and infinite laterals is presented. Diffraction integrals and irradiance distribution of the mentioned types of apertures are found and discussed for both Fresnel and Fraunhofer diffraction.

1. *Introduction*

Among the variety of shapes of diffraction apertures, the rectangular aperture and the slit has attracted a considerable attention[1-3]. The finite cross-shaped aperture treated here, can be considered as a combination of two overlapping rectangular apertures. The results obtained in this paper for the diffraction amplitude and the irradiance distribution for both the Fresnel and the Fraunhofer diffraction can be used in cases when the near and far field diffraction effects of optical systems involving rectangular mirrors or apertures are studied. Also, the cross-shaped diffraction aperture of infinite laterals is an analogue of the intersecting slit like objects, subject to diffraction examination.

2. The Fresnel diffraction due to a cross-shaped aperture

An opaque screen, situated in the (ξ, η)-plane, has a cross-shaped aperture whose laterals are of different lengths and widths. (Fig. 1). The receiving (x, y)-plane is a distance b apart from the aperture plane. On the oposite side, a distance a apart from the aperture plane, a point source of monochromatic light of wavelenght λ is situated. The origin of the aperture plane coincides with the intersection of the laterals axes.

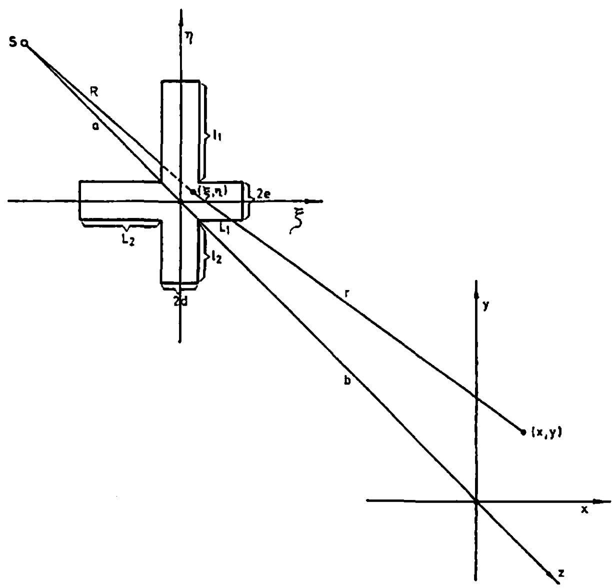

Fig. 1. Considered configuration

The diffracted wave amplitude at point (x, y, b) of the receiving plane, according to the Kirchhoff aproximation, is given by the diffraction integral

$$g(x, y, b) = -\,\mathrm{i}/(2\lambda ab) \int\!\!\int_{\sigma} G(\xi, \eta) \exp\,[\mathrm{i}k\,(R + r)]\; \mathrm{d}\xi\, \mathrm{d}\eta \tag{1}$$

where k is a propagation number $k = 2\,\pi/\lambda$, $G(\xi, \eta)$ is the aperture transmittance function

$$G(\xi, \eta) = \begin{cases} 1 & \text{on } \sigma \\ 0 & \text{elsewhere} \end{cases} \tag{2}$$

and the integration goes over the surface σ of the aperture. If $(\xi, \eta, 0)$ is one of the points in the aperture plane, then it's distance R from the point source, and it's distance r from an arbitrary point (x, y, b) in the receiving plane, as it can be seen from Fig. 1, are given by

$$R = [a^2 + \xi^2 + \eta^2]^{1/2}$$
$$r = [b^2 + (x - \xi)^2 + (y - \eta)^2]^{1/2}.$$

Providing that the dimensions of the aperture are small compared to the distance a and b, and that we examine the field distribution close to the geometrical shadow of the aperture in the receiving plane, we use the so called Fresnel aproximation for the distances R and r

$$R \cong a + (\xi^2 + \eta^2)/2a \qquad |\xi| \,;\, |\eta| \ll a$$
$$r \cong b + [(x - \xi)^2 + (y - \eta)^2]/2b \qquad |(x - \xi)| \,;\, |(y - \eta)| \ll b \tag{3}$$

consisting of up to the square power aproximation of the binomial series expansion of the expressions for the distances R and r.

