

## QUANTIZATION OF THE MASSLESS DIRAC FIELD IN THE NEW CANONICAL FORMULATION\*

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The new canonical formulation of the massless Dirac field is applied in quantization of the field. It is found that the canonical pairs satisfy commutation relations. The scalar, vector and second order tensor constants of motion are evaluated and their physical interpretation is given. The case of the real field is also considered.

### *1. Introduction*

Correct canonical formulation of the Dirac field has been recently developed<sup>1,2,3</sup>. In this formulation the standard Dirac field appears as momentum of a canonical pair. This momentum satisfies the Dirac's equation. In this paper we apply this formulation to the quantization of the massless Dirac field. We show that final results are quite different from the standard theory. The procedure is, however, completely consistent and physically clear.

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In Section 2 we reproduce in a short form a new canonical formulation of the massless Dirac field. General solution of the Lagrange's and the canonical equations is given in Section 3. The quantization of the massless Dirac field in the new canonical formulation is presented in Section 4. The scalar, energy-momentum and angular momentum constants of motion are evaluated in Section 5. In Section 6 the real field is considered. Dual symmetry is analyzed in Section 7. Conclusions are given in Section 8.

## 2. Canonical formulation of the massless Dirac field

A new canonical formulation of the Dirac field has been developed in analogy with the electromagnetic field<sup>4, 5)</sup>.

Starting from the Lagrangian density for the massless Dirac field<sup>5, 3)</sup>

$$\mathcal{L} = K (\partial_\mu \bar{\Phi} \gamma^\mu) (\partial_\nu \gamma^\nu \Phi), \quad (1)$$

where  $\Phi$ ,  $\Phi^\dagger$  are the Lagrange's variables of the field,  $K$  is a real constant, and the Lagrange's equation

$$\frac{\partial \mathcal{L}}{\partial u} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu u)} = 0, \quad (2)$$

follow the Lagrange's equations for  $\Phi$ ,  $\Phi^\dagger$ ;

$$- \partial^\mu \partial_\mu \Phi = 0, \quad (3)$$

$$- \partial^\mu \partial_\mu \Phi^\dagger = 0. \quad (4)$$

The canonical conjugate momenta to  $\Phi$ ,  $\Phi^\dagger$  are\*

$$H_\Phi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = K (\partial_\mu \bar{\Phi} \gamma^\mu) \gamma^0 \equiv K \Psi^\dagger, \quad (5)$$

$$H_{\Phi^\dagger} = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^\dagger} = K (\partial_\nu \gamma^\nu \Phi) \equiv K \Psi. \quad (6)$$

The Hamiltonian density

$$\mathcal{H} = \dot{\Phi}^\dagger H_{\Phi^\dagger} + H_\Phi \dot{\Phi} - \mathcal{L}, \quad (7)$$

is then

$$\mathcal{H} = \frac{1}{K} H_\Phi \gamma^0 H_{\Phi^\dagger} - \partial_t \Phi^\dagger \alpha^i H_{\Phi^\dagger} - H_\Phi \partial_t \alpha^i \Phi. \quad (8)$$

According to

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}, \quad (9)$$

where  $\{F, H\}$  is the Poisson bracket, the canonical equations are

$$\dot{\Phi} = \{\Phi, H\} = \frac{\delta H}{\delta \Pi_{\Phi}} = \frac{1}{K} \gamma^0 \Pi_{\Phi^{\dagger}} - \partial_i \alpha^i \Phi, \quad (10)$$

$$\dot{\Phi}^{\dagger} = \{\Phi^{\dagger}, H\} = \frac{\delta H}{\delta \Pi_{\Phi^{\dagger}}} = \frac{1}{K} \Pi_{\Phi} \gamma^0 - \partial_i \Phi^{\dagger} \alpha^i,$$

$$\dot{\Pi}_{\Phi^{\dagger}} = \{\Pi_{\Phi^{\dagger}}, H\} = -\frac{\delta H}{\delta \Phi^{\dagger}} = -\partial_i \alpha^i \Pi_{\Phi^{\dagger}} \quad (11)$$

$$\dot{\Pi}_{\Phi} = \{\Pi_{\Phi}, H\} = -\frac{\delta H}{\delta \Phi} = -\partial_i \Pi_{\Phi} \alpha^i.$$

By making use of Eqs. (5) and (6), these equations can be written in the form

$$\partial_{\nu} \gamma^{\nu} \Phi = \Psi, \quad (12)$$

$$\partial_{\nu} \Phi^{\dagger} \gamma^0 \gamma^{\nu} \gamma^0 = \Psi^{\dagger},$$

$$\dot{\Psi} + \partial_i \alpha^i \Psi = 0, \quad (13)$$

$$\dot{\Psi}^{\dagger} + \partial_i \Psi^{\dagger} \alpha^i = 0.$$

Eqs. (13) are the massless Dirac field equations and Eqs. (12) are equivalent to the definition of the canonical conjugate momenta to  $\Phi$ ,  $\Phi^{\dagger}$  (Eqs. (5) and (6)). By this correct canonical as well as the Lagrange's formulation is established.

We use the coordinates  $x^{\mu} = (t, \vec{x})$  with the metric tensor

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1, \quad g_{\alpha\beta} = 0, \quad \alpha \neq \beta$$

and units  $c = \hbar = 1$ .

\* The field  $\Psi$  one may also introduce by the definition  $\Pi_{\Phi^{\dagger}} = -iK\Psi$ . In this case one has  $\Psi = i\partial_{\nu} \gamma^{\nu} \Phi$ .

### 3. Solutions of the Lagrange's and canonical equations

We seek solutions of the Lagrange's equations (3) in the form

$$\Phi(\vec{x}, t) = \frac{1}{L^{3/2}} \sum_{\vec{k}} \begin{bmatrix} a_{\vec{k}}^1(t) \\ a_{\vec{k}}^2(t) \\ a_{\vec{k}}^3(t) \\ a_{\vec{k}}^4(t) \end{bmatrix} e^{i\vec{k}\vec{x}} \quad (14)$$

After substitution of (14) into (3) one gets

$$\ddot{a}_{\vec{k}}^i + k^2 a_{\vec{k}}^i = 0, \quad i = 1, 2, 3, 4, \quad (15)$$

where  $k = |\vec{k}|$ .

General solution of Eq. (15) is

$$a_{\vec{k}}^i(t) = a_{\vec{k}1}^i e^{-ikt} + a_{\vec{k}2}^i e^{ikt}. \quad (16)$$

The general solution of Eq. (3) is then

$$\Phi(\vec{x}, t) = \frac{1}{L^{3/2}} \sum_{\vec{k}} \left\{ \begin{bmatrix} 1 \\ a_{\vec{k}1}^1 \\ a_{\vec{k}1}^2 \\ a_{\vec{k}1}^3 \\ a_{\vec{k}1}^4 \\ a_{\vec{k}1}^1 \end{bmatrix} e^{i(\vec{k}\vec{x}-kt)} + \begin{bmatrix} 1 \\ a_{\vec{k}2}^1 \\ a_{\vec{k}2}^2 \\ a_{\vec{k}2}^3 \\ a_{\vec{k}2}^4 \\ a_{\vec{k}2}^1 \end{bmatrix} e^{i(\vec{k}\vec{x}+kt)} \right\}. \quad (17)$$

Let us notice that all components  $\Phi^i$  are independent in this solution.

One can get solutions of the canonical equations directly or they can be obtained by making use of the solution of the Lagrange's equations and the connection (12). We follow the last choice.

