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DUAL SYMMETRY AND THE DIRAC FIELD THEORY*

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A new theory of the Dirac field is developed in which both the Dirac field equations, with positive and negative mass term, describe a physical object. The theory is given in correct Lagrange's and canonical language.

1. Introduction

In several recent papers[1-4] we have analysed the canonical formulation of the Dirac field in which the Dirac equation was considered as a canonical equation and not a Lagrange's equation.

These analyses gave fruitful results in various aspects. Some questions, however, remained open. Among them there is the appearance of singularities in the canonical quantization of the Dirac field with a mass term. Although the physical quantities do not contain these singularities and have clear physical interpretation, it still indicates another approach to the Dirac field. This another approach we consider in this paper.

One of properties of the new canonical formulations of massless physical fields[5,6] is explicite appearance of dual symmetry of the electromagnetic field and corresponding symmetry of other fields. The Dirac field with a mass term does not exhibit a such symmetry. This fact, seems to us, requires appropriate attention.

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In Section 2 we analyse the symmetry properties of the Dirac field in this sense. We find that the Dirac's equations with positive and negative mass term have such a symmetry. We take it as a starting point and in the next sections develop a new theory of the Dirac field.

In Section 3 we define the new Dirac field in correct Lagrange's and canonical language. This may be taken also as the beginning of the theory. We show that the Dirac's equations with positive and negative mass term are equivalent to the canonical equations of a bispinor field.

We give general solution of the field equations in Section 4. The canonical quantization of the field is performed in Section 5. The scalar, energy-momentum and angular momentum constants of motion are evaluated in Section 6. The physical meaning of the field is considered in Section 7 and 8. Some historical remarks on works related to this subject are given in Section 9. Conclusions are given in Section 10.

2. *Dual symmetry of the Dirac field*

The electromagnetic free field exhibits the following symmetry

$$\vec{E} \to \vec{H},$$

$$\vec{H} \to -\vec{E}$$

and it is called dual symmetry. This symmetry is evident if one writes the Maxwell's equations for the free field in the form

$$\begin{aligned} \partial_\alpha F^{\alpha\beta} &= 0, \\ \partial_\alpha \tilde{F}^{\alpha\beta} &= 0, \end{aligned} \tag{1}$$

where $F^{\alpha\beta}$ is the electromagnetic field tensor and $\tilde{F}^{\alpha\beta}$ is its dual tensor, i. e.

$$\tilde{F}^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta}{}_{\mu\nu} F^{\mu\nu}, \tag{2}$$

where $\varepsilon^{\alpha\beta\mu\nu}$ is the totally antisymmetric unit tensor. The substitution $F^{\alpha\beta} \to \tilde{F}^{\alpha\beta}$ does not change field equations.

Similarly to the electromagnetic field the Dirac massless field

$$\partial_\alpha \gamma^\alpha \Psi = 0 \tag{3}$$

has the symmetry property to the transformation[7]

$$\Psi \to \gamma^5 \Psi, \qquad \gamma^5 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

The dual symmetry of the electromagnetic field and the »γ^5« symmetry of the Dirac massless field are special cases of a more general symmetry, e. g. chiral symmetry[7-11]. It is worthwhile to look at this symmetry more carefully. But for the purpose of this paper it will not be necessary to go further in this direction.

We use the coordinates $x^\alpha = (x^0, x^1, x^2, x^3)$, the metric

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1, \qquad g_{\alpha\beta} = 0, \qquad \alpha \neq \beta,$$

the representation of the γ^μ matrices

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3,$$

and units $c = \hbar = 1$.

The transformation (4) changes the first pair of the Dirac field components with the second one and takes the opposite sign:

$$\gamma^5 \Psi = \begin{pmatrix} -\Psi_3 \\ -\Psi_4 \\ -\Psi_2 \\ -\Psi_1 \end{pmatrix}.$$

If one writes Eq. (3) in explicite component form

$$\begin{aligned}
\partial_0 \Psi_1 + \quad & \partial_z \Psi_3 + (\partial_x - i\partial_y)\Psi_4 = 0, \\
\partial_0 \Psi_2 + \quad & (\partial_x + i\partial_y)\Psi_3 - \partial_z \Psi_4 = 0, \\
-\partial_z \Psi_1 + \quad & (-\partial_x + i\partial_y)\Psi_2 - \partial_0 \Psi_3 = 0, \\
(-\partial_x - i\partial_y)\Psi_1 + \quad & \partial_z \Psi_2 - \partial_0 \Psi_4 = 0
\end{aligned}$$

one sees that these equations can be written also in the form

$$A\Psi = 0, \qquad A\tilde{\Psi} = 0, \qquad \tilde{\Psi} = \gamma^5 \Psi, \tag{5}$$

where

$$A = \begin{pmatrix} \partial_0 & 0 & \partial_z & \partial_x - i\partial_y \\ 0 & \partial_0 & \partial_x + i\partial_y & -\partial_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5'}$$

Tnhis form explicitly manifest the symmetry (4) and corresponds to the electromag e tic field equations (1).

The Dirac equation with a mass term

$$(i\,\partial_a\,\gamma^a - \varkappa)\,\Psi = 0, \tag{6}$$

however, does not have this symmetry. It is useful also to write this equation in the component form

$$\begin{aligned}
(i\,\partial_0 - \varkappa)\,\Psi_1 + \quad & i\,\partial_z\,\Psi_3 + \quad (i\,\partial_x + \partial_y)\,\Psi_4 = 0,\\
(i\,\partial_0 - \varkappa)\,\Psi_2 + \quad & (i\,\partial_x - \partial_y)\,\Psi_3 - \quad i\,\partial_z\,\Psi_4 = 0,\\
-\,i\,\partial_z\,\Psi_1 + \quad & (-\,i\,\partial_x - \partial_y)\,\Psi_2 + \quad (-\,i\,\partial_0 - \varkappa)\,\Psi_3 = 0,\\
(-\,i\,\partial_x + \partial_y)\,\Psi_1 + \quad & i\,\partial_z\,\Psi_2 + \quad (-\,i\,\partial_0 - \varkappa)\,\Psi_4 = 0.
\end{aligned} \tag{7-}$$

Performing the transformation (4) ($\Psi \to \gamma^5\,\Psi \equiv \tilde{\Psi}$) these equations go over into

$$\begin{aligned}
(-\,i\,\partial_0 + \varkappa)\,\tilde{\Psi}_3 - \quad & i\,\partial_z\,\tilde{\Psi}_1 + \quad (-\,i\,\partial_x - \partial_y)\,\tilde{\Psi}_2 = 0,\\
(-\,i\,\partial_0 + \varkappa)\,\tilde{\Psi}_4 + \quad & (-\,i\,\partial_x + \partial_y)\,\tilde{\Psi}_1 + \quad i\,\partial_z\,\tilde{\Psi}_2 = 0,\\
i\,\partial_z\,\tilde{\Psi}_3 + \quad & (i\,\partial_x + \partial_y)\,\tilde{\Psi}_4 + \quad (i\,\partial_0 + \varkappa)\,\tilde{\Psi}_1 = 0,\\
(i\,\partial_x - \partial_y)\,\tilde{\Psi}_3 - \quad & i\,\partial_z\,\tilde{\Psi}_4 + \quad (i\,\partial_0 + \varkappa)\,\tilde{\Psi}_2 = 0.
\end{aligned} \tag{8}$$

These are the Dirac field equations with opposite mass term:

$$(i\,\partial_a\,\gamma^a + \varkappa)\,\tilde{\Psi} = 0. \tag{9}$$

Therefore, denoting the field given by Eq. (6) with $\Psi_{\rm I}$ and the field given by Eq. (9) with $\Psi_{\rm II}$, we have the case

$$\Psi_{\rm II} = \gamma^5\,\Psi_{\rm I}. \tag{10}$$

For $\varkappa = 0$ Eq. (10) expresses the dual symmetry of the massless Dirac field.

From here we see that the Dirac field equation (6) alone and equation (9) also do not have the dual symmetry. However, Eq. (10) shows that the field composed of $\Psi_{\rm I}$ and $\Psi_{\rm II}$ will have such a symmetry.

Indeed, writing

$$\Psi = \begin{pmatrix}\Psi_{\rm I}\\ \Psi_{\rm II}\end{pmatrix} \tag{11}$$

and

$$\begin{pmatrix} i\,\partial_a\,\gamma^a - \varkappa & 0\\ 0 & i\,\partial_a\,\gamma^a + \varkappa\end{pmatrix}\Psi = 0 \tag{12}$$

Eq. (12) has the following symmetry property

$$\Psi \to \begin{pmatrix} 0 & \gamma^5 \\ \gamma^5 & 0 \end{pmatrix} \Psi. \tag{13}$$

The right hand side written in the components of $\Psi_{\rm I}$ and $\Psi_{\rm II}$ reads

$$\begin{pmatrix} 0 & \gamma^5 \\ \gamma^5 & 0 \end{pmatrix} \Psi = \begin{pmatrix} \gamma^5 \, \Psi_{\rm II} \\ \gamma^5 \, \Psi_{\rm I} \end{pmatrix} \tag{14}$$

After substitution in (12) we find

$$(i\,\partial_a \gamma^a + \varkappa)\,\Psi_{\rm II} = 0, \tag{15}$$

$$(i\,\partial_a \gamma^a - \varkappa)\,\Psi_{\rm I} = 0.$$

Therefore, Eq. (12) has not been changed by the transformation (13).

