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TRANSFORMATION PROPERTIES OF CANONICAL MOMENTA

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Critical remarks on the recently formulated »correct canonical formalism« are raised with the aim of demonstrating physical counter-arguments and mathematical restrictions on the proposed sheme.

In a recently published paper[1] the authors claim the necessity of a requirement by which in field theory the members of a canonical pair, i. e. field components and their canonical momenta, must possess equal transformation properties. The proclaimed principle is then implicitly taken as the underlying motive for a detailed critical analysis of the canonical formulation of classical and quantum field theory and also as a stepping stone for attempts to reformulate and physically reinterpret various classical fields[2, 3]. The procedure is refered to as »the correct canonical formalism«.

While the physically reinterpreted fields may merit attention, in particular the Dirac field, the requirement of equal transformation properties suffers from severe shortcomings. These may be grouped in two classes:

I. The requirement that the canonical momenta transform by the same representation as the corresponding fields leads to intolerably strange physical implications. In case of a scalar field $\Phi(x)$, for instance, if the canonical momentum $\pi_\Phi(x)$ were a Lorentz scalar the Hamiltonian could no longer be the time component of a fourvector, since from

$$H = \int d^3 x \dot{\Phi} \pi - \int d^3 x \mathscr{L}, \tag{1}$$

the first term on the right-hand side would behave as a scalar and the second as the time component of a fourvector. Similarly

$$P_k = - \int d^3 x \pi \partial_k \Phi \tag{2}$$

could no longer be the space component.

Consequently, the concept of the fourmomentum of the field where the momentum comprises the space components and the energy of the field the timecomponent, would then be violated and the identification of energy and momentum with the Poincare group translation generators made impossible.

In the quantized version the requirement of the equal transformation properties would lead to contradictions in the basic commutation relations since

$$[\Phi (x), \pi (y)]_{x_0 = y_0} = i \, \delta^{(3)} (\vec{x} - \vec{y}) \tag{3}$$

would have the left hand side transform as a scalar and the right-hand side as the time component of a fourvector on account of the equality

$$\delta^{(3)} (\vec{x} - \vec{y}) = - \partial_0 \Delta (x - y)/_{x_0 = y_0}. \tag{4}$$

The scalar field can most easily demonstrate our point but the same criticism applies to fields satisfying any Lorentz group representation. Even more generally, including internal symmetries, the Noether theorem gives the generalized charges

$$Q_a = - \mathrm{i} \int d^3 x \, \pi_A^{(l)} (x) \, F_{AB}^{(l)} \, \Phi_B^{(l)} (x), \tag{5}$$

where the requirement that $\pi_A^{(l)} (x)$ transforms in the same way as $\Phi_A^{(l)} (x)$ prevents Q_A from being Lorentz scalars.

All these examples present a serious threat to the foundations of Special relativity.

2. The requirement that the canonical momenta transform by the same, or equivalent, representation as the corresponding fields is mathematically extremely restrictive. To show this we start with the assumptions, shared by the authors of »the correct canonical formalism«, that $\Phi_A (x)$ be components of a field which transform by some finite dimensional representation S of the Lorentz group $LG_+^\uparrow$, that the Lagrangean density $\mathscr{L}$ is a Lorentz scalar and that the canonical momenta are defined as

$$\pi_A (x) = \frac{\partial \mathscr{L}}{\partial \dot{\Phi}_A (x)}. \tag{6}$$

About the functional form of the Lagrangean density $\mathscr{L}$ we make no assumptions making thereby the forthcoming conclusions valid for any Lagrangean.

Introducing auxiliary quantities

$$\pi_{A\mu} = \frac{\partial \mathscr{L}}{\partial \partial_\mu \Phi_A}, \tag{7}$$

with $\pi_A = -\,\mathrm{i}\,\pi_{A4}$, one readily deduces their transformation properties as

$$\pi'_{A\mu} = \widetilde{L}_{\mu\nu} \widetilde{S}_{AB}\, \pi_{B\nu}, \tag{8}$$

where L denotes the fundamental (vector) representation $\left(\frac{1}{2}, \frac{1}{2}\right)$ of $LG_+^\uparrow$, and the wiggle »$\sim$« denotes the transposed inverse matrix.