Using the standard representation of the matrices  $\gamma^\mu$ , we have

$$\begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \\ \Psi^4 \end{pmatrix} = \begin{pmatrix} \partial_0 & 0 & \partial_z & \partial_x - i\partial_y \\ 0 & \partial_0 & \partial_x + i\partial_y & -\partial_z \\ -\partial_z & -\partial_x + i\partial_y & -\partial_0 & 0 \\ -\partial_x - i\partial_y & \partial_z & 0 & -\partial_0 \end{pmatrix} \begin{pmatrix} \Phi^1 \\ \Phi^2 \\ \Phi^3 \\ \Phi^4 \end{pmatrix} \quad (18)$$

or in two-component form

$$\begin{pmatrix} \Psi_I \\ \Psi_{II} \end{pmatrix} = \begin{pmatrix} \partial_0 & \vec{\nabla} \vec{\sigma} \\ -\vec{\nabla} \vec{\sigma} & -\partial_0 \end{pmatrix} \begin{pmatrix} \Phi_I \\ \Phi_{II} \end{pmatrix}, \quad (19)$$

where

$$\Psi_I = \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix}, \quad \Psi_{II} = \begin{pmatrix} \Psi^3 \\ \Psi^4 \end{pmatrix}, \quad (20)$$

$$\Phi_I = \begin{pmatrix} \Phi^1 \\ \Phi^2 \end{pmatrix}, \quad \Phi_{II} = \begin{pmatrix} \Phi^3 \\ \Phi^4 \end{pmatrix}. \quad (21)$$

For the sake of later calculations we also write  $\Psi$  in  $\vec{k}$ -space:

$$\Psi(\vec{x}, t) = \frac{1}{L^{3/2}} \sum_{\vec{k}} \begin{bmatrix} A_{\vec{k}}^1(t) \\ A_{\vec{k}}^2(t) \\ A_{\vec{k}}^3(t) \\ A_{\vec{k}}^4(t) \end{bmatrix} e^{i\vec{k}\vec{x}}. \quad (22)$$

Introducing the following notations

$$\begin{pmatrix} A_{\vec{k}}^1(t) \\ A_{\vec{k}}^2(t) \end{pmatrix} = A_{\vec{k}}^{\rightarrow}(t), \quad \begin{pmatrix} A_{\vec{k}}^3(t) \\ A_{\vec{k}}^4(t) \end{pmatrix} = B_{\vec{k}}^{\rightarrow}(t),$$

$$\begin{pmatrix} a_{\vec{k}1}^1 \\ a_{\vec{k}1}^2 \end{pmatrix} = \alpha_{\vec{k}1}^{\rightarrow}, \quad \begin{pmatrix} a_{\vec{k}1}^3 \\ a_{\vec{k}1}^4 \end{pmatrix} = \beta_{\vec{k}1}^{\rightarrow}, \quad (23)$$

$$\begin{pmatrix} a_{\vec{k}2}^1 \\ a_{\vec{k}2}^2 \end{pmatrix} = \alpha_{\vec{k}2}^{\rightarrow}, \quad \begin{pmatrix} a_{\vec{k}2}^3 \\ a_{\vec{k}2}^4 \end{pmatrix} = \beta_{\vec{k}2}^{\rightarrow},$$

$$\alpha_{\vec{k}}^{\rightarrow}(t) = \begin{pmatrix} a_{\vec{k}}^1(t) \\ a_{\vec{k}}^2(t) \end{pmatrix} = \alpha_{\vec{k}1}^{\rightarrow} e^{-ikt} + \alpha_{\vec{k}2}^{\rightarrow} e^{ikt}, \quad (24)$$

$$\beta_{\vec{k}}^{\rightarrow}(t) = \begin{pmatrix} a_{\vec{k}}^3(t) \\ a_{\vec{k}}^4(t) \end{pmatrix} = \beta_{\vec{k}1}^{\rightarrow} e^{-ikt} + \beta_{\vec{k}2}^{\rightarrow} e^{ikt},$$

the functions  $\Psi$  and  $\Phi$  read

$$\Psi = \begin{pmatrix} \Psi_I \\ \Psi_{II} \end{pmatrix} = \frac{1}{L^{3/2}} \sum_{\vec{k}} \begin{pmatrix} A_{\vec{k}}^{\rightarrow}(t) \\ B_{\vec{k}}^{\rightarrow}(t) \end{pmatrix} e^{i\vec{k}\vec{x}}, \quad (25)$$

$$\Phi = \begin{pmatrix} \Phi_I \\ \Phi_{II} \end{pmatrix} = \frac{1}{L^{3/2}} \sum_{\vec{k}} \left\{ \begin{pmatrix} \alpha_{\vec{k}1}^{\rightarrow} \\ \beta_{\vec{k}1}^{\rightarrow} \end{pmatrix} e^{i(\vec{k}\vec{x}-kt)} + \begin{pmatrix} \alpha_{\vec{k}2}^{\rightarrow} \\ \beta_{\vec{k}2}^{\rightarrow} \end{pmatrix} e^{i(\vec{k}\vec{x}+kt)} \right\}. \quad (26)$$

After substitution of (25) and (26) into (18) one gets equations for the coefficients  $A_{\vec{k}}^{\rightarrow}(t)$  and  $B_{\vec{k}}^{\rightarrow}(t)$ :

$$\begin{pmatrix} A_{\vec{k}}^{\rightarrow}(t) \\ B_{\vec{k}}^{\rightarrow}(t) \end{pmatrix} = \begin{pmatrix} -ik & i\vec{k}\vec{\sigma} \\ -i\vec{k}\vec{\sigma} & ik \end{pmatrix} \begin{pmatrix} \alpha_{\vec{k}1}^{\rightarrow} \\ \beta_{\vec{k}1}^{\rightarrow} \end{pmatrix} e^{-ikt} + \begin{pmatrix} ik & i\vec{k}\vec{\sigma} \\ -i\vec{k}\vec{\sigma} & -ik \end{pmatrix} \begin{pmatrix} \alpha_{\vec{k}2}^{\rightarrow} \\ \beta_{\vec{k}2}^{\rightarrow} \end{pmatrix} e^{ikt}. \quad (27)$$

This is equation (18) in  $\vec{k}$ -space.  $\Phi$  given by (17) and  $\Psi$  given by (18) or (27) are general solution of the canonical equations.

The coefficients  $\alpha_{\vec{k}}^i(t)$ ,  $\beta_{\vec{k}}^i(t)$  and  $A_{\vec{k}}^i(t)$ ,  $B_{\vec{k}}^i(t)$  are proportional to canonical pairs in  $\vec{k}$ -space of the field. In the next Section it will be useful to have inverse of the relations (24) and (27).

Writing  $\Phi$  in the form

$$\begin{pmatrix} \alpha_{\vec{k}}^{\rightarrow}(t) \\ \beta_{\vec{k}}^{\rightarrow}(t) \end{pmatrix} = \begin{pmatrix} \alpha_{\vec{k}1}^{\rightarrow} \\ \beta_{\vec{k}1}^{\rightarrow} \end{pmatrix} e^{-ikt} + \begin{pmatrix} \alpha_{\vec{k}2}^{\rightarrow} \\ \beta_{\vec{k}2}^{\rightarrow} \end{pmatrix} e^{ikt}, \quad (28)$$

multiplying it by the matrices

$$\Gamma_1 = \begin{pmatrix} ik & i\vec{k}\vec{\sigma} \\ -i\vec{k}\vec{\sigma} & -ik \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} -ik & i\vec{k}\vec{\sigma} \\ -i\vec{k}\vec{\sigma} & ik \end{pmatrix}, \quad (29)$$

and subtracting from (27), one gets, respectively,

$$\begin{pmatrix} A_{\vec{k}}^{\rightarrow}(t) \\ B_{\vec{k}}^{\rightarrow}(t) \end{pmatrix} - \Gamma_1 \begin{pmatrix} \alpha_{\vec{k}}^{\rightarrow}(t) \\ \beta_{\vec{k}}^{\rightarrow}(t) \end{pmatrix} = -2ik \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_{\vec{k}1}^{\rightarrow} \\ \beta_{\vec{k}1}^{\rightarrow} \end{pmatrix} e^{-ikt}, \quad (30)$$

$$\begin{pmatrix} A_{\vec{k}}^{\rightarrow}(t) \\ B_{\vec{k}}^{\rightarrow}(t) \end{pmatrix} - \Gamma_2 \begin{pmatrix} a_{\vec{k}}^{\rightarrow}(t) \\ \beta_{\vec{k}}^{\rightarrow}(t) \end{pmatrix} = 2i\vec{k} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_{\vec{k}2}^{\rightarrow} \\ \beta_{\vec{k}2}^{\rightarrow} \end{pmatrix} e^{ikt}. \quad (31)$$