From this consideration we conclude that both Dirac field equations, (6) and (9), are important for description of a physical object. We take it as a starting point and in the next sections define and investigate this object.

3. *A new Dirac field*

Equations (15) as the canonical equations may be equations for the momenta alone and for the field variables and their momenta together. Investigations of the Ref. [4] show that the first possibility would have singularities. Due to this reason one may expect that these equations are a linear combination of the field variables and their momenta.

Let us denote the Lagrange's variables of the field by Φ, $\Phi^\dagger$.

The requirement that the Lagrange's equations are of the Klein-Gordon type leads to the Lagrange's density

$$\mathscr{L} = K\,[(-\,i\,\partial_a \overline{\Phi}\,\gamma^a)\,(i\,\partial_\beta\,\gamma^\beta\,\Phi) - \varkappa^2\,\overline{\Phi}\,\Phi], \tag{16}$$

where K and $\varkappa^2$ are real constants. This density might also include the term

$$b^*\,(-\,i\,\partial_a\,\overline{\Phi}\,\gamma^a)\,\Phi + b\,\overline{\Phi}\,(i\,\partial_a\,\gamma^a\,\Phi).$$

But for the Klein-Gordon equations $b^* = -\,b$ and then it becomes the fourdivergence of $\overline{\Phi}\,\gamma^a\,\Phi$.

This place may be beginning of the theory, i. e. we might forget everything what we have said so far and take Eq. (16) as the definition of the bispinor field Φ, $\Phi^\dagger$.

The Lagrange's equations

$$\frac{\partial \mathscr{L}}{\partial \chi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial (\partial_\mu \chi)} = 0, \tag{17}$$

for Φ and $\Phi^\dagger$ are

$$\begin{aligned} K[-\varkappa^2 \Phi^\dagger - i\,\partial_\nu (-i\,\partial_\mu \Phi^\dagger \gamma^{\mu\dagger})\,\gamma^{\nu\dagger}] &= 0, \\ K[-\varkappa^2 \Phi - (-i\,\partial_\nu \gamma^\nu)(i\,\partial_\mu \gamma^\mu \Phi)] &= 0, \end{aligned} \tag{18}$$

or

$$\begin{aligned} \partial_a \partial^a \Phi^\dagger + \varkappa^2 \Phi^\dagger &= 0, \\ \partial_a \partial^a \Phi + \varkappa^2 \Phi &= 0. \end{aligned} \tag{19}$$

The conjugate momenta to Φ, $\Phi^\dagger$, according to

$$\Pi_\chi = \frac{\partial \mathscr{L}}{\partial \dot{\chi}},$$

are

$$\begin{aligned} \Pi_\Phi &= K(-i\,\partial_\nu \Phi^\dagger \gamma^{\nu\dagger})\,i, \\ \Pi_{\Phi^\dagger} &= K(-i)(i\,\partial_\mu \gamma^\mu \Phi), \end{aligned} \tag{20}$$

or

$$\begin{aligned} \frac{1}{K}\Pi_{\Phi^\dagger} &= (\partial_0 \Phi^\dagger \gamma^0 - \partial_i \Phi^\dagger \gamma^i), \\ \frac{1}{K}\Pi_{\Phi^\dagger} &= \partial_0 \gamma^0 \Phi + \partial_i \gamma^i \Phi. \end{aligned} \tag{21}$$

The Hamiltonian density

$$\mathscr{H} = \Pi_\Phi \dot{\Phi} + \dot{\Phi}^\dagger \Pi_{\Phi^\dagger} - \mathscr{L}, \tag{22}$$

is then

$$\mathscr{H} = \frac{1}{K}\Pi_\Phi \gamma^0 \Pi_{\Phi^\dagger} - \Pi_\varphi \partial_i \alpha^i \Phi - \partial_i \Phi^\dagger \alpha^i \Pi_{\Phi^\dagger} + K(\varkappa^2 \Phi^\dagger \gamma^0 \Phi), \tag{23}$$

and the canonical equations, according to

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \{F, H\}, \tag{24}$$

are

$$\dot{\Phi} = \{\Phi, H\} = \frac{\delta H}{\delta \Pi_\Phi} = \frac{1}{K} \gamma^0 \Pi_{\Phi^\dagger} - \partial_l \alpha^l \Phi, \tag{25}$$

$$\dot{\Phi}^\dagger = \{\Phi^\dagger, H\} = \frac{\delta H}{\delta \Pi_{\Phi^\dagger}} = \frac{1}{K} \Pi_\Phi \gamma^0 - \partial_l \Phi^\dagger \alpha^l, \tag{26}$$

$$\dot{\Pi}_\Phi = \{\Pi_\varphi, H\} = -\frac{\delta H}{\delta \Phi} = - \partial_l \Pi_\Phi \alpha^l - K \varkappa^2 \Phi^+ \gamma^0, \tag{27}$$

$$\dot{\Pi}_{\Phi^\dagger} = \{\Pi_{\Phi^\dagger}, H\} = - \frac{\delta H}{\delta \Phi^\dagger} = - \partial_l \alpha^l \Pi_{\Phi^\dagger} - K \varkappa^2 \gamma^0 \Phi, \tag{28}$$

$$\frac{1}{K} \Pi_\Phi \gamma^0 - \partial_\nu \bar{\Phi} \gamma^\nu = 0, \tag{29}$$

$$\begin{aligned} &\frac{1}{K} \Pi_{\Phi^\dagger} - \partial_\mu \gamma^\mu \Phi = 0, \\ &\partial_\nu (\Pi_\Phi \gamma^0) \gamma^\nu + K \varkappa^2 \Phi = 0, \\ &\partial_\nu \gamma^\nu \Pi_{\Phi^\dagger} + K \varkappa^2 \Phi = 0, \end{aligned} \tag{30}$$

where we have made use of $\Pi_\Phi = (\Pi_{\Phi^\dagger})^\dagger$.

Let us concentrate our attention to the canonical equations for Φ and $\Pi_{\Phi^\dagger}$:

$$\begin{aligned} &\frac{1}{K} \Pi_{\Phi^\dagger} - \partial_\nu \gamma^\nu \Phi = 0, \\ &\partial_\nu \gamma^\nu \Pi_{\Phi^\dagger} + K \varkappa^2 \Phi = 0. \end{aligned} \tag{31}$$

We can write these equations in the matrix form

$$\begin{pmatrix} i \partial_\nu \gamma^\nu & - \varkappa \\ - \varkappa & i \partial_\nu \gamma^\nu \end{pmatrix} \begin{pmatrix} \varkappa \Phi \\ \frac{i}{K} \Pi_{\Phi^\dagger} \end{pmatrix} = 0. \tag{32}$$

Performing the unitary transformation

$$S \begin{pmatrix} \varkappa \Phi \\ \frac{i}{K} \Pi_{\Phi^\dagger} \end{pmatrix} = \begin{pmatrix} \Psi_{\mathrm{I}} \\ \Psi_{\mathrm{II}} \end{pmatrix}, \tag{33}$$

where

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S^{-1} = S^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{33'}$$

Eq (32) becomes

$$S \begin{pmatrix} i\,\partial_\nu \gamma^\nu & -\varkappa \\ -\varkappa & i\,\partial_\nu \gamma^\nu \end{pmatrix} S^{-1} \begin{pmatrix} \Psi_{\rm I} \\ \Psi_{\rm II} \end{pmatrix} = 0 \tag{34}$$

and

$$\begin{pmatrix} i\,\partial_\nu \gamma^\nu - \varkappa & 0 \\ 0 & i\,\partial_\nu \gamma^\nu + \varkappa \end{pmatrix} \begin{pmatrix} \Psi_{\rm I} \\ \Psi_{\rm II} \end{pmatrix} = 0 \tag{35}$$

or written in the component form

$$(i\,\partial_\nu \gamma^\nu - \varkappa)\,\Psi_{\rm I} = 0, \tag{36}$$

$$(i\,\partial_\nu \gamma^\nu + \varkappa)\,\Psi_{\rm II} = 0. \tag{37}$$

The same procedure for other two of equations (29) and (30) gives

$$\begin{aligned} -i\,\partial_\nu \overline{\Psi}_{\rm I}\,\gamma^\nu - \varkappa\,\overline{\Psi}_{\rm I} &= 0, \\ -i\,\partial_\nu \overline{\Psi}_{\rm II}\,\gamma^\nu + \varkappa\,\overline{\Psi}_{\rm II} &= 0. \end{aligned} \tag{38}$$

From here we conclude that the canonical system (29) and (30) is equivalent to the Dirac's equations with negative and positive mass term.