Considering L_1, L_2, l_1, l_2 as the lenghts of the cross laterals, and d, e as their widths, and introducing expressions (3) and (2) in (1), we get for the diffracted wave amplitude in Fresnel-Kirchhoff aproximation

$$g \cong [-\mathrm{i}\exp(\mathrm{i}\delta)/2\lambda ab]\,[A + B + D] \tag{4}$$

where

$$\delta = k\,[a + b + (x^2 + y^2)/2b] \tag{5}$$

and

$$A = \int_{-(L_2+e)}^{-d} F(\xi)\,\mathrm{d}\xi \int_{-e}^{e} F(\eta)\,\mathrm{d}\eta \;;\; B = \int_{-d}^{d} F(\xi)\,\mathrm{d}\xi \int_{-(l_2+e)}^{(l_1+e)} F(\eta)\,\mathrm{d}\eta$$
$$D = \int_{d}^{L_1+d} F(\xi)\,\mathrm{d}\xi \int_{-e}^{e} F(\eta)\,\mathrm{d}\eta \tag{6}$$

with

$$F(\xi) = \exp\{\mathrm{i}k\,[(\xi^2/2)\,(l/a + l/b) - x\,\xi/b]\}$$
$$F(\eta) = \exp\{\mathrm{i}k\,[\eta^2/2)\,(l/a + \iota/b) - y\,\eta/b]\}. \tag{7}$$

Solution of the integrals (6) leads to the following expression for the wave amplitude of the diffracted light by the cross-shaped aperture

$$g \cong [-\mathrm{i}\exp(\mathrm{i}\Delta)/(2\lambda abp^2)]\,[(A_1 A_3 + B_1 B_3 + D_1 D_3 - A_2 A_4 - B_2 B_4 - D_2 D_4) + \mathrm{i}\,(A_1 A_4 + A_2 A_3 + B_1 B_4 + B_2 B_3 + D_1 D_4 + D_2 D_3)] \tag{8}$$

n which

$$\Delta = \delta - k\,[(q/p)^2 + (t/p)^2]$$

and notations A_j, B_j, D_j $(j = 1, 2, 3, 4)$ stand for

$$A_1 = C\,[p\,(L_2 + d) + q/p] - C\,[pd + q/p]$$
$$A_2 = S\,[p\,(L_2 + d) + q/p] - S\,[p\mathrm{a} + q/p]$$
$$A_3 = C\,[pd + t/p] + C\,[pd - t/p]$$
$$A_4 = S\,[pd + t/p] + S\,[pd - t/p]$$

$$
\begin{aligned}
B_1 &= C\,[pd - q/p] + C\,[pd + q/p] \\
B_2 &= S\,[pd - q/p] + S\,[pd + q/p] \\
B_3 &= C\,[p\,(l_1 + e) - t/p] + C\,[p\,(l_2 + e) + t/p] \\
B_4 &= S\,[p\,(l_1 + e) - t/p] + S\,[p\,(l_2 + e) + t/p] \\
D_1 &= C\,[p\,(L_1 + d) - q/p] - C\,[pd - q/p] \\
D_2 &= S\,[p\,(L_1 + d) - q/p] - S\,[pd - q/p] \\
D_3 &= C\,[pe - t/p] + C\,[pe + t/p] \\
D_4 &= S\,[pe - t/p] + S\,[pe + t/p]
\end{aligned}
\tag{9}
$$

with

$$
p^2 = \frac{2}{\lambda}\left(\frac{1}{a} + \frac{1}{a}\right) \;;\; q = \frac{2\,x}{\lambda\,b} \;;\; t = \frac{2\,y}{\lambda\,b} \tag{10}
$$

$C\,(u)$ and $S\,(u)$ appearing in the expressions (9) are the Fresnel integrals.