From here, after multiplication by  $\mp (2i\vec{k})^{-1} \gamma^0 e^{\pm ikt}$ , respectively, follows

$$\begin{pmatrix} \alpha_{\vec{k}1}^{\rightarrow} \\ \beta_{\vec{k}1}^{\rightarrow} \end{pmatrix} = -\frac{1}{2i\vec{k}} \left[ \begin{pmatrix} A_{\vec{k}}^{\rightarrow}(t) \\ -B_{\vec{k}}^{\rightarrow}(t) \end{pmatrix} - \begin{pmatrix} i\vec{k} & i\vec{k}\vec{\sigma} \\ i\vec{k}\vec{\sigma} & i\vec{k} \end{pmatrix} \begin{pmatrix} \alpha_{\vec{k}}^{\rightarrow}(t) \\ \beta_{\vec{k}}^{\rightarrow}(t) \end{pmatrix} \right] e^{ikt}, \quad (32)$$

$$\begin{pmatrix} \alpha_{\vec{k}2}^{\rightarrow} \\ \beta_{\vec{k}2}^{\rightarrow} \end{pmatrix} = \frac{1}{2i\vec{k}} \left[ \begin{pmatrix} A_{\vec{k}}^{\rightarrow}(t) \\ -B_{\vec{k}}^{\rightarrow}(t) \end{pmatrix} - \begin{pmatrix} -i\vec{k} & i\vec{k}\vec{\sigma} \\ i\vec{k}\vec{\sigma} & -i\vec{k} \end{pmatrix} \begin{pmatrix} \alpha_{\vec{k}}^{\rightarrow}(t) \\ \beta_{\vec{k}}^{\rightarrow}(t) \end{pmatrix} \right] e^{-ikt}, \quad (33)$$

or

$$\begin{aligned} \alpha_{\vec{k}1}^{\rightarrow} &= \frac{1}{2\vec{k}} [iA_{\vec{k}}^{\rightarrow}(t) + k\alpha_{\vec{k}}^{\rightarrow}(t) + (\vec{k}\vec{\sigma})\beta_{\vec{k}}^{\rightarrow}(t)] e^{ikt}, \\ \beta_{\vec{k}1}^{\rightarrow} &= \frac{1}{2\vec{k}} [-iB_{\vec{k}}^{\rightarrow}(t) + (\vec{k}\vec{\sigma})\alpha_{\vec{k}}^{\rightarrow}(t) + k\beta_{\vec{k}}^{\rightarrow}(t)] e^{ikt}, \\ \alpha_{\vec{k}2}^{\rightarrow} &= -\frac{1}{2\vec{k}} [iA_{\vec{k}}^{\rightarrow}(t) - k\alpha_{\vec{k}}^{\rightarrow}(t) + (\vec{k}\vec{\sigma})\beta_{\vec{k}}^{\rightarrow}(t)] e^{-ikt}, \\ \beta_{\vec{k}2}^{\rightarrow} &= -\frac{1}{2\vec{k}} [-iB_{\vec{k}}^{\rightarrow}(t) + (\vec{k}\vec{\sigma})\alpha_{\vec{k}}^{\rightarrow}(t) - k\beta_{\vec{k}}^{\rightarrow}(t)] e^{-ikt}, \end{aligned} \quad (34)$$

or in explicite component form

$$\begin{aligned} a_{\vec{k}1}^1 &= \frac{1}{2k} [iA_{\vec{k}}^1(t) + k a_{\vec{k}}^1(t) + k_z a_{\vec{k}}^3(t) + (k_x - ik_y) a_{\vec{k}}^4(t)] e^{ikt} \\ a_{\vec{k}1}^2 &= \frac{1}{2k} [iA_{\vec{k}}^2(t) + k a_{\vec{k}}^2(t) - k_z a_{\vec{k}}^4(t) + (k_x + ik_y) a_{\vec{k}}^4(t)] e^{ikt} \\ a_{\vec{k}1}^3 &= \frac{1}{2k} [-iA_{\vec{k}}^3(t) + k_z a_{\vec{k}}^1(t) + (k_x - ik_y) a_{\vec{k}}^2(t) + k a_{\vec{k}}^3(t)] e^{ikt} \\ a_{\vec{k}1}^4 &= \frac{1}{2k} [-iA_{\vec{k}}^4(t) - k_z a_{\vec{k}}^2(t) + (k_x + ik_y) a_{\vec{k}}^1(t) + k a_{\vec{k}}^4(t)] e^{ikt} \end{aligned} \quad (35)$$

$$\begin{aligned}
 a_{kz}^1 &= \frac{1}{2k} [-i A_k^1(t) + k a_k^1(t) - k_z a_k^3(t) - (k_x - i k_y) a_k^4(t)] e^{-ikt} \\
 a_{kz}^2 &= \frac{1}{2k} [-i A_k^2(t) + k a_k^2(t) + k_z a_k^4(t) - (k_x + i k_y) a_k^3(t)] e^{-ikt}, \\
 a_{kz}^3 &= \frac{1}{2k} [i A_k^3(t) - k_z a_k^1(t) - (k_x - i k_y) a_k^2(t) + k a_k^3(t)] e^{-ikt}, \\
 a_{kz}^4 &= \frac{1}{2k} [i A_k^4(t) + k_z a_k^2(t) - (k_x + i k_y) a_k^1(t) + k a_k^4(t)] e^{-ikt}.
 \end{aligned}
 \tag{36}$$

#### 4. Canonical quantization of the massless Dirac field

In this Section we perform the canonical quantization of the massless Dirac field using the results from the previous sections.

We assume, therefore, the following substitution of the Poisson bracket

$$\{A, B\} \rightarrow \frac{1}{i} [A, B], \tag{37}$$

where  $[A, B]$  is the commutator:  $[A, B] = AB - BA$ .

In the classical particle mechanics the canonical equations of the system described by the canonical variables  $\{q_i, p_i\}$  are given by

$$\dot{q}_i = \{q_i, H\}, \quad \dot{p}_i = \{p_i, H\},$$

where  $\{A, B\}$  is the Poisson bracket:

$$\{A, B\} = \sum_i \left( \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i} \frac{\partial A}{\partial p_i} \right).$$

These equations are particular cases of the general dynamical equation

$$\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t}$$

where  $F$  is a function of the canonical variables and time.

The corresponding equation for a field  $\eta(\vec{x}, t)$  is

$$\frac{dF}{dt} = \int \left( \frac{\delta F}{\delta \eta} \frac{\delta H}{\delta \pi} - \frac{\delta H}{\delta \eta} \frac{\delta F}{\delta \pi} + \frac{\partial F}{\partial t} \right) d^3 x,$$

where

$$F = \int \mathcal{F}(\eta, \partial_i \eta, \pi, \partial_i \pi, t) d^3 x,$$

$\pi$  is the conjugate momentum to  $\eta$  and  $\frac{\delta F}{\delta \eta}$  is the functional derivation:

$$\frac{\delta F}{\delta \eta} = \frac{\partial \mathcal{F}}{\partial \eta} - \partial_i \frac{\partial \mathcal{F}}{\partial (\partial_i \eta)}.$$

In comparison to the particle dynamics the Poisson bracket for the field is defined by

$$\{A, B\} = \int \left( \frac{\delta A}{\delta \eta} \frac{\delta B}{\delta \pi} - \frac{\delta B}{\delta \eta} \frac{\delta A}{\delta \pi} \right) d^3 x$$

and the dynamical equation for the field  $\eta$  now reads

Taking 
$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}.$$

one gets

$$F = \int \eta(\vec{x}, t) \delta(\vec{x} - \vec{x}') d^3 x \equiv F_\eta,$$

Similarly, taking

$$\dot{\eta}(\vec{x}, t) = \{\eta(\vec{x}, t), H\} = \frac{\partial \mathcal{H}}{\partial \pi} - \partial_i \frac{\partial \mathcal{H}}{\partial (\partial_i \eta)}.$$

it follows

$$F = \int \pi(\vec{x}, t) \delta(\vec{x} - \vec{x}') d^3 x \equiv F_\pi,$$

$$\dot{\pi}(\vec{x}, t) = \{\pi(\vec{x}, t), H\} = -\frac{\partial \mathcal{H}}{\partial \eta} + \partial_i \frac{\partial \mathcal{H}}{\partial (\partial_i \eta)}.$$

These equations are in accordance with (10) and (11).

The Poisson bracket for  $F_\eta$  and  $F_\pi$  are

$$\{F_{\eta'}, F_{\eta''}\} = 0, \quad \{F_{\pi'}, F_{\pi''}\} = 0,$$

$$\{F_{\eta'}, F_{\pi''}\} = \int \delta(\vec{x} - \vec{x}') \delta(\vec{x} - \vec{x}'') d^3 x = \delta(\vec{x}' - \vec{x}'').$$

On the other hand

$$F_{\eta'} = \eta(\vec{x}', t), \quad F_{\pi'} = \pi(\vec{x}', t).$$

So, we have

$$\{\eta(\vec{x}, t), \eta(\vec{x}', t)\} = \{\pi(\vec{x}, t), \pi(\vec{x}', t)\} = 0,$$

$$\{\eta(\vec{x}, t), \pi(\vec{x}', t)\} = \delta(\vec{x} - \vec{x}').$$

The substitution (37) we mean in this sense.