Rephrasing this conclusion we may say that the Dirac's equations with negative and positive mass term are equivalent to the canonical system (29) and (30) or to the Lagrange's system (19) for the field $(\Phi, \Phi^{\dagger})$ which is defined by the Lagrangian (16). This we take as the definition of a new Dirac field, but the difference from the standard Dirac field has to be kept on mind.

Explicit connection of the Dirac field and the canonical variables of the field $(\Phi, \Phi^{\dagger})$, according to (33), is

$$\begin{aligned} \Psi_{\rm I} &= \frac{1}{\sqrt{2}}\left(\varkappa\,\Phi + \frac{i}{K}\,\Pi_{\Phi^{\dagger}}\right), & \Phi &= \frac{1}{\varkappa\sqrt{2}}(\Psi_{\rm I} + \Psi_{\rm II}), \\ \Psi_{\rm II} &= \frac{1}{\sqrt{2}}\left(\varkappa\,\Phi - \frac{i}{K}\,\Pi_{\Phi^{\dagger}}\right), & \Pi_{\varphi^{\dagger}} &= \frac{K}{i\sqrt{2}}(\Psi_{\rm I} - \Psi_{\rm II}). \end{aligned} \tag{39}$$

4. *Solutions of the Lagrange's and canonical equations*

We seek solutions of the Lagrange's equations (19) in the form

$$\Phi(\vec{x}, t) = \frac{1}{L^{3/2}} \sum_{\vec{k}} \begin{bmatrix} a^{1}_{\vec{k}}(t) \\ a^{2}_{\vec{k}}(t) \\ a^{3}_{\vec{k}}(t) \\ a^{4}_{\vec{k}}(t) \end{bmatrix} e^{i\vec{k}\vec{x}}. \tag{40}$$

After substitution of (40) into (19) one gets

$$\ddot{a}^{i}_{\vec{k}}(t) + k^2\, a^{i}_{\vec{k}}(t) = 0, \qquad i = 1, 2, 3, 4, \tag{41}$$

where

$$k_0^2 = \varkappa^2 + \vec{k}^2. \tag{42}$$

The general solution of Eq. (41) is

$$a^{i}_{\vec{k}}(t) = a^{i}_{\vec{k}1}\, e^{-ik_0 t} + a^{i}_{\vec{k}2}\, e^{ik_0 t}, \qquad i = 1, 2, 3, 4, \tag{43}$$

where $k_0 = +\sqrt{\varkappa^2 + \vec{k}^2}$.

The general solution of Eq. (19) is then

$$\Phi(\vec{x}, t) = \frac{1}{L^{3/2}} \sum_{\vec{k}} \left\{ \begin{bmatrix} a^{1}_{\vec{k}1} \\ a^{2}_{\vec{k}1} \\ a^{3}_{\vec{k}1} \\ a^{4}_{\vec{k}1} \end{bmatrix} e^{i(\vec{k}\vec{x} - k_0 t)} + \begin{bmatrix} a^{1}_{\vec{k}2} \\ a^{2}_{\vec{k}2} \\ a^{3}_{\vec{k}2} \\ a^{4}_{\vec{k}2} \end{bmatrix} e^{i(\vec{k}\vec{x} + k_0 t)} \right\}. \tag{44}$$

Let us notice that all components Φ^i are independent.

Solutions of the canonical equations (29) and (30) we find by making use of Eqs. (44) and (29):

$$\Pi_{\Phi^\dagger} = \frac{1}{L^{3/2}} \sum_{\vec{k}} \begin{bmatrix} A^{1}_{\vec{k}}(t) \\ A^{2}_{\vec{k}}(t) \\ A^{3}_{\vec{k}}(t) \\ A^{4}_{\vec{k}}(t) \end{bmatrix} e^{i\vec{k}\vec{x}} = K\, \partial_\mu\, \gamma^\mu\, \Phi =$$

$$= \frac{K}{L^{3/2}} \sum_{\vec{k}} \begin{bmatrix} \partial_0 & 0 & i k_z & i k_x + k_y \\ 0 & \partial_0 & i k_x - k_y & - i k_z \\ - i k_z & - i k_x - k_y & - \partial_0 & 0 \\ - i k_x + k_y & i k_z & 0 & - \partial_0 \end{bmatrix} \begin{bmatrix} a^1_{\vec{k}}(t) \\ a^2_{\vec{k}}(t) \\ a^3_{\vec{k}}(t) \\ a^4_{\vec{k}}(t) \end{bmatrix} e^{i \vec{k}\vec{x}}. \tag{45}$$

From here and taking into account (43), we find

$$\begin{bmatrix} A^1_{\vec{k}}(t) \\ A^2_{\vec{k}}(t) \\ A^3_{\vec{k}}(t) \\ A^4_{\vec{k}}(t) \end{bmatrix} = K \left\{ \begin{bmatrix} - i k_0 & 0 & i k_z & i k_x + k_y \\ 0 & - i k_0 & i k_x - k_y & - i k_z \\ - i k_z & - i k_x - k_y & i k_0 & 0 \\ - i k_x + k_y & i k_z & 0 & i k_0 \end{bmatrix} \begin{bmatrix} a^1_{\vec{k}1} \\ a^2_{\vec{k}1} \\ a^3_{\vec{k}1} \\ a^4_{\vec{k}1} \end{bmatrix} e^{-ik_0 t} + \right.$$

$$\left. + \begin{bmatrix} i k_0 & 0 & i k_z & i k_x + k_y \\ 0 & i k_0 & i k_x - k_y & - i k_z \\ - i k_z & - i k_x - k_y & - i k_0 & 0 \\ - i k_x + k_y & i k_z & 0 & - i k_0 \end{bmatrix} \begin{bmatrix} a^1_{\vec{k}2} \\ a^2_{\vec{k}2} \\ a^3_{\vec{k}2} \\ a^4_{\vec{k}2} \end{bmatrix} e^{ik_0 t} \right\}. \tag{46}$$

$\Phi, \Pi_{\Phi^\dagger}$ and their h. c. are required solutions.

In the next section we will need the inverse of the system (43) and (46). Thus' we also give it here:

$$a^1_{\vec{k}1} = \frac{1}{2 k_0} \left[\frac{i}{K} A^1_{\vec{k}}(t) + k_0 a^1_{\vec{k}}(t) + k_z a^3_{\vec{k}}(t) + (k_x - \mathrm{i}\, k_y) a^4_{\vec{k}}(t) \right] e^{ik_0 t},$$

$$a^2_{\vec{k}1} = \frac{1}{2 k_0} \left[\frac{i}{K} A^1_{\vec{k}}(t) + k_0 a^2_{\vec{k}}(t) + (k_x + k_y) a^3_{\vec{k}}(t) - k_z a^4_{\vec{k}}(t) \right] e^{ik_0 t},$$

$$a^3_{\vec{k}1} = \frac{1}{2 k_0} \left[\frac{-i}{K} A^3_{\vec{k}}(t) + k_z a^1_{\vec{k}}(t) + (k_x - i k_y) a^2_{\vec{k}}(t) + k_0 a^3_{\vec{k}}(t) \right] e^{ik_0 t}, \tag{47}$$

$$a^4_{\vec{k}1} = \frac{1}{2 k_0} \left[\frac{-i}{K} A^4_{\vec{k}}(t) + (k_x + \mathrm{i}\, k_y) a^1_{\vec{k}}(t) - k_z a^2_{\vec{k}}(t) + k_0 a^4_{\vec{k}}(t) \right] e^{ik_0 t},$$

$$a^1_{\vec{k}2} = \frac{1}{2 k_0} \left[- \frac{i}{K} A^1_{\vec{k}}(t) + k_0 a^1_{\vec{k}}(t) - k_z a^3_{\vec{k}}(t) - (k_x - i k_y) a^4_{\vec{k}}(t) \right] e^{-ik_0 t},$$

$$a^{2}_{\vec{k}2} = \frac{1}{2\,k_0}\left[-\frac{i}{K}A^{2}_{\vec{k}}(t) + k_0\,a^{2}_{\vec{k}}(t) - (k_x + i\,k_y)\,a^{3}_{\vec{k}}(t) + k_z\,a^{4}_{\vec{k}}(t)\right]e^{-ik_0 t},$$