The function of the diffracted irradiance distribution is found by multiplying the expression (8) with its conjugate. Therefore

$$
I\,(x, y, b) = g\,g^* \cong \frac{1}{16\,(a + b)^2}\Big\{[A_1\,A_3 + B_1\,B_3 + D_1\,D_3 - A_2\,A_4 - B_2\,B_4 - D_2\,D_4]^2 + [A_1\,A_4 + A_2\,A_3 + B_1\,B_4 + B_2\,B_3 + D_1\,D_4 - D_2\,D_3]^2\Big\}. \tag{11}
$$

This general expression is specialised for the case of aperture in the form of equilateral cross. Taking for this particular case

$$
e = d;\quad l_1 = l_2 = L_1 = L_2 = l \tag{12}
$$

it is easy to verify that

$$
A_3 = D_3 \quad \text{and} \quad A_4 = D_4 \tag{13}
$$

and therefore

$$
g\,g^* \cong \frac{1}{16\,(a + b)^2}\Big\{[A_3\,(A_1 + D_1) + B_1\,B_3 - A_4\,(A_2 + D_2) - B_2\,B_4]^2 + [A_3\,(A_2 + D_2) + A_4\,(A_1 + D_1) + B_1\,B_4 + B_2\,B_3]^2\Big\}. \tag{14}
$$

It is also evident that

$$
I\,(0, t, b) = I\,(q, 0, b) \tag{15}
$$

i. e. the symmetrical cross-aperture, gives a symmetrical diffraction distribution. Therefore it is sufficient to know the irradiance distribution in the first quadrangle of the (x, y)-plane, and have it repeated in the other three, since

$$
I\,(q, t) = I\,(q, -t) = I\,(-q, t) = I\,(-q, -t). \tag{16}
$$

On the z-axis we put $q = t = 0$ and get the following irradiance distribution

$$I_0 = \frac{1}{(a+b)^2} \left[C^2 (pd) + S^2 (pd) \right] \left\{ \left[2\, C \left[p\,(l+d) \right] - C\,(pd) \right]^2 + \right.$$

$$\left. + \left[2\, S \left[p\,(l+d) \right] - S\,(pd) \right]^2 \right\}. \qquad (17)$$

The first multiplyer in this expression $\frac{1}{(a+b)^2} \left[C_2 (pd) + S^2 (pd) \right]$ is recognised as the axial irradiance distribution of a slit of width d[2]. It determines the position of the focal planes of the aperture. (They are found for those values pd, for which the distance between the origin and a point on cornu spiral of parameter (pd) reaches its maximum). The second multiplyer comes as a consequence of the form and the finite dimensions of the cross-shaped slits.