The Hamiltonian of the field becomes then the operator

$$H = \int \mathcal{H} d^3 x = \int \left\{ \frac{1}{K} \Pi_\phi \gamma^0 \Pi_{\phi^\dagger} - \partial_i \Phi \alpha^i \Pi_{\phi^\dagger} - \Pi_\phi \partial_i \alpha^i \Phi \right\} d^3 x \quad (38)$$

and the dynamical equation (9) goes over into

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{1}{i} [F, H]. \quad (39)$$

The commutation relations are

$$[\Phi_a(\vec{x}, t), \Phi_b(\vec{y}, t)] = [\Phi_a(\vec{x}, t), \Phi_b^\dagger(\vec{y}, t)] = [\Phi_a^\dagger(\vec{x}, t), \Phi_b^\dagger(\vec{y}, t)] = 0,$$

$$[\Pi_{\phi_a}(\vec{x}, t), \Pi_{\phi_b}(\vec{y}, t)] = [\Pi_{\phi_a}(\vec{x}, t), \Pi_{\phi_b^\dagger}(\vec{y}, t)] = [\Pi_{\phi_a^\dagger}(\vec{x}, t), \Pi_{\phi_b^\dagger}(\vec{y}, t)] = 0,$$

$$[\Phi_{\phi_a}(\vec{x}, t), \Pi_{\phi_b^\dagger}(\vec{y}, t)] = [\Phi_a^\dagger(\vec{x}, t), \Pi_{\phi_b}(\vec{y}, t)] = 0, \quad (40)$$

$$[\Phi_a(\vec{x}, t), \Pi_{\phi_b}(\vec{y}, t)] = i \delta_{ab} \delta(\vec{x} - \vec{y}),$$

$$[\Phi_a^\dagger(\vec{x}, t), \Pi_{\phi_b^\dagger}(\vec{y}, t)] = i \delta_{ab} \delta(\vec{x} - \vec{y}), \quad a, b = 1, 2, 3, 4.$$

Let us mention that Eqs. (40) must be the commutation relations, not anti-commutation relations. The commutation relations only give correct canonical equations in accordance with (39). We think that this must be clearly emphasized.

From the Hamiltonian (38), the dynamical equation (39) and the commutation rules (40) follow the equation of motions for the operators  $\Phi, \Phi^\dagger, \Pi_\phi, \Pi_{\phi^\dagger}$ :

$$\dot{\Phi} = -i [\Phi, H] = \frac{1}{K} \gamma^0 \Pi_{\phi^\dagger} - \partial_i \alpha^i \Phi,$$

$$\dot{\Phi}^\dagger = -i [\Phi^\dagger, H] = \frac{1}{K} \Pi_\phi \gamma^0 - \partial_i \Phi^\dagger a^i, \quad (41)$$

$$\dot{\Pi}_\phi = -i [\Pi_\phi, H] = -\partial_i \Pi_\phi a^i,$$

$$\dot{\Pi}_{\phi^\dagger} = -i [\Pi_{\phi^\dagger}, H] = -\partial_i a^i \Pi_{\phi^\dagger}.$$

These equations are in agreement with Eqs. (12) and (13).

At the present stage of the theory we prefer further analysis in the  $\vec{k}$ -space.

By making use of (14) and (22), i. e.

$$\begin{aligned} \Phi^i(\vec{x}, t) &= \frac{1}{L^{3/2}} \sum_{\vec{k}} a_{\vec{k}}^i(t) e^{i\vec{k}\vec{x}}, \\ \Phi^{i\dagger}(\vec{x}, t) &= \frac{1}{L^{3/2}} \sum_{\vec{k}} a_{\vec{k}}^{i\dagger}(t) e^{-i\vec{k}\vec{x}}, \\ \Pi_{\phi^i} &= K \Psi^{i\dagger}(\vec{x}, t) = K \frac{1}{L^{3/2}} \sum_{\vec{k}} A_{\vec{k}}^{i\dagger}(t) e^{-i\vec{k}\vec{x}}, \\ \Pi_{\phi^{i\dagger}} &= K \Psi^i(\vec{x}, t) = K \frac{1}{L^{3/2}} \sum_{\vec{k}} A_{\vec{k}}^i(t) e^{i\vec{k}\vec{x}}, \end{aligned} \quad (42)$$

the commutation rules (40) in the  $\vec{k}$ -space for equal times read

$$\begin{aligned} [a_{\vec{k}}^i, a_{\vec{k}'}^j] &= [a_{\vec{k}}^i, a_{\vec{k}'}^{j\dagger}] = [a_{\vec{k}}^{i\dagger}, a_{\vec{k}'}^{j\dagger}] = 0, \\ [A_{\vec{k}}^i, A_{\vec{k}'}^j] &= [A_{\vec{k}}^i, A_{\vec{k}'}^{j\dagger}] = [A_{\vec{k}}^{i\dagger}, A_{\vec{k}'}^{j\dagger}] = 0, \\ [a_{\vec{k}}^i, A_{\vec{k}'}^j] &= [a_{\vec{k}}^{i\dagger}, A_{\vec{k}'}^{j\dagger}] = 0, \\ [a_{\vec{k}}^i, A_{\vec{k}'}^{j\dagger}] &= \frac{i}{K} \delta_{ij} \delta_{\vec{k}\vec{k}'}, \\ [a_{\vec{k}}^{i\dagger}, A_{\vec{k}'}^j] &= \frac{i}{K} \delta_{ij} \delta_{\vec{k}\vec{k}'}, \quad i, j = 1, 2, 3, 4. \end{aligned} \quad (43)$$

Having these commutation rules we now evaluate the commutation rules for the time independent coefficients  $(a_{\vec{k}1}^i, a_{\vec{k}2}^i, a_{\vec{k}1}^{i\dagger}, a_{\vec{k}1}^{i\dagger})$ :

$$[a_{\vec{k}1}^i, a_{\vec{k}'1}^{j\dagger}] = \frac{1}{2kK} \delta^{ij} \delta_{\vec{k}\vec{k}'}, \quad (44)$$

$$[a_{\vec{k}1}^n, a_{\vec{k}'1}^{m\dagger}] = -\frac{1}{2kK} \delta^{nm} \delta_{\vec{k}\vec{k}'}, \quad (45)$$

$$[a_{\vec{k}2}^i, a_{\vec{k}'2}^{j\dagger}] = -\frac{1}{2kK} \delta^{ij} \delta_{\vec{k}\vec{k}'}, \quad (46)$$

$$[a_{\vec{k}2}^n, a_{\vec{k}'2}^{m\dagger}] = \frac{1}{2kK} \delta^{nm} \delta_{\vec{k}\vec{k}'}, \quad (47)$$

$$i, j = 1, 2, \quad n, m = 3, 4,$$

and all other commutators are zero.