(48)

$$a^{3}_{\vec{k}2} = \frac{1}{2\,k_0}\left[\frac{i}{K}A^{3}_{\vec{k}}(t) - k_z\,a^{1}_{\vec{k}}(t) - (k_x - i\,k_y)\,a^{2}_{\vec{k}}(t) + k_0\,a^{3}_{\vec{k}}(t)\right]e^{-ik_0 t},$$

$$a^{4}_{\vec{k}2} = \frac{1}{2\,k_0}\left[\frac{i}{K}A^{4}_{\vec{k}2}(t) - (k_x + i\,k_y)\,a^{1}_{\vec{k}}(t) + k_z\,a^{2}_{\vec{k}}(t + k_0)\,a^{4}_{\vec{k}}(t)\right]e^{-ik_0 t}.$$

5. *Canonical quantization of the field*

We assume the canonical quantization in the form[3]

$$\{\ ,\ \}_{Poisson} \rightarrow \frac{1}{i}\,[\ ,\ ]_{Quantum}. \tag{49}$$

The Hamiltonian of the field then becomes the operator

$$H = \int \mathscr{H}\,d^3 x = \int\left\{\frac{1}{K}\Pi_\Phi\,\gamma^0\,\Pi_{\Phi^\dagger} - \Pi_\Phi\,\partial_i\,\alpha^i\,\Phi - \partial_i\,\Phi^\dagger\,\alpha^i\,\Pi_\Phi + \right.$$
$$\left. + K\,(\varkappa^2\Phi^\dagger\,\gamma^0\,\Phi)\right\}d^3 x \tag{50}$$

and the dynamical equation (24) goes over into

$$\frac{d F}{d t} = \frac{\partial F}{\partial t} + \frac{1}{i}\,[F, H]. \tag{51}$$

The canonical conditions are

$$[\Phi_a(\vec{x}, t), \Phi_b(\vec{y}, t)] = [\Phi_a(\vec{x}, t), \Phi^\dagger_b(\vec{y}, t)] = [\Phi^\dagger_a(\vec{x}, t), \Phi^\dagger_b(\vec{y}, t)] = 0,$$

$$[\Pi_{\Phi_a}(\vec{x}, t), \Pi_{\Phi_b}(\vec{y}, t)] = [\Pi_{\Phi^\dagger_a}(\vec{x}, t), \Pi_{\Phi^\dagger_b}(\vec{y}, t)] = [\Pi_{\Phi^\dagger_a}(\vec{x}, t), \Pi_{\Phi^\dagger_b}(\vec{y}, t)] = 0,$$

$$[\Phi_a(\vec{x}, t), \Pi_{\Phi^\dagger_b}(\vec{y}, t)] = [\Phi^\dagger_a(\vec{x}, t), \Pi_\Phi(\vec{y}, t)] = 0, \tag{52}$$

$$[\Phi_a(\vec{x}, t), \Pi_{\Phi_b}(\vec{y}, t)] = i\,\delta_{ab}\,\delta(\vec{x} - \vec{y}),$$

$$[\Phi^\dagger_a(\vec{x}, t), \Pi_{\Phi^\dagger_b}(\vec{y}, t)] = i\,\delta_{ab}\,\delta(\vec{x} - \vec{y}), \qquad a, b = 1, 2, 3, 4.$$

Let us mention that Eqs. (52) must be the commutation relations, not anticommutation relations. The commutation relations only give correct canonical equations in accordance with (51) and (49).

From the Hamiltonian (50), the dynamical equation (51) and the commutation rules (52) follow the equations of motion for the operators Φ, $\Phi^\dagger$, Π_Φ, $\Pi_{\Phi^\dagger}$:

$$\begin{aligned}
\dot{\Phi} &= -i\,[\Phi, H] = \frac{1}{K}\gamma^0\,\Pi_{\Phi^\dagger} - \partial_l\, a^l\,\Phi,\\
\dot{\Phi}^\dagger &= -i\,[\Phi^\dagger, H] = \frac{1}{K}\Pi_\Phi\,\gamma^0 - \partial_l\,\Phi^\dagger\,a^l,\\
\dot{\Pi}_\Phi &= -i\,[\Pi_\Phi, H] = -\,\partial_l\,\Pi_\Phi\,a^l - K\,\varkappa^2\,\Phi^\dagger\,\gamma^0,\\
\dot{\Pi}_{\Phi^\dagger} &= -i\,[\Pi_{\Phi^\dagger}, H] = -\,\partial_l\,a^l\,\Pi_{\Phi^\dagger} - K\,\varkappa^2\,\gamma^0\,\Phi.
\end{aligned} \tag{53}$$

These equations are in agreement with Eqs. (25—28).

We perform further analysis in the $\vec{k}$-space. By making use of (40) and (45) the quantum rules (52) in the $\vec{k}$-space are

$$\begin{aligned}
&\left[a^i_{\vec{k}}(t), a^j_{\vec{k}'}(t)\right] = \left[a^i_{\vec{k}}(t), a^{j\dagger}_{\vec{k}'}(t)\right] = \left[a^{i\dagger}_{\vec{k}}(t), a^{j\dagger}_{\vec{k}'}(t)\right] = 0,\\
&\left[A^i_{\vec{k}}(t), A^j_{\vec{k}'}(t)\right] = \left[A^i_{\vec{k}}(t), A^{j\dagger}_{\vec{k}'}(t)\right] = \left[A^{i\dagger}_{\vec{k}}(t), A^{j\dagger}_{\vec{k}'}(t)\right] = 0,\\
&\left[a^i_{\vec{k}}(t), A^j_{\vec{k}'}(t)\right] = \left[a^{i\dagger}_{\vec{k}}(t), A^{j\dagger}_{\vec{k}'}(t)\right] = 0,\\
&\left[a^i_{\vec{k}}(t), A^{j\dagger}_{\vec{k}'}(t)\right] = \frac{i}{K}\,\delta_{ij}\,\delta_{\vec{k}\vec{k}'},\\
&\left[a^{i\dagger}_{\vec{k}}(t), A^i_{\vec{k}'}(t)\right] = \frac{i}{K}\,\delta_{ij}\,\delta_{\vec{k}\vec{k}'}, \qquad i, j = 1, 2, 3, 4.
\end{aligned} \tag{54}$$

Having these commutation rules we now evaluate the commutation rules for the time independent coefficients $(a^i_{\vec{k}1}, a^j_{\vec{k}2})$:

$$\left[a^i_{\vec{k}1}, a^{j\dagger}_{\vec{k}'1}\right] = \frac{1}{2K k_0}\,\delta^{ij}\,\delta_{\vec{k}\vec{k}'},$$

$$\left[a^n_{\vec{k}1}, a^{m\dagger}_{\vec{k}'1}\right] = -\,\frac{1}{2K k_0}\,\delta^{nm}\,\delta_{\vec{k}\vec{k}'},$$

$$\left[a^{i}_{\vec{k}2}, a^{j\dagger}_{\vec{k}'2}\right] = -\frac{1}{2K k_0}\delta^{ij}\delta_{\vec{k}\vec{k}'},$$

$$\left[a^{n}_{\vec{k}2}, a^{m\dagger}_{\vec{k}'2}\right] = \frac{1}{2K k_0}\delta^{nm}\delta_{\vec{k}\vec{k}'}, \quad i, j = 1, 2, \quad n, m = 3, 4, \tag{55}$$

all other commutators are zero.

Introducing the new coefficients according to

$$a^{i}_{\vec{k}1} = \sqrt{\frac{1}{2K k_0}}\, A^{i}_{\vec{k}1},$$

$$a^{i}_{\vec{k}2} = \sqrt{\frac{1}{2K k_0}}\, A^{i\dagger}_{\vec{k}2},$$

$$a^{i+2}_{\vec{k}1} = \sqrt{\frac{1}{2K k_0}}\, B^{i\dagger}_{\vec{k}1},$$

$$a^{i+2}_{\vec{k}2} = \sqrt{\frac{1}{2K k_0}}\, B^{i}_{\vec{k}2}, \qquad i = 1, 2, \tag{56}$$

the commutation relations (55) become

$$\left[A^{i}_{\vec{k}n}, A^{j\dagger}_{\vec{k}'m}\right] = \delta_{nm}\,\delta^{ij}\,\delta_{\vec{k}\vec{k}'},$$

$$\left[B^{i}_{\vec{k}n}, B^{j\dagger}_{\vec{k}'m}\right] = \delta_{nm}\,\delta^{ij}\,\delta_{\vec{k}\vec{k}'}, \qquad i, j, n, m = 1, 2, \tag{57}$$

and all other commutators are zero.