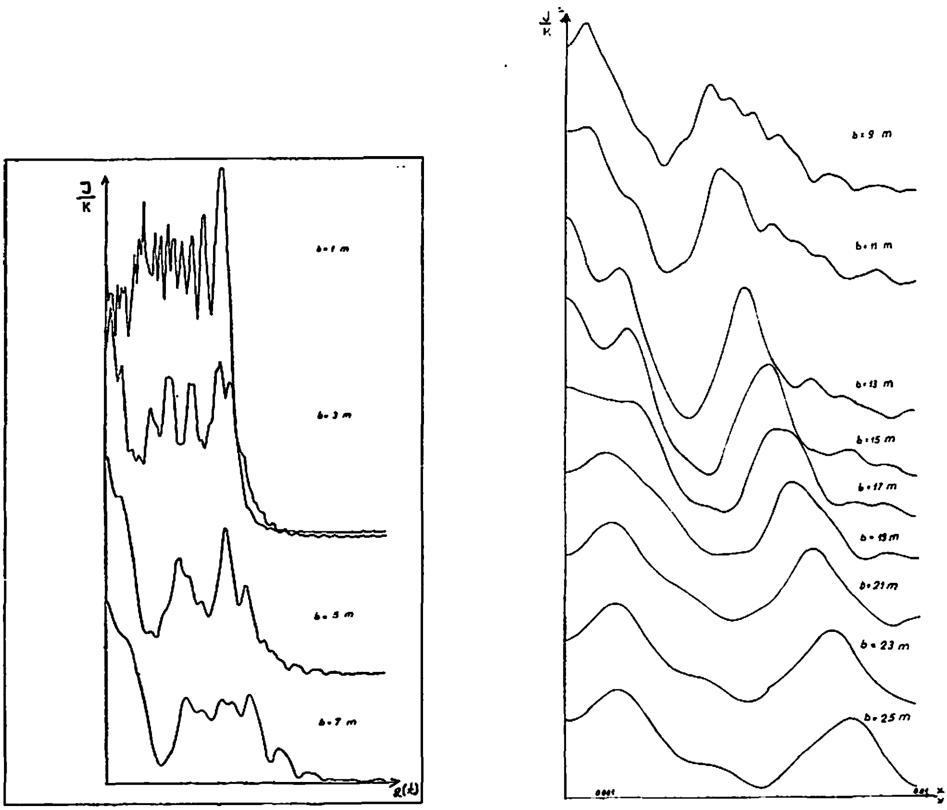

Fig. 2. Plots of the irradiance distribution along one of the axes in the receiving plane, due to Fresnel diffraction, calculated by Eq. (14) by means of a computer. The corresponding value of b joins the curves.

In order to get an impression of the irradiance distribution of the cross-like apertures, on Fig. 2 we give computer calculated plots of irradiance on the q or t (x or y) axis, when, the aperture is an equilateral cross with $l = d = 1$ mm. The distance $a = 15$ m, while each plot is joined by the corresponding value of b. The plots are done for $\lambda = 632{,}8$ nm.

3. Fresnel diffraction due to crossed slits

The form of the aperture situated in the (ξ, η)-plane (which except along the slits is opaque) is given on Fig. 3. The aperture is bounded by the lines 1, 2, 3, 4 whoose equations correspondingly are given by

1. $\xi = -a$ (18)
2. $\xi = a$
3. $\xi \cos\alpha + \eta \sin\alpha - e = 0$ (19)
4. $\xi \cos\alpha + \eta \sin\alpha + e = 0.$

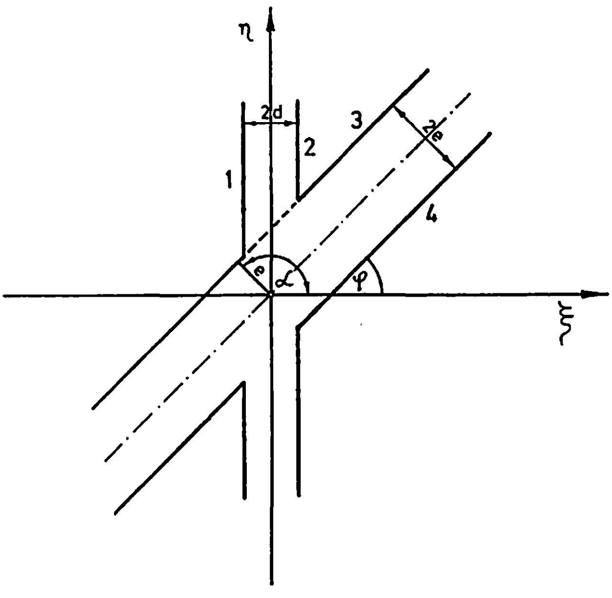

Fig. 3. Geometry of the crossed-slits aperture.

In (19) α is the angle which the normal taken from the origin towards the lines 3 and 4 makes with the positive ξ-axis.

$$\alpha = \frac{\pi}{2} + \varphi \qquad (20)$$

$\left(\frac{\pi}{2} - \varphi\right)$ being the angle of intersection of the slits.