It is easy to see that matrices $\widetilde{S}$ comprise a representation of $LG_+^\uparrow$, equivalent to representation S. To this end it suffices to examine the corresponding expressions for Casimir operators. One has

$$\widetilde{C}_1 = \widetilde{M}_{\alpha\beta} \widetilde{M}_{\alpha\beta} = C_1^T, \tag{9 a}$$

$$\widetilde{C}_2 = \varepsilon_{\alpha\beta\mu\nu}\, \widetilde{M}_{\alpha\beta} \widetilde{M}_{\mu\nu} = C_2^T, \tag{9 b}$$

and since the finite-dimensional representations of the Lorentz group[4] are fully reducible, the Casimir operators form diagonal matrices giving

$$\widetilde{C}_1 = C_1, \qquad \widetilde{C}_2 = C_2, \tag{10}$$

which implies the equivalence of S and $\widetilde{S}$.

Consequently, the transformation property (8), actually states that $\pi_{A\mu}$ transform according to the direct product repreasentation $L \otimes S$ of $LG_+^\uparrow$. The Clebsch-Gordan expansion of the reducible representation $L \otimes S$ indicates then the allowed transformation representations for π_A. The analysis is straightforward:

(a) Scalar field $\Phi(x)$ implies $S = (0, 0)$. We have

$$\left(\frac{1}{2}, \frac{1}{2}\right) \otimes (0, 0) = \left(\frac{1}{2}, \frac{1}{2}\right), \tag{11}$$

and, hence, π_μ transform by the irreducible vector representation. Canonical momentum π has to be the time component of a fourvector regardless of the form of the Lagrangean.

(b) Vector field $A_\mu(x)$ implies $S = \left(\frac{1}{2}, \frac{1}{2}\right)$.

We have $$\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\frac{1}{2}, \frac{1}{2}\right) = (0, 0) \oplus (1, 0) \oplus (0, 1) \oplus (1, 1). \qquad (12)$$

The reduced form contains a scalar (0, 0), an antisymmetric tensor (1, 0) $\oplus$ (0, 1), a symmetric traceless tensor (1, 1), but no vector representation $\left(\frac{1}{2}, \frac{1}{2}\right)$. Consequently the canonical momenta can not transform as vectors and any attempt to reformulate the vector field in accordance with »the correct canonical formalism« is futile.

(c) Antisymmetric tensor field $A_{\alpha\beta}$ implies $S = (1, 0) \oplus (0, 1)$ and so

$$L \otimes S = \left(\frac{1}{2}, \frac{1}{2}\right) \oplus \left(\frac{3}{2}, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, \frac{3}{2}\right). \qquad (13)$$

(d) Symmetric tensor field $S_{\alpha\beta}$ implies $S = (1, 1)$ and so

$$L \otimes S = \left(\frac{1}{2}, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, \frac{3}{2}\right) \oplus \left(\frac{3}{2}, \frac{1}{2}\right) \oplus \left(\frac{3}{2}, \frac{3}{2}\right). \qquad (14)$$

Relations (13) and (14) again show that S does not reappear in $L \otimes S$ and therefore π_A cannot transform as Φ_A.

(e) The Dirac field $\Psi(x)$ transforms according to $S = \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$, and we have

$$L \otimes S = \left(1, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 1\right) \oplus \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right), \qquad (15)$$

where the last two terms witness the reappearance of S in $L \otimes S$.

The Dirac field, therefore, has the unique property that by a suitable reformulation of the Lagrangean one may hope to satisfy the requirement of »the correct canonical formalism«. The importance that one attaches to this circumstance, however, is largely a matter of esthetcial domain.

Taken from the positive attitude this may possibly indicate, in the light of Eq. (15), an explanation for the fact that basic matter fields in Nature, quarks and leptons, are associated with the Dirac field. However, the gauge fields, gluons, photons and heavy intermediate bosons, which come into existence by the »global-to-local« mechanism, should then be dissociated from the canonical procedure.

References

1) K. Ljolje, S. Vobornik, FIZIKA **11** (1979) 171;
2) J. Brana, K. Ljolje, FIZIKA **10** (1978) 85;
3) K. Ljolje, N. Marjanović-Gabela, FIZIKA **11** (1979) 181;
4) M. A. Najmark, Linejnie predstavlenija grupi Lorenca, Fizmatgiz (1958), Moskva.

TRANSFORMACIONA SVOJSTVA KANONSKIH IMPULSA

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Originalni znanstveni rad

U radu se prezentira kritički osvrt na nedavno formulirani »ispravni kanonski formalizam«, demostriraju fizikalni kontraargumenti i analiziraju matematička ograničenja predložene sheme.