The field $\Phi(\vec{x}, t)$ written in the new coefficients is

$$\Phi(\vec{x}, t) = \frac{1}{L^{3/2}}\sum_{\vec{k}}\sqrt{\frac{1}{2Kk_0}}\left\{\begin{bmatrix} A^{1}_{\vec{k}1} \\ A^{2}_{\vec{k}1} \\ B^{1\dagger}_{\vec{k}1} \\ B^{2\dagger}_{\vec{k}1} \end{bmatrix} e^{i(\vec{k}\vec{x} - k_0 t)} + \begin{bmatrix} A^{1\dagger}_{\vec{k}2} \\ A^{2\dagger}_{\vec{k}2} \\ B^{1}_{\vec{k}2} \\ B^{2}_{\vec{k}2} \end{bmatrix} e^{i(\vec{k}\vec{x} + k_0 t)}\right\}. \tag{58}$$

6. Constants of motion

The scalar constant of motion, which follows from

$$j^\mu = i\left(\Phi^\dagger \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^\dagger)} - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \Phi\right), \quad \partial_\mu j^\mu = 0, \tag{59}$$

is given by

$$Q = \text{const} \int j^0 \, \mathrm{d}^3 x = \text{const}\, \mathrm{i} \int (\Phi^\dagger \Pi_{\Phi^\dagger} - \Pi_\Phi \Phi) \, \mathrm{d}^3 x. \tag{60}$$

Using the expressions (40—46) and (56) it becomes

$$Q = \text{const} \sum_{\vec{k}} \{(A^{i\dagger}_{\vec{k}1} A^{i}_{\vec{k}1} - B^{i}_{\vec{k}1} B^{i\dagger}_{\vec{k}1}) + (- A^{i}_{\vec{k}2} A^{i\dagger}_{\vec{k}2} + B^{i\dagger}_{\vec{k}1} B^{i}_{\vec{k}2})\} \tag{61}$$

or written in the particle number operators form

$$Q = \text{const} \sum_{\substack{\vec{k} \\ i=1,2}} \{(A^{i\dagger}_{\vec{k}1} A^{i}_{\vec{k}1} - B^{i\dagger}_{\vec{k}1} B^{i}_{\vec{k}1}) - (A^{i\dagger}_{\vec{k}2} A^{i}_{\vec{k}2} - B^{i\dagger}_{\vec{k}2} B^{i}_{\vec{k}2}) - 4\}. \tag{62}$$

The energy-momentum constant of motion follows from

$$T_\alpha{}^\beta = \partial_\alpha \Phi^\dagger \frac{\partial \mathcal{L}}{\partial (\partial_\beta \Phi^\dagger)} + \frac{\partial \mathcal{L}}{\partial (\partial_\beta \Phi)} \partial_\alpha \Phi - \delta_\alpha{}^\beta \mathcal{L}, \quad \partial_\beta T_\alpha{}^\beta = 0, \tag{63}$$

and for the Lagrangian (16) reads

$$P_0 = \text{const} \int \{\dot{\Phi}^\dagger \Pi_{\Phi^\dagger} + \Pi_\Phi \dot{\Phi} - (\Pi_\Phi \gamma^0 \Pi_{\Phi^\dagger} - \Phi^\dagger \gamma^0 \Phi)\} \, \mathrm{d}^3 x, \tag{64}$$

$$P_j = \text{const} \int (\partial_j \Phi^\dagger \Pi_{\Phi^\dagger} + \Pi_\Phi \partial_j \Phi) \, \mathrm{d}^3 x. \tag{65}$$

Using the expressions (40—46) and (56) it becomes

$$P_0 = \text{const} \sum_{\substack{\vec{k} \\ i=1,2}} k_0 \{(A^{i\dagger}_{\vec{k}1} A^{i}_{\vec{k}1} + A^{i\dagger}_{\vec{k}2} A^{i}_{\vec{k}2}) - (B^{i\dagger}_{\vec{k}1} B^{i}_{\vec{k}1} + B^{i\dagger}_{\vec{k}2} B^{i}_{\vec{k}2})\}, \tag{66}$$

$$P_j = \text{const} \sum_{\substack{\vec{k} \\ i=1,2}} k_j \{(A^{i\dagger}_{\vec{k}1} A^{i}_{\vec{k}1} - A^{i\dagger}_{\vec{k}2} A^{i}_{\vec{k}2}) - (B^{i\dagger}_{\vec{k}1} B^{i}_{\vec{k}1} - B^{i\dagger}_{\vec{k}2} B^{i}_{\vec{k}2})\}. \tag{67}$$

The angular momentum density tensor

$$M^{\alpha,\beta\gamma} = (x^\gamma T^{\beta\lambda} - x^\beta T^{\gamma\alpha}) - \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \Phi)} \left(\frac{i}{2}\sigma^{\beta\gamma}\right) \Phi - \Phi^\dagger \left(-\frac{i}{2}\sigma^{\beta\gamma}\right) \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \Phi^\dagger)} \quad (68)$$

for the Lagrangian (16) is

$$M^{\alpha,\beta\gamma} = (x^\gamma T^{\beta\alpha} - x^\beta T^{\gamma\alpha}) - K(\partial_\nu \overline{\Phi} \gamma^\nu) \gamma^\alpha \left(\frac{i}{2}\sigma^{\beta\gamma}\right) \Phi - \\ - K \Phi^\dagger \left(-\frac{i}{2}\sigma^{\beta\gamma}\right) (\gamma^0 \gamma^\alpha)(\partial_\mu \gamma^\mu \Phi). \quad (69)$$

From $\partial_\alpha M^{\alpha,\beta\gamma} = 0$ follows the constant of motion

$$M^{\beta\gamma} = \text{const} \{ \int (x^\gamma T^{\beta 0} - x^\beta T^{\gamma 0}) d^3x + \\ + \frac{K}{2} \int [(- i \partial_\nu \overline{\Phi} \gamma^\nu) \gamma^0 \sigma^{\beta\gamma} \Phi + \Phi^\dagger \sigma^{\beta\gamma} (i \partial_\mu \gamma^\mu \Phi)] \, \mathrm{d}^3x \}. \quad (70)$$

The spin pseudovector

$$S_r = \frac{1}{2} \varepsilon_{ijr} S^{ij},$$

where

$$S^{ij} = \text{const} \frac{K}{2} \int \{(- i \partial_\nu \overline{\Phi} \gamma^\nu) \gamma^0 \sigma^{ij} \Phi + \Phi^\dagger \sigma^{ij} (i \partial_\nu \gamma^\nu \Phi)\} \, \mathrm{d}^3x, \quad (71)$$

and using

$$\sigma^{ij} = \frac{i}{2}(\gamma^i \gamma^j - \gamma^j \gamma^i) = \varepsilon^{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \equiv \varepsilon^{ijk} \tilde{\sigma}_k,$$

is then

$$S_r = \text{const} \frac{K}{2} \int \left\{ \Phi^\dagger \begin{pmatrix} \sigma_r & 0 \\ 0 & \sigma_r \end{pmatrix} (i \partial_\nu \gamma^\nu \Phi) + (- i \partial_\nu \overline{\Phi} \gamma^\nu) \gamma^0 \begin{pmatrix} \sigma_r & 0 \\ 0 & \sigma_r \end{pmatrix} \Phi \right\} \mathrm{d}^3x. \quad (72)$$

In the $\vec{k}$-space, according to (40—46), it is

$$S_r = \text{const} \sum_{\vec{k}} \frac{K}{2} \{ (\alpha_1^\dagger \beta_1^\dagger) \begin{pmatrix} 2\sigma_r k_0 & -[\sigma_r, \vec{k}\vec{\sigma}] \\ [\sigma_r, \vec{k}\vec{\sigma}] & -2\sigma_r k_0 \end{pmatrix} \begin{pmatrix} \alpha^0 \\ \beta_1 \end{pmatrix} + \\ + (\alpha_2^\dagger \beta_2^\dagger) \begin{pmatrix} -2\sigma_r k_0 & -[\sigma_r, \vec{k}\vec{\sigma}] \\ [\sigma_r, \vec{k}\vec{\sigma}] & 2\sigma_r k_0 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} + \quad (73)$$

$$+ (\alpha_2^\dagger\ \beta_2^\dagger) \begin{pmatrix} 0 & -[\sigma_r, \vec{k}\vec{\sigma}] \\ [\sigma_r, \vec{k}\vec{\sigma}] & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} e^{-2ik_0 t} +$$

$$+ (\alpha_1^\dagger\ \beta_1^\dagger) \begin{pmatrix} 0 & -[\sigma_r, \vec{k}\vec{\sigma}] \\ [\sigma_r, \vec{k}\vec{\sigma}] & 0 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} e^{2ik_0 t}\},$$

where

$$[\sigma_r, \vec{k}\vec{\sigma}] = \sigma_r (\vec{k}\vec{\sigma}) - (\vec{k}\vec{\sigma})\sigma_r, \tag{74}$$

$$\alpha_1 = \begin{pmatrix} a^1_{\vec{k}1} \\ a^2_{\vec{k}1} \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} a^1_{\vec{k}2} \\ a^2_{\vec{k}2} \end{pmatrix},$$

$$\beta_1 = \begin{pmatrix} a^3_{\vec{k}1} \\ a^4_{\vec{k}1} \end{pmatrix}, \qquad \beta_2 = \begin{pmatrix} a^3_{\vec{k}2} \\ a^4_{\vec{k}2} \end{pmatrix}. \tag{75}$$