The diffraction integral for this case is given by

$$g(\xi, \eta, b) = \frac{-\,\mathrm{i}\, e^{\mathrm{i}\delta}}{2\lambda a b}\left\{\int\limits_{-d}^{d} F(\xi)\,\mathrm{d}\xi \int\limits_{-\infty}^{\infty} F(\eta)\,\mathrm{d}\eta + \int\limits_{-\infty}^{-d} F(\xi)\,\mathrm{d}\xi \int\limits_{\eta_4}^{\eta_3} F(\eta)\,\mathrm{d}\eta + \right.$$

$$\left. + \int\limits_{d}^{\infty} F(\xi)\,\mathrm{d}\xi \int\limits_{\eta_4}^{\eta_3} F(\eta)\,\mathrm{d}\eta \right\} \qquad (21)$$

where the boundaries η_3 and η_4 are calculated from (19)

$$\eta_3 = \frac{e}{\sin\alpha} - \xi\,\mathrm{ctg}\alpha \qquad \eta_4 = -\frac{e}{\sin\alpha} - \xi\,\mathrm{ctg}\alpha. \qquad (22)$$

We shall perform the integration concerning the slits intersecting each other under an angle of 90° i. e. $\alpha = \pi/2$ $(\varphi = 0)$

$$g = -\frac{\mathrm{i}\, e^{\mathrm{i}\Delta}}{2\,a\,b\,p^2}\left\{ (1+\mathrm{i})\,\{[C(pd+q/p) + C(pd-q/p)] + \mathrm{i}\,[S(pd+q/p) + \right.$$

$$+ S(pd-q/p)]\} + \{[S(pe-t/p) + C(pe+t/p)] + \mathrm{i}\,[S(pe-t/p) +$$

$$+ S(pe+t/p)]\}\,\{[1/2 - C(pd+q/p)] + \mathrm{i}\,[1/2 - S(pd+q/p)]\} +$$

$$+ \{[C(pe-t/p) + C(pe+t/p)] + \mathrm{i}\,[S(pe-t/p) + S(pe+t/p)]\}\,\{[1/2 -$$

$$\left. - C(pd-q/p)] + \mathrm{i}\,[1/2 - S(pd-q/p)]\}\right\}. \qquad (23)$$

Now, if in addition we take that the slits are of equal widths $e = d$, we shall get the following expression for the irradiance distribution

$$I(q, t, b) = \frac{1}{16\,(a+b)^2}\left\{\Big[[C(pd+t/p) + C(pd-t/p)]\,[1 - C(pd+q/p) - \right.$$

$$- C(pd-q/p)] + [C(pd+q/p) + C(pd-q/p)] - [S(pd+t/p) +$$

$$+ S(pd-t/p)]\,[1 - S(pd+q/p) - S(pd-q/p)] - [S(pd+q/p) +$$

$$+ S(pd-q/p)]\Big]^2 + \Big[[C(pd+t/p) + C(pd-t/p)]\,[1 - S(pd +$$

$$+ q/p) - S(pd-q/p)] + [S(pd+t/p) + S(pd-t/p)]\,[1 - C(pd +$$

$$+ q/p) - C(pd-q/p)] + [C(pd+q/p) + C(pd-q/p)] + [S(pd +$$

$$\left. + q/p) + S(pd-q/p)]\Big]^2\right\}. \qquad (24)$$

We could get the expression (24) by using the expression (14) and puting $l \to \infty$. By doing the same specialisation in (17), or simply by puting $q = t = 0$ in (24), we fiind that the irradiance distribution on the z-axsis now is given by

$$I(0, 0, b) = \frac{1}{(a+b)^2} [C_2(pd) + S^2(pd)] \{[C(pd) - 1]^2 + [S(pd) - 1]^2\} \quad (25)$$

whose main focal properties are the same as those of the finite symmetrical cross-shaped aperture.

For the distributions in the axial planes of the slits we get

$$I(0, t, b) = I(q, 0, b) = \Bigg\{\Bigg[\frac{1}{(a+b)^2} C(pd) - S(pd) + C(pd + q/p) + C(pd - q/p)] [1/2 - C(pd)] - [S(pd + q/p) + S(pd - q/p)] [1/2 - S(pd)]\Bigg]^2 + \Bigg[C(pd) + S(pd) + [C(pd + q/p) + C(pd - q/p)] [1/2 - S(pd)] + [S(pd + q/p) + S(pd - q/p)] [1/2 - C(pd)]\Bigg]^2\Bigg\}. \quad (26)$$

It could be also verified that

$$I(q, q) = I(q, -q) \quad (26)$$

which should be expected due to the symmetry of the slits.