Introducing the new coefficients according to

$$a_{\vec{k}j}^i = \sqrt{\frac{1}{2kK}} a_{\vec{k}jN}^i \quad (48)$$

the commutation relations (44—47) become

$$\begin{aligned} [a_{\vec{k}1N}^i, a_{\vec{k}'1N}^{j\dagger}] &= \delta^{ij} \delta_{\vec{k}\vec{k}'}, \\ [a_{\vec{k}1N}^n, a_{\vec{k}'1N}^{m\dagger}] &= -\delta^{nm} \delta_{\vec{k}\vec{k}'}, \\ [a_{\vec{k}2N}^i, a_{\vec{k}'2N}^{j\dagger}] &= -\delta^{ij} \delta_{\vec{k}\vec{k}'}, \\ [a_{\vec{k}2N}^n, a_{\vec{k}'2N}^{m\dagger}] &= \delta^{nm} \delta_{\vec{k}\vec{k}'}. \end{aligned} \quad (49)$$

In these relations the signs of the right hand sides are not the same. We can make them equal by the new notation

$$\begin{aligned} a_{\vec{k}1N}^n &= b_{\vec{k}1N}^{n\dagger}, & a_{\vec{k}1N}^{n\dagger} &= b_{\vec{k}1N}^n, & n &= 3, 4, \\ a_{\vec{k}2N}^i &= b_{\vec{k}2N}^{i\dagger}, & a_{\vec{k}2N}^{i\dagger} &= b_{\vec{k}2N}^i, & i &= 1, 2. \end{aligned} \quad (50)$$

Then we have

$$\begin{aligned}
 \left[ a_{k1N}^{\rightarrow i}, a_{k'1N}^{\rightarrow j\dagger} \right] &= \delta^{ij} \delta_{kk'}, \\
 \left[ b_{k1N}^{\rightarrow n}, b_{k'1N}^{\rightarrow m\dagger} \right] &= \delta^{nm} \delta_{kk'}, \\
 \left[ b_{k2N}^{\rightarrow i}, b_{k'2N}^{\rightarrow j\dagger} \right] &= \delta^{ij} \delta_{kk'}, \\
 \left[ a_{k2N}^{\rightarrow n}, a_{k'2N}^{\rightarrow m\dagger} \right] &= \delta^{nm} \delta_{kk'}, \\
 i, j &= 1, 2, \quad n, m = 3, 4.
 \end{aligned} \tag{51}$$

The field  $\Phi(\vec{x}, t)$  in the new notation is given by

$$\Phi(\vec{x}, t) = \frac{1}{L^{3/2}} \sum_{\vec{k}} \sqrt{\frac{1}{2kK}} \left\{ \begin{bmatrix} a_{k1N}^{\rightarrow 1} \\ a_{k1N}^{\rightarrow 2} \\ b_{k1N}^{\rightarrow 3\dagger} \\ b_{k1N}^{\rightarrow 4\dagger} \end{bmatrix} e^{i(\vec{kx}-kt)} + \begin{bmatrix} b_{k2N}^{\rightarrow 1\dagger} \\ b_{k2N}^{\rightarrow 2\dagger} \\ a_{k2N}^{\rightarrow 3} \\ a_{k2N}^{\rightarrow 4} \end{bmatrix} e^{i(\vec{kx}+kt)} \right\}. \tag{52}$$

(The calculation could start with this expression dropping also the index  $N$ .)

By this we have completed the quantization of the massless Dirac field.

### 5. Constants of motion

In order to see consequences of the new quantization of the massless Dirac field we evaluate the main constants of motion. In the Lagrange's language they are given by the Noether theorem and in the canonical language by the equations

$$\frac{\partial F}{\partial t} = 0, \quad \{F, H\} = 0. \tag{53}$$

The scalar constant of motion, which follows from

$$j^\mu = i \left( \Phi^\dagger \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^\dagger)} - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \Phi \right), \quad \partial_\mu j^\mu = 0, \tag{54}$$

is given by

$$Q = \text{const} \int j^0 d^3 x = \text{const} \int (\Phi^\dagger \Psi - \Psi^\dagger \Phi) d^3 x. \tag{55}$$

Using the expressions (17), (22), (23), (27), (48) and (50) it becomes

$$Q = \text{const} \sum_{\vec{k}} \{ (a_{\vec{k}1N}^{1\dagger} a_{\vec{k}1N}^1 + a_{\vec{k}1N}^{2\dagger} a_{\vec{k}1N}^2) - (b_{\vec{k}1N}^3 b_{\vec{k}1N}^{3\dagger} + b_{\vec{k}1N}^4 b_{\vec{k}1N}^{4\dagger}) - \\ - (b_{\vec{k}2N}^1 b_{\vec{k}2N}^{1\dagger} + b_{\vec{k}2N}^2 b_{\vec{k}2N}^{2\dagger}) + (a_{\vec{k}2N}^{3\dagger} a_{\vec{k}2N}^3 + a_{\vec{k}2N}^{4\dagger} a_{\vec{k}2N}^4) \} \quad (56)$$

or written in the particle number operator form

$$Q = \text{const} \sum_{\vec{k}} \{ (a_{\vec{k}1N}^{1\dagger} a_{\vec{k}1N}^1 + a_{\vec{k}1N}^{2\dagger} a_{\vec{k}1N}^2) - (b_{\vec{k}1N}^{3\dagger} b_{\vec{k}1N}^3 + b_{\vec{k}1N}^{4\dagger} b_{\vec{k}1N}^4) - \\ - (b_{\vec{k}2N}^{1\dagger} b_{\vec{k}2N}^1 + b_{\vec{k}2N}^{2\dagger} b_{\vec{k}2N}^2) + (a_{\vec{k}2N}^{3\dagger} a_{\vec{k}2N}^3 + a_{\vec{k}2N}^{4\dagger} a_{\vec{k}2N}^4) - 4 \}. \quad (57)$$

The energy-momentum constant of motion follows from

$$T_{\alpha\beta} = \partial_{\alpha} \Phi^{\dagger} \frac{\partial \mathcal{L}}{\partial (\partial_{\beta} \Phi^{\dagger})} + \frac{\partial \mathcal{L}}{\partial (\partial_{\beta} \Phi)} \partial_{\alpha} \Phi - \delta_{\alpha}^{\beta} \mathcal{L}, \quad \partial_{\beta} T_{\alpha}^{\beta} = 0 \quad (58)$$

and for the Lagrangian (1) reads

$$P_0 = \text{const} \int (\dot{\Phi}^{\dagger} \Psi + \Psi^{\dagger} \dot{\Phi} - \Psi^{\dagger} \gamma^0 \Psi) d^3 x, \\ P_i = \text{const} \int (\partial_i \Phi^{\dagger} \Psi + \Psi^{\dagger} \partial_i \Phi) d^3 x. \quad (59)$$

By making use of Eqs. (17), (22), (23), (48) and (50) it becomes

$$P_0 = \text{const} \sum_{\vec{k}} k \{ (a_{\vec{k}1N}^{1\dagger} a_{\vec{k}1N}^1 + a_{\vec{k}1N}^{2\dagger} a_{\vec{k}1N}^2) + (b_{\vec{k}2N}^1 b_{\vec{k}2N}^{1\dagger} + b_{\vec{k}2N}^2 b_{\vec{k}2N}^{2\dagger}) - \\ - (b_{\vec{k}1N}^3 b_{\vec{k}1N}^{3\dagger} + b_{\vec{k}1N}^4 b_{\vec{k}1N}^{4\dagger}) - (a_{\vec{k}2N}^{3\dagger} a_{\vec{k}2N}^3 + a_{\vec{k}2N}^{4\dagger} a_{\vec{k}2N}^4) \}, \quad (60)$$

$$P_i = \text{const} \sum_{\vec{k}} k_i \{ (a_{\vec{k}1N}^{1\dagger} a_{\vec{k}1N}^1 + a_{\vec{k}1N}^{2\dagger} a_{\vec{k}1N}^2) - (b_{\vec{k}1N}^3 b_{\vec{k}1N}^{3\dagger} + b_{\vec{k}1N}^4 b_{\vec{k}1N}^{4\dagger}) - \\ - (b_{\vec{k}2N}^1 b_{\vec{k}2N}^{1\dagger} + b_{\vec{k}2N}^2 b_{\vec{k}2N}^{2\dagger}) + (a_{\vec{k}2N}^{3\dagger} a_{\vec{k}2N}^3 + a_{\vec{k}2N}^{4\dagger} a_{\vec{k}2N}^4) \} \quad (61)$$

or

$$P_0 = \text{const} \sum_{\vec{k}} k \{ (a_{\vec{k}1N}^{1\dagger} a_{\vec{k}1N}^1 + a_{\vec{k}1N}^{2\dagger} a_{\vec{k}1N}^2) + (b_{\vec{k}2N}^{1\dagger} b_{\vec{k}2N}^1 + b_{\vec{k}2N}^{2\dagger} b_{\vec{k}2N}^2) - \\ (62)$$

$$\begin{aligned}
& - (b_{k1N}^{3\uparrow} b_{k1N}^3 + b_{k1N}^{4\uparrow} b_{k1N}^4) - (a_{k2N}^{3\uparrow} a_{k2N}^3 + a_{k2N}^{4\uparrow} a_{k2N}^4)\}, \\
P_i = \text{const} \sum_{\vec{k}} k_i \{ & (a_{k1N}^{1\uparrow} a_{k1N}^1 + a_{k1N}^{2\uparrow} a_{k1N}^2) - (b_{k1N}^{3\uparrow} b_{k1N}^3 + b_{k1N}^{4\uparrow} b_{k1N}^4) \\
& - (b_{k2N}^{1\uparrow} b_{k2N}^1 + b_{k2N}^{2\uparrow} b_{k2N}^2) + (a_{k2N}^{3\uparrow} a_{k2N}^3 + a_{k2N}^{4\uparrow} a_{k2N}^4)\}.
\end{aligned} \tag{63}$$