For given $\vec{k}$ the projection $\frac{\vec{k}\vec{S}}{k}$ is

$$\frac{\vec{k}\vec{S}}{k} = \text{const}\,\frac{\varkappa \vec{K}}{2} \{\alpha_1^\dagger \sigma_z \alpha_1 - \beta_1^\dagger \sigma_z \beta_1 - \alpha_2^\dagger\ \alpha_2 + \beta_2^\dagger \sigma_z \beta_2)\}. \tag{76}$$

For the polar axis in the $\vec{k}$ direction it becomes

$$\frac{\vec{k}\vec{S}}{k} = \text{const}\,\frac{1}{2}\ \{[(A^{1\dagger}_{\vec{k}1} A^1_{\vec{k}1} - A^{2\dagger}_{\vec{k}1} A^2_{\vec{k}1}) - (A^1_{\vec{k}2} A^{1\dagger}_{\vec{k}2} - A^2_{\vec{k}2} A^{2\dagger}_{\vec{k}2}) -$$
$$- (B^1_{\vec{k}1} B^{1\dagger}_{\vec{k}1} - B^2_{\vec{k}1} B^{2\dagger}_{\vec{k}1}) - (B^{1\dagger}_{\vec{k}2} B^1_{\vec{k}2} - B^{2\dagger}_{\vec{k}2} B^2_{\vec{k}2})]\}, \tag{77}$$

where we have used also Eq. (56).

7. Statistical interpretation of the field

What does the new Dirac field physically mean? We investigate this question in this Section.

The constant of motion operators (62), (66), (67) and (77) contain only the particle number operators. From here it follows that these physical quantities can be interpreted in terms of particles. Let us start with this interpretation.

We select the undefined constants in Eqs. (62), (66), (67) and (77) then to be

$$\text{const}_Q = q, \qquad \text{const}_P = 1, \qquad \text{const}_M = 1, \tag{78}$$

where indices denote to which quantity a constant refers.

According to (66) the total energy may have arbitrary positive as well as negative values. This is not satisfactory result. However, this difficulty can be easily eliminated.

The solution (44) we may write in the form

$$\Phi = \begin{pmatrix} \Phi^1 \\ \Phi^2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Phi^3 \\ \Phi^4 \end{pmatrix} \equiv \Phi_n + \Phi_d, \tag{79}$$

$$\Phi_n = \frac{1}{L^{3/2}} \sum_{\vec{k}} \left\{ \begin{bmatrix} a^1_{\vec{k}1} \\ a^2_{\vec{k}1} \\ 0 \\ 0 \end{bmatrix} e^{i(\vec{k}\vec{x} - k_0 t)} + \begin{bmatrix} a^1_{\vec{k}2} \\ a^2_{\vec{k}2} \\ 0 \\ 0 \end{bmatrix} e^{i(\vec{k}\vec{x} + k_0 t)} \right\}, \tag{80}$$

$$\Phi_d = \frac{1}{L^{3/2}} \sum_{\vec{k}} \left\{ \begin{bmatrix} 0 \\ 0 \\ a^3_{\vec{k}1} \\ a^4_{\vec{k}1} \end{bmatrix} e^{i(\vec{k}\vec{x} - k_0 t)} + \begin{bmatrix} 0 \\ 0 \\ a^3_{\vec{k}2} \\ a^4_{\vec{k}2} \end{bmatrix} e^{i(\vec{k}\vec{x} + k_0 t)} \right\}. \tag{81}$$

The momentum $\Pi_{\Phi^\dagger}$ is then

$$\Pi_{\Phi^\dagger} = (\Pi_{\Phi^\dagger})_n + (\Pi_{\Phi^\dagger})_d, \tag{82}$$

$$(\Pi_{\Phi^\dagger})_n = \frac{iK}{L^{3/2}} \sum_{\vec{k}} \left\{ \begin{bmatrix} -k_0 a^1_{\vec{k}1} \\ -k_0 a^2_{\vec{k}1} \\ -k_z a^1_{\vec{k}1} - (k_x - i k_y) a^2_{\vec{k}1} \\ -(k_x + i k_y) a^1_{\vec{k}1} + k_z a^2_{\vec{k}1} \end{bmatrix} e^{i(\vec{k}\vec{x} - k_0 t)} + \right.$$
$$\left. + \begin{bmatrix} k_0 a^1_{\vec{k}2} \\ k_0 a^2_{\vec{k}2} \\ -k_z a^1_{\vec{k}2} - (k_x - i k_y) a^2_{\vec{k}2} \\ -(k_x + i k_y) a^1_{\vec{k}2} + k_z a^2_{\vec{k}2} \end{bmatrix} e^{i(\vec{k}\vec{x} + k_0 t)} \right\}, \tag{83}$$

$$(\Pi_{\Phi}\dagger)_d = \frac{i\,K}{L^{3/2}} \sum_{\vec{k}} \left\{ \begin{bmatrix} k_z \overset{3}{a}_{\vec{k}1} + (k_x - i\,k_y)\overset{4}{a}_{\vec{k}1} \\ (k_x + i\,k_y)\,\overset{3}{a}_{\vec{k}1} - k_z\,\overset{4}{a}_{\vec{k}1} \\ k_0\,\overset{3}{a}_{\vec{k}1} \\ k_0\,\overset{4}{a}_{\vec{k}1} \end{bmatrix} e^{i\,(\vec{k}\vec{x} - k_0 t)} + \right.$$
$$\left. + \begin{bmatrix} k_z\,\overset{3}{a}_{\vec{k}2} + (k_x - i\,k_y)\,\overset{4}{a}_{\vec{k}2} \\ (k_x + i\,k_y)\,\overset{3}{a}_{\vec{k}2} - k_z\,\overset{4}{a}_{\vec{k}2} \\ -k_0\,\overset{3}{a}_{\vec{k}2} \\ -k_0\,\overset{4}{a}_{\vec{k}2} \end{bmatrix} e^{i\,(\vec{k}\vec{x} + k_0 t)} \right\} \tag{84}$$

and

$$\Psi = \begin{pmatrix} \Psi_{\text{I}} \\ \Psi_{\text{II}} \end{pmatrix} = \begin{pmatrix} \Psi_{\text{I}} \\ \Psi_{\text{II}} \end{pmatrix}_n + \begin{pmatrix} \Psi_{\text{I}} \\ \Psi_{\text{II}} \end{pmatrix}_d, \tag{85}$$

$$\begin{pmatrix} \Psi_{\text{I}} \\ \Psi_{\text{II}} \end{pmatrix}_n = \frac{1}{\sqrt{2}} \frac{1}{L^{3/2}} \sum_{\vec{k}} \left\{ \begin{bmatrix} (\varkappa + k_0)\,\overset{1}{a}_{\vec{k}1} \\ (\varkappa + k_0)\,\overset{2}{a}_{\vec{k}1} \\ k_z\,\overset{1}{a}_{\vec{k}1} + (k_x - i\,k_y)\,\overset{2}{a}_{\vec{k}1} \\ (k_x + i\,k_y)\,\overset{1}{a}_{\vec{k}1} - k_z\,\overset{2}{a}_{\vec{k}1} \\ (\varkappa - k_0)\,\overset{1}{a}_{\vec{k}1} \\ (\varkappa - k_0)\,\overset{2}{a}_{\vec{k}1} \\ -k_z\,\overset{1}{a}_{\vec{k}1} - (k_x - i\,k_y)\,\overset{2}{a}_{\vec{k}1} \\ -(k_x + i\,k_y)\,\overset{1}{a}_{\vec{k}1} + k_z\,\overset{2}{a}_{\vec{k}1} \end{bmatrix} e^{i\,(\vec{k}\vec{x} - k_0 t)} + \right.$$
$$\left. + \begin{bmatrix} (\varkappa - k_0)\,\overset{1}{a}_{\vec{k}2} \\ (\varkappa - k_0)\,\overset{2}{a}_{\vec{k}2} \\ k_z\,\overset{1}{a}_{\vec{k}2} + (k_x - i\,k_y)\,\overset{2}{a}_{\vec{k}2} \\ (k_x + i\,k_y)\,\overset{1}{a}_{\vec{k}2} - k_z\,\overset{2}{a}_{\vec{k}2} \\ (\varkappa + k_0)\,\overset{1}{a}_{\vec{k}2} \\ (\varkappa + k_0)\,\overset{2}{a}_{\vec{k}2} \\ -k_z\,\overset{1}{a}_{\vec{k}2} - (k_x - i\,k_y)\,\overset{2}{a}_{\vec{k}2} \\ -(k_x + i\,k_y)\,\overset{1}{a}_{\vec{k}2} + k_z\,\overset{2}{a}_{\vec{k}2} \end{bmatrix} e^{i\,(\vec{k}\vec{x} + k_0 t)} \right\}, \tag{86}$$