The expression (24) has a 4-fold symmetry in the receiving plane. Namely it is evident that

$$I(q, t, b) = I(q, -t, b) = I(-q, t, b) = I(q, -t, b) = I(t, q, b) = I(t, -q, b) = I(-t, q, b) = I(-t, -q, b)$$

and, as in the case of the symmetrical cross with finite dimensions, it is sufficient to examine the irradiance in one of the receiving plane quadrangles.

4. Fraunhofer diffraction due to cross-shaped aperture

To complete the study of diffraction on cross-shaped apertures, we shall discuss the case of Fraunhofer diffraction on symmetrical cross-shaped aperture. Now $b = f$ is the focal value of the lense used, and as it is usually done in this type of diffraction, we omit the squre terms in (7). Therefore the diffraction integral is given by

$$g(x,y) = K\left\{ \int_{-(l+d)}^{-d} e^{-is\xi}\, d\xi \int_{-d}^{d} e^{-it\eta}\, d\eta + \int_{-d}^{d} e^{-is\xi}\, d\xi \int_{-(l+d)}^{l+d} e^{-it\eta}\, d\eta + \int_{d}^{l+d} e^{-is\xi}\, d\xi \int_{-d}^{d} e^{-it\eta}\, d\eta \right\} \tag{27}$$

here

$$s = kx/f; \qquad t = ky/f \qquad \text{and} \qquad K = \text{const.} \tag{28}$$

Performing the integration, we get the following irradiance distribution

$$I = g\,g^{\star} = 16\,K^2\,\{d\,(l+d)\,[\mathrm{sinc}\,(td)\,\mathrm{sinc}\,[s\,(l+d)] + \mathrm{sinc}\,(sd)\,\mathrm{sinc}\,[t\,(l + d)]] - d^2\,\mathrm{sinc}\,(td)\,\mathrm{sinc}\,(sd)\}^2. \tag{29}$$

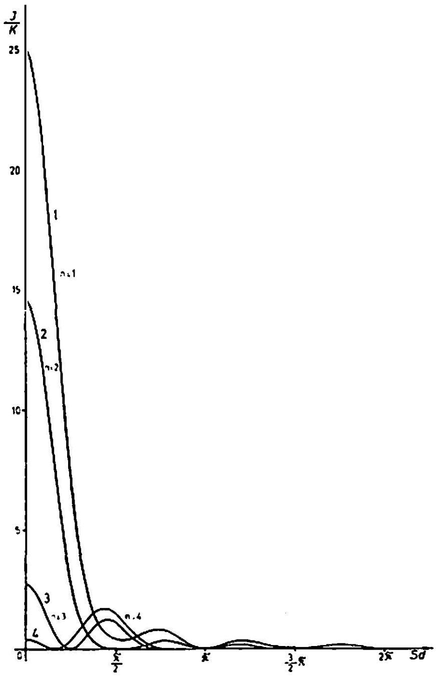

Fig. 4. Fraunhofer diffraction plots of the irradiance versus relative distance (sd) for $l + d = 3$ mm, $d = 1$ mm, calculated by Eq. (29). The n-value attached to the curves is a multiple of $\pi/6$ taken as a constant value for the (st) coordinate.

Since the function sinc $(x) = \sin x/x =$ sinc $(-x)$ is a symmetrical one, it follows that the expression (29) satisfies

$$I\,(s,\,t) = I\,(s,\,-t) = I\,(-s,\,t) = I\,(-s,\,-t) \tag{30}$$

It also does not change its form when s and t interchange their positions

$$I(s, t) = I(t, s).$$

Since the variables s and t equally contribute to the expression (29), it is evident that if m is any constant value

$$I(m, s) = I(t, m).$$

For this reason the diagrams in Figs. 4, 5, 6 and 7 are done with one of the variables taken constant, for example t or $y = 0, \pi/6, 2\pi/6, \ldots n\pi/6$. The number