The angular momentum density tensor

$$M^{\alpha,\beta\gamma} = (x^\gamma T^{\beta\alpha} - x^\beta T^{\gamma\alpha}) - \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \Phi)} \left( \frac{i}{2} \sigma^{\beta\gamma} \right) \Phi - \Phi^\dagger \left( -\frac{i}{2} \sigma^{\beta\gamma} \right) \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \Phi^\dagger)} \tag{64}$$

for the Lagrangian (1) reads

$$\begin{aligned}
M^{\alpha,\beta\gamma} = K \{ & x^\gamma (\partial^\beta \bar{\Phi} \gamma^\alpha \Psi + \bar{\Psi} \gamma^\alpha \partial^\beta \Phi - g^{\alpha\beta} \bar{\Psi} \Psi) - \\
& - x^\beta (\partial^\gamma \bar{\Phi} \gamma^\alpha \Psi + \bar{\Psi} \gamma^\alpha \partial^\gamma \Phi - g^{\alpha\beta} \bar{\Psi} \Psi) - \\
& - \bar{\Psi} \gamma^\alpha \left( \frac{i}{2} \sigma^{\beta\gamma} \right) \Phi - \bar{\Phi} \left( -\frac{i}{2} \sigma^{\beta\gamma} \right) \gamma^\alpha \Psi \}.
\end{aligned} \tag{65}$$

From here the constant of motion is

$$\begin{aligned}
M^{\beta\gamma} = \text{const} \int \{ & x^\gamma (\partial^\beta \Phi^\dagger \Psi + \Psi^\dagger \partial^\beta \Phi - g^{0\gamma} \bar{\Psi} \Psi) - \\
& - x^\beta (\partial^\gamma \Phi^\dagger \Psi + \Psi^\dagger \partial^\gamma \Phi - g^{0\beta} \bar{\Psi} \Psi) - \\
& - \frac{i}{2} (\Psi^\dagger \sigma^{\beta\gamma} \Phi - \bar{\Phi} \gamma^0 \sigma^{\beta\gamma} \gamma^0 \Psi) \} d^3 x.
\end{aligned} \tag{66}$$

The spin part of this constant, using Eqs. (17) and (22), reads

$$S^{IJ} = \text{const} \sum_{\vec{k}} \left[ A_{\vec{k}}^{m\uparrow} (\sigma^{IJ})_{ml} a_{\vec{k}}^l - a_{\vec{k}}^{m\uparrow} (\sigma^{IJ})_{ml} A_{\vec{k}}^l \right]. \tag{67}$$

The spin pseudovector is then

$$S_r = \frac{1}{2} \varepsilon_{ijr} S^{ij}$$

and using the expressions (20), (26), (27), (28), (31) and (32)

$$\begin{aligned}
 S_r = \text{const} \sum_{\vec{k}} & \left\{ (a_{\vec{k}1}^\dagger \beta_{\vec{k}1}^\dagger) \begin{pmatrix} k\sigma_r & -\frac{1}{2}[\sigma_r, \vec{k}\vec{\sigma}] \\ \frac{1}{2}[\sigma_r, \vec{k}\vec{\sigma}] & -k\sigma_r \end{pmatrix} \begin{pmatrix} a_{\vec{k}1} \\ \beta_{\vec{k}1} \end{pmatrix} + \right. \\
 & + (a_{\vec{k}2}^\dagger \beta_{\vec{k}2}^\dagger) \begin{pmatrix} -k\sigma_r & -\frac{1}{2}[\sigma_r, \vec{k}\vec{\sigma}] \\ \frac{1}{2}[\sigma_r, \vec{k}\vec{\sigma}] & k\sigma_r \end{pmatrix} \begin{pmatrix} a_{\vec{k}2} \\ \beta_{\vec{k}2} \end{pmatrix} + \\
 & + (a_{\vec{k}2}^\dagger \beta_{\vec{k}2}^\dagger) \begin{pmatrix} 0 & -\frac{1}{2}[\sigma_r, \vec{k}\vec{\sigma}] \\ \frac{1}{2}[\sigma_r, \vec{k}\vec{\sigma}] & 0 \end{pmatrix} \begin{pmatrix} a_{\vec{k}1} \\ \beta_{\vec{k}1} \end{pmatrix} e^{-2ikt} + \\
 & \left. + (a_{\vec{k}1}^\dagger \beta_{\vec{k}1}^\dagger) \begin{pmatrix} 0 & -\frac{1}{2}[\sigma_r, \vec{k}\vec{\sigma}] \\ \frac{1}{2}[\sigma_r, \vec{k}\vec{\sigma}] & 0 \end{pmatrix} \begin{pmatrix} a_{\vec{k}2} \\ \beta_{\vec{k}2} \end{pmatrix} e^{2ikt} \right\}, \tag{68}
 \end{aligned}$$

where  $[\sigma_r, \vec{k}\vec{\sigma}] = \sigma_r (\vec{k}\vec{\sigma}) - (\vec{k}\vec{\sigma}) \sigma_r$ .

For one particle state with the momentum  $\vec{k}$  the projection  $\frac{\vec{k}\vec{S}}{k}$  is

$$\begin{aligned}
 \frac{\vec{k}\vec{S}}{k} = \text{const} & \left\{ \left[ a_{\vec{k}1}^\dagger (\vec{k}\vec{\sigma}) a_{\vec{k}1} - \beta_{\vec{k}1} (\vec{k}\vec{\sigma}) \beta_{\vec{k}1} \right] - \right. \\
 & \left. - \left[ a_{\vec{k}2}^\dagger (\vec{k}\vec{\sigma}) a_{\vec{k}2} - \beta_{\vec{k}2}^\dagger (\vec{k}\vec{\sigma}) \beta_{\vec{k}2} \right] \right\}. \tag{69}
 \end{aligned}$$

For polar axis in  $\vec{k}$  direction it becomes

$$\begin{aligned}
 \frac{\vec{k}\vec{S}}{k} = \text{const} k & \left\{ (a_{\vec{k}1}^{1\dagger} a_{\vec{k}1}^1 - a_{\vec{k}1}^{2\dagger} a_{\vec{k}1}^2) - (a_{\vec{k}1}^{3\dagger} a_{\vec{k}1}^3 - a_{\vec{k}1}^{4\dagger} a_{\vec{k}1}^4) - \right. \\
 & \left. - (a_{\vec{k}2}^{1\dagger} a_{\vec{k}2}^1 - a_{\vec{k}2}^{2\dagger} a_{\vec{k}2}^2) + (a_{\vec{k}2}^{3\dagger} a_{\vec{k}2}^3 - a_{\vec{k}2}^{4\dagger} a_{\vec{k}2}^4) \right\} \tag{70}
 \end{aligned}$$

or

$$\begin{aligned}
 \frac{\vec{k}\vec{S}}{k} = \text{const} \frac{1}{2} & \left\{ (a_{\vec{k}1N}^{1\dagger} a_{\vec{k}1N}^1 - a_{\vec{k}1N}^{2\dagger} a_{\vec{k}1N}^2) - (b_{\vec{k}1N}^3 b_{\vec{k}1N}^{3\dagger} - b_{\vec{k}1N}^4 b_{\vec{k}1N}^{4\dagger}) - \right. \\
 & \left. - (b_{\vec{k}2N}^1 b_{\vec{k}2N}^{1\dagger} - b_{\vec{k}2N}^2 b_{\vec{k}2N}^{2\dagger}) + (a_{\vec{k}2N}^{3\dagger} a_{\vec{k}2N}^3 - a_{\vec{k}2N}^{4\dagger} a_{\vec{k}2N}^4) \right\}. \tag{71}
 \end{aligned}$$

## 6. Real field

The commutation rules (51) show a similarity of the field with the electromagnetic field. The electromagnetic field is a real field. Consequently, it is of interest to look for real solutions of the massless Dirac field equations. For this purpose we write the canonical equations (12—13) and the Lagrange's equations (3—4) in the form

$$i \partial_\nu \gamma^\nu \Psi = 0,$$

$$\Psi = i \partial_\nu \gamma^\nu (-i \Phi), \quad (72)$$

$$(i \partial_\nu \gamma^\nu) (i \partial_\nu \gamma^\nu) \Phi = 0,$$

and c. c..