$$\begin{pmatrix}\Psi_{\mathrm{I}}\\ \Psi_{\mathrm{II}}\end{pmatrix}_d = \frac{1}{\sqrt{2}}\frac{1}{L^{3/2}}\sum_{\vec{k}}\left\{\begin{bmatrix} -k_z a^3_{\vec{k}1} - (k_x - i k_y) a^4_{\vec{k}1} \\ -(k_x + i k_y) a^3_{\vec{k}1} + k_z a^4_{\vec{k}1} \\ (-k_0 + \varkappa) a^3_{\vec{k}1} \\ (-k_0 + \varkappa) a^4_{\vec{k}1} \\ k_z a^3_{\vec{k}1} + (k_x - i k_y) a^4_{\vec{k}1} \\ (k_x + i k_y) a^3_{\vec{k}1} - k_z a^4_{\vec{k}1} \\ (k_0 + \varkappa) a^3_{\vec{k}1} \\ (k_0 + \varkappa) a^4_{\vec{k}1} \end{bmatrix} e_i^{(\vec{k}\vec{x} - k_0 t)} + \right.$$

$$\left. + \begin{bmatrix} -k_z a^3_{\vec{k}2} - (k_x - i k_y) a^4_{\vec{k}2} \\ -(k_x + i k_y) a^3_{\vec{k}2} + k_z a^4_{\vec{k}2} \\ (k_0 + \varkappa) a^3_{\vec{k}2} \\ (k_0 + \varkappa) a^4_{\vec{k}2} \\ k_z a^3_{\vec{k}2} + (k_x - i k_y) a^4_{\vec{k}2} \\ (k_x + i k_y) a^3_{\vec{k}2} - k_z a^4_{\vec{k}2} \\ (\varkappa - k_0) a^3_{\vec{k}2} \\ (\varkappa - k_0) a^4_{\vec{k}2} \end{bmatrix} e^{i\,(\vec{k}\vec{x} + k_0 t)}\right\}. \tag{87}$$

Comparing Eqs. (86) and (87) we find

$$\begin{pmatrix}\Psi_{\mathrm{I}}\\ \Psi_{\mathrm{II}}\end{pmatrix}_d = \begin{pmatrix} 0 & \gamma^5 \\ \gamma^5 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\frac{1}{L^{3/2}}\sum_{\vec{k}}\left\{\begin{bmatrix} (\varkappa + k_0) a^3_{\vec{k}1} \\ (\varkappa + k_0) a^4_{\vec{k}1} \\ k_z a^3_{\vec{k}1} + (k_x - i k_y) a^4_{\vec{k}1} \\ (k_x + i k_y) a^3_{\vec{k}1} - k_z a^4_{\vec{k}1} \\ (\varkappa - k_0) a^3_{\vec{k}1} \\ (\varkappa - k_0) a^4_{\vec{k}1} \\ -k_z a^3_{\vec{k}1} - (k_x - i k_y) a^4_{\vec{k}1} \\ -(k_x + i k_y) a^3_{\vec{k}1} + k_z a^4_{\vec{k}1} \end{bmatrix} e^{i\,(\vec{k}\vec{x} - k_0 t)} + \right.$$

$$+\begin{bmatrix} (\varkappa - k_0)\, \overset{3}{a}_{\vec{k}2} \\ (\varkappa - k_0)\, \overset{4}{a}_{\vec{k}2} \\ k_z\, \overset{3}{a}_{\vec{k}2} + (k_x - i\, k_y)\, \overset{4}{a}_{\vec{k}2} \\ (k_x + i\, k_y)\, \overset{3}{a}_{\vec{k}2} - k_z\, \overset{4}{a}_{\vec{k}2} \\ (\varkappa + k_0)\, \overset{3}{a}_{\vec{k}2} \\ (\varkappa + k_0)\, \overset{4}{a}_{\vec{k}2} \\ - k_z\, \overset{3}{a}_{\vec{k}2} - (k_x - i\, k_y)\, \overset{4}{a}_{\vec{k}2} \\ - (k_x + i\, k_y)\, \overset{3}{a}_{\vec{k}2} + k_z\, \overset{4}{a}_{\vec{k}2} \end{bmatrix} e^{i\,(\vec{k}\vec{x} + k_0 t)} \Bigg\}. \tag{88}$$

One gets the solution »d« by the »γ« transformation of the »n« type solution. In the electromagnetic field theory the dual solution is associated with the magnetic monopoles[11] and due to this reason it can be eliminated from the final results. We take the same position here and eliminate also the solution »d«. This restriction has the meaning:

$$\overset{i\dagger}{B}_{\vec{k}j} \overset{i}{B}_{\vec{k}j} \,|\rangle = 0, \qquad i, j = 1, 2, \tag{89}$$

or in the classical case $\overset{i\dagger}{B}_{\vec{k}j} \overset{i}{B}_{\vec{k}j} = 0$.

After elimination of the »dual solution«, the constants of motion become

$$\begin{aligned} Q &= q \sum_{\substack{\vec{k} \\ i=1,2}} (\overset{i\dagger}{A}_{\vec{k}1} \overset{i}{A}_{\vec{k}1} - \overset{i}{A}_{\vec{k}2} \overset{i\dagger}{A}_{\vec{k}2}), \\ P_0 &= \sum_{\substack{\vec{k} \\ i=1,2}} k_0\, (\overset{i\dagger}{A}_{\vec{k}1} \overset{i}{A}_{\vec{k}1} + \overset{i\dagger}{A}_{\vec{k}2} \overset{i}{A}_{\vec{k}2}), \\ P_j &= \sum_{\substack{\vec{k} \\ i=1,2}} k_j\, (\overset{i\dagger}{A}_{\vec{k}1} \overset{i}{A}_{\vec{k}1} - \overset{i\dagger}{A}_{\vec{k}2} \overset{i}{A}_{\vec{k}2}), \\ \frac{\vec{k}\,\vec{S}}{k} &= \frac{1}{2} \{ (\overset{1\dagger}{A}_{\vec{k}1} \overset{1}{A}_{\vec{k}1} - \overset{2\dagger}{A}_{\vec{k}1} \overset{2}{A}_{\vec{k}1}) - (\overset{1}{A}_{\vec{k}2} \overset{1\dagger}{A}_{\vec{k}2} - \overset{2}{A}_{\vec{k}2} \overset{2\dagger}{A}_{\vec{k}2}) \}. \end{aligned} \tag{90}$$

The problem of the negative energies has disappeared and Eqs. (90) have no more difficulties in this sense. Let us notice that the infinity energy term of the standard theory does not appear. The infinity term does only appear in the scalar constant Q and we will pay attention to it later on.

The eigenvalues of the particle number operators are 0, 1, 2, ... The spin of the particles is 1/2, according to the fourth of Eqs. (90). Thus, we have here a second problem in the theory. This problem is much more serious then the previous one. It comes from the commutation rules (52) and these rules, as we have said, cannot be substituted by the anticommutation rules because the anticommutation rules are not it agreement with the canonical equations. One may try to make a selcction of the states, similarly as one does in the nonrelativistic systems of identical particles. However, one cannot see the basis for a such selection even if it would be possible.

There is another approach to the Dirac field equation in which the field describes one particle. In this interpretation the field is not quantized. However, let us see whether this possibility is present in Eqs. (90) (where constants are not necessary as in (78)). Let it be a particle in a given spin state, for example $\overset{1\dagger}{A}_{\vec{k}1}\overset{1}{A}_{\vec{k}1} \neq 0$, $\overset{2\dagger}{A}_{\vec{k}1}\overset{2}{A}_{\vec{k}1} = \overset{1\dagger}{A}_{\vec{k}2}\overset{1}{A}_{\vec{k}2} = \overset{2\dagger}{A}_{\vec{k}2}\overset{2}{A}_{\vec{k}2} = 0$. Writing the energy operator in the nonrelativistic limit, $|\vec{k}| \ll \varkappa$,

$$P_0 \doteq \text{const}\; \varkappa \sum_{\vec{k}}{}' \overset{1\dagger}{A}_{\vec{k}1}\overset{1}{A}_{\vec{k}1} + \text{const} \sum_{\vec{k}}{}' \varepsilon_k \overset{1\dagger}{A}_{\vec{k}1}\overset{1}{A}_{\vec{k}1}, \tag{91}$$

where the apostrophe denotes restriction on $|\vec{k}| \ll \varkappa$ and $\varepsilon_{\vec{k}} = k^2/2\,\varkappa$, the first term must be the rest energy of the nonrelativistic particle. From here we have

$$\text{const} \sum_{\vec{k}}{}' \overset{1\dagger}{A}_{\vec{k}1}\overset{1}{A}_{\vec{k}1} = 1. \tag{92}$$

This operator equation determines the states for which the requirement of the rest energy is fulfilled:

$$\begin{gathered}\text{const} \sum_{\vec{k}}{}' \overset{1\dagger}{A}_{\vec{k}1}\overset{1}{A}_{\vec{k}1} \,|\, a \rangle = 1 \,|\, a \rangle, \\ |\, a \rangle = |\, \overset{1}{n}_{\vec{k}_1 1}, \overset{1}{n}_{\vec{k}_2 1}, \dots \rangle \\ \text{const} = \frac{1}{\sum_{\vec{k}}{}' \overset{1}{n}_{\vec{k}1}}.\end{gathered} \tag{93}$$

The last equation implies that the states $|a\rangle$ are restricted to a total sum of the integers $\overset{1}{n}_{\vec{k}1}$ $(\sum_{\vec{k}}{}' \overset{1}{n}_{\vec{k}1} = N)$.