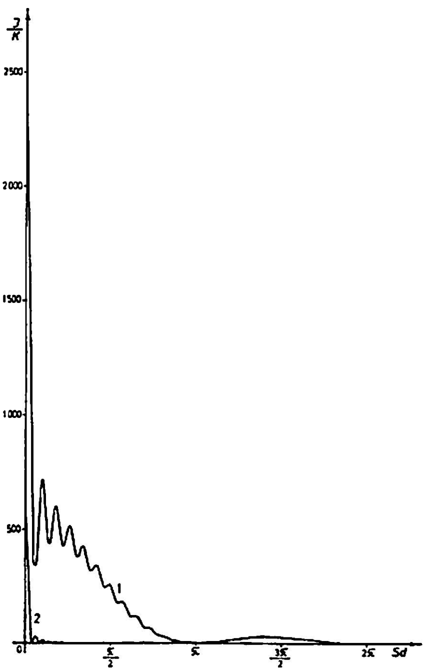

Fig. 5. Fraunhofer diffraction plots of the irradiance distribution versus the relative distance (sd) for $l + d = 25$ mm, $d = 1$ mm, calculated by Eq. (29).

$n = 1, 2, 3, \ldots$ is attached to the corresponding curve. The diagrams are done for apertures having the following dimensions: $l + d = 3$ mm, $l + d = 25$ mm, $l + d = 50$ mm and $l + d = 100$ mm, $d = 1$ mm.

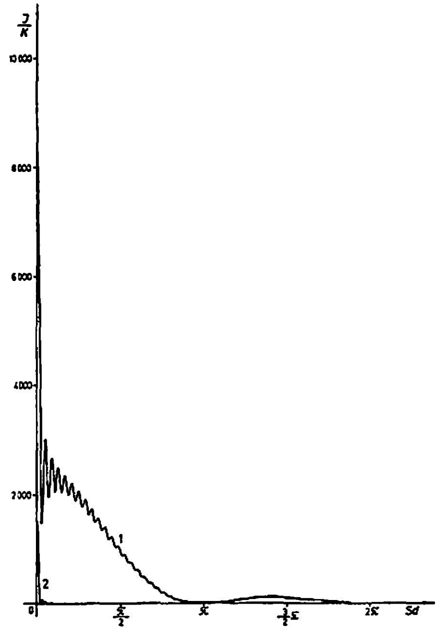

Fig. 6. Fraunhofer diffraction plots of the irradiance distribution versus the relative distance(sd) for $l + d = 50$ mm, $d = 1$ mm, calculated by Eq. (29).

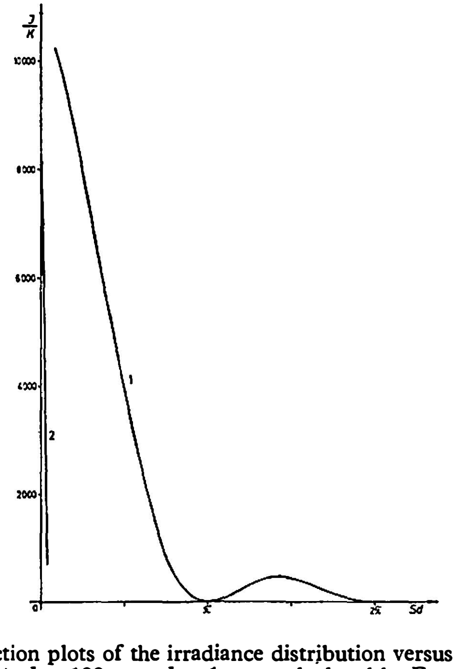

Fig. 7. Fraunhofer diffraction plots of the irradiance distribution versus the relative distance (sd) for $l + d = 100$ mm, $d = 1$ mm, calculated by Eq. (29).