The unitary transformation

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \sigma_2 \\ -\sigma_2 & 1 \end{pmatrix}, \quad U^\dagger = U^{-1},$$

where  $\sigma_i$  are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

changes the operator  $i \partial_\nu \gamma^\nu$  into the real one<sup>7)</sup>. Indeed, the new field functions and  $\gamma^\nu$  matrices, given by

$$\tilde{\Psi} = U \Psi, \quad \Psi = U^{-1} \tilde{\Psi}, \quad (73)$$

$$\tilde{\gamma}^\nu = U \gamma^\nu U^{-1}, \quad (74)$$

$$\gamma^\nu = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are

$$\tilde{\Psi} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_1 - i \Psi_4 \\ \Psi_2 + i \Psi_3 \\ \Psi_3 + i \Psi_2 \\ \Psi_4 - i \Psi_1 \end{pmatrix}, \quad (75)$$

$$\tilde{\gamma}^0 = \begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \tilde{\gamma}^1 = \begin{pmatrix} i \sigma_3 & 0 \\ 0 & i \sigma_3 \end{pmatrix}, \quad \tilde{\gamma}^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \tilde{\gamma}^3 = \begin{pmatrix} -i \sigma_1 & 0 \\ 0 & -i \sigma_1 \end{pmatrix}. \quad (76)$$

The field equations then become

$$i \partial_\nu \tilde{\gamma}^\nu \tilde{\Psi} = 0, \quad (77)$$

$$\tilde{\Psi} = i \partial_\nu \tilde{\gamma}^\nu (-i \tilde{\Phi}),$$

$$(i \partial_\nu \tilde{\gamma}^\nu)(i \partial_\nu \tilde{\gamma}^\nu) \tilde{\Phi} = 0. \quad (78)$$

Now, we state the reality condition. We require that the canonical pairs are real:

$$\Psi^* = \Psi,$$

$$\Phi^* = \Phi.$$

The second of Eqs. (77) shows that this condition cannot be fulfilled. From here we conclude that there are no real solutions in this realization of the canonical formulation of the massless Dirac field<sup>6)</sup>.

### 7. Dual symmetry of the field

The transformation<sup>8-12)</sup>

$$\Psi' = \gamma^5 \Psi, \quad \gamma^5 = - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (79)$$

leaves Eqs. (13) unchanged. From here it follows that if  $\Psi$  is a solution of Eqs. (13) then  $\Psi'$  is also a solution. Explicitly,

$$\begin{aligned} \Psi^1 &= -\Psi'^3, \\ \Psi^2 &= -\Psi'^4, \\ \Psi^3 &= -\Psi'^1, \\ \Psi^4 &= -\Psi'^2. \end{aligned} \quad (80)$$

This dual symmetry can be used in decomposition of the field variables and corresponding physical quantities.

The substitution of (17) into (12) gives<sup>5)</sup>

$$\Psi = \sum_{\vec{k}} \left\{ \left[ a_{\vec{k}1}^1 \begin{pmatrix} k \\ 0 \\ k_z \\ k_x + i k_y \end{pmatrix} + a_{\vec{k}1}^2 \begin{pmatrix} 0 \\ k \\ k_x - i k_y \\ -k_z \end{pmatrix} + a_{\vec{k}1}^3 \begin{pmatrix} -k_z \\ -k_x - i k_y \\ -k \\ 0 \end{pmatrix} \right] + \right.$$

$$\begin{aligned}
& + a_{k1}^4 \begin{pmatrix} -k_x + i k_y \\ k_z \\ 0 \\ -k \end{pmatrix} \Big] e^{i(\vec{k}\vec{x} - kt)} + \\
& + \left[ a_{k2}^1 \begin{pmatrix} -k \\ 0 \\ k_z \\ k_x + i k_y \end{pmatrix} + a_{k2}^2 \begin{pmatrix} 0 \\ -k \\ k_x - i k_y \\ -k_z \end{pmatrix} + a_{k2}^3 \begin{pmatrix} -k_x - i k_y \\ k \\ 0 \end{pmatrix} + \right. \\
& \left. + a_{k2}^4 \begin{pmatrix} -k_x + i k_y \\ k_z \\ 0 \\ k \end{pmatrix} \right] e^{i(\vec{k}\vec{x} + kt)}.
\end{aligned} \tag{81}$$

One can easily see that two columns in the first row are linearly dependent on other two. This is also the case in the second row. Taking  $a_{k1}^3 = a_{k1}^4 = a_{k2}^3 = a_{k2}^4 = 0$  one gets the general solution of Eqs. (13). Let us denote it by  $\Psi_n$ :

$$\begin{aligned}
\Psi_n = \sum_{\vec{k}} \left\{ \left[ a_{k1}^1 \begin{pmatrix} k \\ 0 \\ k_z \\ k_x + i k_y \end{pmatrix} + a_{k1}^2 \begin{pmatrix} 0 \\ k \\ k_x - i k_y \\ -k_z \end{pmatrix} \right] e^{i(\vec{k}\vec{x} - kt)} + \right. \\
\left. + \left[ a_{k2}^1 \begin{pmatrix} -k \\ 0 \\ k_z \\ k_x + i k_y \end{pmatrix} + a_{k2}^2 \begin{pmatrix} 0 \\ -k \\ k_x - i k_y \\ -k_z \end{pmatrix} \right] e^{i(\vec{k}\vec{x} + kt)} \right\}.
\end{aligned} \tag{82}$$

Taking now  $a_{k1}^1 = a_{k1}^2 = a_{k2}^1 = a_{k2}^2 = 0$  one gets also the general solution of Eqs. (13). Let us denote it by  $\Psi_d$ :

$$\begin{aligned}
\Psi_d = \sum_{\vec{k}} \left\{ \left[ a_{k1}^3 \begin{pmatrix} -k_z \\ -k_x - i k_y \\ -k \\ 0 \end{pmatrix} + a_{k1}^4 \begin{pmatrix} -k_x + i k_y \\ k_z \\ 0 \\ -k \end{pmatrix} \right] e^{i(\vec{k}\vec{x} - kt)} + \right. \\
\left. + \left[ a_{k2}^3 \begin{pmatrix} -k_z \\ -k_x - i k_y \\ k \\ 0 \end{pmatrix} + a_{k2}^4 \begin{pmatrix} -k_x + i k_y \\ k_z \\ 0 \\ k \end{pmatrix} \right] e^{i(\vec{k}\vec{x} + kt)} \right\}.
\end{aligned} \tag{83}$$

These two solutions are in dual relation given by (79). Therefore, we can write

$$\Psi = \Psi_n + \Psi_d. \tag{84}$$

This decomposition of  $\Psi$  corresponds to the following decomposition of  $\Phi$ , according to (12):

$$\Phi = \Phi_n + \Phi_d, \quad (85)$$

where

$$\Phi_n = \frac{1}{L^{3/2}} \sum_{\vec{k}} \left\{ \begin{bmatrix} 1 \\ a_{\vec{k}1}^1 \\ a_{\vec{k}1}^2 \\ 0 \\ 0 \end{bmatrix} e^{i(\vec{k}\vec{x}-kt)} + \begin{bmatrix} 1 \\ a_{\vec{k}2}^1 \\ a_{\vec{k}2}^2 \\ 0 \\ 0 \end{bmatrix} e^{i(\vec{k}\vec{x}+kt)} \right\}, \quad (86)$$

$$\Phi_d = \frac{1}{L^{3/2}} \sum_{\vec{k}} \left\{ \begin{bmatrix} 0 \\ 0 \\ a_{\vec{k}1}^3 \\ a_{\vec{k}1}^4 \\ a_{\vec{k}2}^1 \end{bmatrix} e^{i(\vec{k}\vec{x}-kt)} + \begin{bmatrix} 0 \\ 0 \\ a_{\vec{k}2}^3 \\ a_{\vec{k}2}^4 \\ a_{\vec{k}2}^1 \end{bmatrix} e^{i(\vec{k}\vec{x}+kt)} \right\}. \quad (87)$$