5. *Fraunhofer diffraction due to crossed slits*

Having the same geometry of the aperture as on Fig. 3, we start with the diffraction integral

$$g = K \left\{ \int_{-d}^{d} e^{-is\xi}\, d\xi \int_{\infty -}^{\infty} e^{-it\eta}\, d\eta + \int_{-\infty}^{\infty} e^{-is\xi}\, d\xi \int_{\eta_4}^{\eta_3} e^{-it\eta}\, d\eta - \int_{-d}^{d} e^{-is\xi}\, d\xi \int_{\eta_4}^{\eta_3} e^{-it\eta}\, d\eta \right\} \quad (31)$$

where η_3 and η_4 are given as in (22). Using the well known definition of the Dirac delta function

$$\int_{-\infty}^{\infty} e^{-iax}\, dx = 2\,\pi\,\delta\,(a) \quad (32)$$

we find that the diffracted wave amplitude is given by

$$g = K\,d \left\{ \pi\,\delta\,(t)\,\mathrm{sinc}\,(sd) + \frac{\pi}{\sin\alpha}\,\mathrm{sinc}\left(\frac{t\,d}{\sin\alpha}\right)\delta\,(s - \mathrm{tctg}\,\alpha) - \right.$$

$$\left. - \frac{1}{\sin\alpha}\,\mathrm{sinc}\left(\frac{t\,d}{\sin\alpha}\right)\mathrm{sinc}\,[d\,(s - \mathrm{tctg}\,\alpha)] \right\} \quad (33)$$

while the irradiance is the square of (33)

$$I = K^2\,d^2\,\pi^2 \left\{ \delta\,(t)\,\mathrm{sinc}\,(sd) + \frac{1}{\sin\alpha}\,\mathrm{sinc}\left(\frac{t\,d}{\sin\alpha}\right)\delta\,(s - \mathrm{tctg}\,\alpha) - \right.$$

$$\left. - \frac{1}{\pi\sin\alpha}\,\mathrm{sinc}\left(\frac{t\,d}{\sin\alpha}\right)\mathrm{sinc}\,[d\,(s - \mathrm{tctg}\,\alpha)] \right\}^2 . \quad (34)$$

The first two terms in the bracket contribute to the irradiance only allong the lines $t = 0$ and $s - t\,\mathrm{ctg}\,\alpha = 0$. The rest of the space of the receiving plane is covered with the diffraction fringes of the third term, i. e. for $t \neq 0$ and $s - t\,\mathrm{ctg}\,\alpha \neq 0$

$$g\,g^* = \frac{K^2\,d^2}{\sin^2\alpha}\,\mathrm{sinc}^2\left(\frac{t\,d}{\sin\alpha}\right)\mathrm{sinc}^2\,[d\,(\mathrm{s} - t\,\mathrm{ctg}\,\alpha)] \quad (35)$$

which is recognised as the diffracted irradiance distribution by a rhomboidal aperture.

The orthogonally intersecting slits exhibit an irradiance

$$I \doteq K\,d^2\,\pi^2 \left\{ \delta\,(t)\,\mathrm{sinc}\,(sd) + \mathrm{sinc}\,(td)\,\delta\,(s) - \frac{1}{\pi}\,\mathrm{sinc}\,(td)\,\mathrm{sinc}\,(sd) \right\}^2 . \quad (36)$$

The receiving plane, except allong $t = 0$, and $s = 0$ is covered by the diffracted irradiance distribution due to squrae shaped aperture

$$g\,g^* = K^2\,d^2\,\mathrm{sinc}^2\,(td)\cdot\mathrm{sinc}^2\,(sd) \qquad (37)$$

which is familiar from any textbook on diffraction.

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DIFRAKCIJA NA KRSTOLIKOM OTVORU I NA UKRŠTENIM PUKOTINAMA

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Dato je teoretsko rešenje problema Fresnelove i Fraunhoferove difrakcije kod krstolikih difrakcionih otvora sa konačnim, simetričnim i beskonačno dugim kracima. Nađeni su difrakcioni integrali i raspored iradijance kod gore spomenutih tipova difrakcionih apertura, i diskutovane su njihove vrednosti kod oba tipa difrakcija — Fresnelove i Fraunhoferove.