We have then

$$\Psi_n = \partial_\alpha \gamma^\alpha \Phi_n, \quad (88)$$

$$\Psi_d = \partial_\alpha \gamma^\alpha \Phi_d.$$

Using this decomposition the constants of motion (56), (62), (63) and the spin projection (71) become:

$$Q = Q_n + Q_d, \quad (89)$$

$$P_\alpha = P_{\alpha n} - P_{\alpha d}, \quad (90)$$

$$\frac{\vec{k} \vec{S}}{k} = \frac{(\vec{k} \vec{S})}{k} n - \frac{(\vec{k} \vec{S})}{k} d, \quad (91)$$

where

$$Q_n = \text{const} \sum_{\vec{k}} \{ (a_{\vec{k}1N}^{1\dagger} a_{\vec{k}1N}^1 + a_{\vec{k}1N}^{2\dagger} a_{\vec{k}1N}^2) - (b_{\vec{k}2N}^{1\dagger} b_{\vec{k}2N}^{1\dagger} + b_{\vec{k}2N}^2 b_{\vec{k}2N}^{2\dagger}) \}, \quad (92)$$

$$Q_d = \text{const} \sum_{\vec{k}} \{ (a_{\vec{k}2N}^{3\dagger} a_{\vec{k}2N}^3 + a_{\vec{k}1N}^{4\dagger} a_{\vec{k}2N}^4) - (b_{\vec{k}1N}^{3\dagger} b_{\vec{k}1N}^{3\dagger} + b_{\vec{k}1N}^4 b_{\vec{k}1N}^{4\dagger}) \}, \quad (93)$$

$$P_{0n} = \text{const} \sum_{\vec{k}} k \{ (a_{\vec{k}1N}^{1\dagger} a_{\vec{k}1N}^1 + a_{\vec{k}1N}^{2\dagger} a_{\vec{k}1N}^2 + (b_{\vec{k}2N}^{1\dagger} b_{\vec{k}2N}^1 + b_{\vec{k}2N}^{2\dagger} b_{\vec{k}2N}^2) \}, \quad (94)$$

$$P_{0d} = \text{const} \sum_{\vec{k}} k \{ (a_{\vec{k}2N}^{3\dagger} a_{\vec{k}2N}^3 + a_{\vec{k}2N}^{4\dagger} a_{\vec{k}2N}^4) + (b_{\vec{k}1N}^{3\dagger} b_{\vec{k}1N}^3 + b_{\vec{k}1N}^{4\dagger} b_{\vec{k}1N}^4) \}, \quad (95)$$

$$P_n = \text{const} \sum_{\vec{k}} k_i \{ (a_{\vec{k}1N}^{1\dagger} a_{\vec{k}1N}^1 + a_{\vec{k}1N}^{2\dagger} a_{\vec{k}1N}^2) - (b_{\vec{k}2N}^{1\dagger} b_{\vec{k}2N}^1 + b_{\vec{k}2N}^{2\dagger} b_{\vec{k}2N}^2) \}, \quad (96)$$

$$P_d = \text{const} \sum_{\vec{k}} k_i \{ (a_{\vec{k}2N}^{3\dagger} a_{\vec{k}2N}^3 + a_{\vec{k}2N}^{4\dagger} a_{\vec{k}2N}^4) - (b_{\vec{k}1N}^{3\dagger} b_{\vec{k}1N}^3 + b_{\vec{k}1N}^{4\dagger} b_{\vec{k}1N}^4) \}, \quad (97)$$

$$\frac{(\vec{k} \vec{S})}{k} n = \text{const} \frac{1}{2} \{ (a_{\vec{k}1N}^{1\dagger} a_{\vec{k}1N}^1 - a_{\vec{k}1N}^{2\dagger} a_{\vec{k}1N}^2) - (b_{\vec{k}2N}^{1\dagger} b_{\vec{k}2N}^1 - b_{\vec{k}2N}^{2\dagger} b_{\vec{k}2N}^2) \}, \quad (98)$$

$$\frac{(\vec{k} \vec{S})}{k} d = \text{const} \frac{1}{2} \{ (a_{\vec{k}2N}^{3\dagger} a_{\vec{k}2N}^3 - a_{\vec{k}2N}^{4\dagger} a_{\vec{k}2N}^4) - (b_{\vec{k}1N}^{3\dagger} b_{\vec{k}1N}^3 - b_{\vec{k}1N}^{4\dagger} b_{\vec{k}1N}^4) \}. \quad (99)$$

Due to the fact that  $\Psi_n$  and  $\Psi_d$  are general solutions of the canonical equations (13) we may restrict our attention to one of these solution. It leads to a selection of states of the system. There is no a priori rule for this selection. One may expect that the correct selection comes from the interaction of this field with other fields. At the present moment the following selection seems to be plausible

$$a_{\vec{k}2N}^{i\dagger} a_{\vec{k}2N}^i | > = b_{\vec{k}1N}^{i\dagger} b_{\vec{k}1N}^i | > = 0, \quad i = 3, 4. \quad (100)$$

It corresponds to the elimination of the magnetic monopoles in the theory of the electromagnetic field.

In this case, changing  $b_{\vec{k}2N}^i \rightarrow b_{-\vec{k}2N}^i$ , the constants of motion become

$$Q = \text{const} \sum_{\vec{k}} \{ (a_{\vec{k}1N}^{1\dagger} a_{\vec{k}1N}^1 + a_{\vec{k}1N}^{2\dagger} a_{\vec{k}1N}^2) - (b_{\vec{k}2N}^{1\dagger} b_{\vec{k}2N}^1 + b_{\vec{k}2N}^{2\dagger} b_{\vec{k}2N}^2) - 4 \}, \quad (101)$$

$$P^a = \text{const} \sum_{\vec{k}} k^a \{ (a_{\vec{k}1N}^{1\dagger} a_{\vec{k}1N}^1 + a_{\vec{k}1N}^{2\dagger} a_{\vec{k}1N}^2) + (b_{\vec{k}2N}^{1\dagger} b_{\vec{k}2N}^1 + b_{\vec{k}2N}^{2\dagger} b_{\vec{k}2N}^2) \}, \quad (102)$$

$$\frac{(\vec{k} \vec{S})}{k} = \text{const} \frac{1}{2} \{ (a_{\vec{k}1N}^{1\dagger} a_{\vec{k}1N}^1 - a_{\vec{k}1N}^{2\dagger} a_{\vec{k}1N}^2) + (b_{\vec{k}2N}^{1\dagger} b_{\vec{k}2N}^1 - b_{\vec{k}2N}^{2\dagger} b_{\vec{k}2N}^2) \}. \quad (103)$$

The particle interpretation of these constants is evident. It is important to notice that the energy of these particles is always positive. Thus, the historical problem of negative energies associated with solutions of the Dirac equation doesn't appear in correct canonical description of the massless Dirac field. There is only one problem present in the infinity constant term in  $Q$ . Although it can be subtracted from  $Q$ , the physical meaning of this infinity is not clear.

According to (104) the spin of the particles is 1/2. Therefore, the quantization of the correct canonical description of the massless Dirac field gives particles with spin 1/2 but with Bose statistics. This is also important result and quite different from the standard theory.

## 8. Conclusion

Consistent and correct canonical quantization of the massless Dirac field leads to the commutation and not anticommutation rules for the canonical pairs. The constants of motion of the field exhibit particle description but with the Bose statistics. The spin of the particles is  $1/2$ .

The energy-momentum vector and the angular momentum tensor do not contain infinities. However, the scalar constant does contain an infinity term.

The appearance of particles with negative energies does not cause difficulties because restriction can be made on states without such particles. The elimination of these states corresponds to the elimination of the field produced by the magnetic monopoles in the electromagnetic field theory.

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## KVANTIZIRANJE DIRACOVOG POLJA BEZ MASE U NOVOJ KANONSKOJ FORMULACIJI

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Na osnovu nove kanonske formulacije Diracovog polja bez mase mirovanja u rđdu je izvršeno konzistentno kanonsko kvantiziranje ovog polja. Nađeno je da je kvantiziranje moguće provesti samo sa komutacionim, a ne i antikomutacionim pravilima. Izračunate konstante kretanja pokazuju čestičnu interpretaciju sa spinom  $1/2$ . Problemi negativnih energija standardne teorije ne pojavljuju se, kao ni beskonačnosti u izrazu za energiju. Skalarna konstanta sadrži jedan beskonačni član i on zasluđuje posebnu pažnju.