The energy eigenvalues for these states are

$$E_0 \doteq \varkappa + \sum_{\vec{k}}{}' \varepsilon_{\vec{k}} \frac{\overset{1}{n}_{\vec{k}1}}{\sum_{\vec{k}} \overset{1}{n}_{\vec{k}1}} \equiv \varkappa + \sum_{\vec{k}}{}' \varepsilon_{\vec{k}}\, w_{\vec{k}}, \tag{94}$$

where

$$w_{\vec{k}} = \frac{\overset{1}{n}_{\vec{k}1}}{\sum_{\vec{k}} \overset{1}{n}_{\vec{k}1}}. \tag{95}$$

This is (almost) just what we have in the nonrelativistic quantum mechanics of one free particle: the rest energy + the average of the free particle energy over a given probability distribution in $\vec{k}$-space. There is, however, one important difference: $w_{\vec{k}}$ is not continuous function for finite N. In order to get the continuous function there must be: $N \to \infty$, $\overset{1}{n}_{\vec{k}1} \to \infty$. But in this case the quantum rules lose power.

Disregarding the quantum rules we may take, in accordance to the first of Eqs. (90),

$$\sum_{\vec{k}} \overset{1*}{A}_{\vec{k}1} \overset{1}{A}_{\vec{k}1} = 1 \tag{96}$$

and make the interpretation of $\overset{1*}{A}_{\vec{k}1} \overset{1}{A}_{\vec{k}1}$ as the occupation probability of the momentum $\vec{k}$. The first three of Eqs. (90) give then the charge, the average energy and the average momentum of the particle and the last one, taken in the form

$$\sum_{\vec{k}} \frac{\vec{k}\vec{S}}{k} = \frac{1}{2}, \tag{97}$$

the spin of the particle in the nonrelativistic limit.

Thus, we come to the conclusion that Eqs. (90) can describe one particle bu without the quantization. The field $(\Phi, \Phi^{\dagger})$ has then the meaning of the probability amplitude for this particle.

The choice $\overset{2}{A}_{\vec{k}1} \neq 0$, $\overset{1}{A}_{\vec{k}1} = \overset{1}{A}_{\vec{k}2} = \overset{2}{A}_{\vec{k}2} = 0$ gives the spin state $-1/2$ of the particle. Other two cases, $\overset{1}{A}_{\vec{k}1} = \overset{2}{A}_{\vec{k}1} = 0$, $\overset{1}{A}_{\vec{k}2} \neq 0$ and $\overset{2}{A}_{\vec{k}2} \neq 0$, describe a particle with the charge $-q$. Therefore, the field $(\Phi, \Phi^{\dagger})$ in whole describes generally the statistical behaviour of a particle with the charge q and the spin $1/2$ and its antiparticle.

This interpretation of the Dirac field has been rejected in the standard theory because of the non positive definite energy. We see that in the correct canonical description of the Dirac field this problem does not exist after elimination of the dual solution. So, this interpretation is acceptable. In the next section we consider this interpretation in a more detail.

8. The field without quantization

Accepting the statistical interpretation from the previous Section we now investigate the field (Φ, Φ^+) without quantization.

(a) *Nonrelativistic limit*

We consider first the nonrelativistic limit. Due to the existence of the constant $\varkappa$ such a limit does exist.

Eqs. (44) and (45—46) give the solution of the canonical equations (25—28). Excluding the dual solution, i. e. taking $\Phi_3 = \Phi_4 = 0$, we have

$$\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{L^{3/2}} \sum_{\vec{k}} \left\{ \begin{bmatrix} \overset{1}{a}_{\vec{k}1} \\ \overset{2}{a}_{\vec{k}1} \\ 0 \\ 0 \end{bmatrix} e^{i(\vec{k}\vec{x} - k_0 t)} + \begin{bmatrix} \overset{1}{a}_{\vec{k}2} \\ \overset{2}{a}_{\vec{k}2} \\ 0 \\ 0 \end{bmatrix} e^{i(\vec{k}\vec{x} + k_0 t)} \right\}, \tag{98}$$

$$\Pi_{\Phi^\dagger} = \frac{K}{L^{3/2}} \sum_{\vec{k}} \left\{ \begin{bmatrix} -i k_0 \overset{1}{a}_{\vec{k}1} \\ -i k_0 \overset{2}{a}_{\vec{k}1} \\ -i k_z \overset{1}{a}_{\vec{k}1} - (i k_x + k_y) \overset{2}{a}_{\vec{k}1} \\ (-i k_x + k_y) \overset{1}{a}_{\vec{k}1} + i k_z \overset{2}{a}_{\vec{k}1} \end{bmatrix} e^{-ik_0 t} + \right. \tag{99}$$

$$\left. + \begin{bmatrix} i k_0 \overset{1}{a}_{\vec{k}1} \\ i k_0 \overset{2}{a}_{\vec{k}1} \\ -i k_z \overset{1}{a}_{\vec{k}1} - (i k_x + k_y) \overset{2}{a}_{\vec{k}1} \\ (-i k_x + k_y) \overset{1}{a}_{\vec{k}1} + i k_z \overset{2}{a}_{\vec{k}1} \end{bmatrix} e^{ik_0 t} \right\} e^{i\vec{k}\vec{x}}.$$

In the nonrelativistic limit, $|\vec{k}| \ll \varkappa$, the last two components of $\Pi_{\Phi^\dagger}$ become small in comparison to the first two. Thus, in the nonrelativistic limit, we have

$$\Pi_{\Phi^\dagger} \doteq \frac{K}{L^{3/2}} \sum_{\vec{k}} \left\{ \begin{bmatrix} -i k_0 \overset{1}{a}_{\vec{k}1} \\ -i k_0 \overset{2}{a}_{\vec{k}1} \\ 0 \\ 0 \end{bmatrix} e^{i(\vec{k}\vec{x} - k_0 t)} + \begin{bmatrix} i k_0 \overset{1}{a}_{\vec{k}2} \\ i k_0 \overset{2}{a}_{\vec{k}2} \\ 0 \\ 0 \end{bmatrix} e^{i(\vec{k}\vec{x} + k_0 t)} \right\} = K \partial_0 \Phi. \tag{100}$$

Eqs. (39) then become

$$\Psi_{\mathrm{I}} \doteq \frac{1}{\sqrt{2}} (\varkappa \Phi + i \partial_0 \Phi), \tag{101}$$

$$\Psi_{\mathrm{II}} \doteq \frac{1}{\sqrt{2}} (\varkappa \Phi - i \partial_0 \Phi).$$

The substitution Φ from (98) into (101) gives

$$\Psi_{\mathrm{I}} \doteq \frac{1}{\sqrt{2}} \frac{1}{L^{3/2}} \sum_{\vec{k}} \left\{ (\varkappa + k_0) \begin{bmatrix} a^1_{\vec{k}1} \\ a^2_{\vec{k}1} \\ 0 \\ 0 \end{bmatrix} e^{i(\vec{k}\vec{x} - k_0 t)} + (\varkappa - k_0) \begin{bmatrix} a^1_{\vec{k}2} \\ a^2_{\vec{k}2} \\ 0 \\ 0 \end{bmatrix} e^{i(\vec{k}\vec{x} + k_0 t)} \right\} \doteq$$

$$\doteq \frac{1}{\sqrt{2}} \frac{1}{L^{3/2}} \sum_{\vec{k}} (\varkappa + k_0) \begin{bmatrix} a^1_{\vec{k}1} \\ a^2_{\vec{k}1} \\ 0 \\ 0 \end{bmatrix} e^{i(\vec{k}\vec{x} - k_0 t)}, \tag{102}$$

$$\Psi_{\mathrm{II}} \doteq \frac{1}{\sqrt{2}} \frac{1}{L^{3/2}} \sum_{\vec{k}} \left\{ (\varkappa - k_0) \begin{bmatrix} a^1_{\vec{k}2} \\ a^2_{\vec{k}2} \\ 0 \\ 0 \end{bmatrix} e^{i(\vec{k}\vec{x} - k_0 t)} + (\varkappa + k_0) \begin{bmatrix} a^1_{\vec{k}2} \\ a^2_{\vec{k}2} \\ 0 \\ 0 \end{bmatrix} e^{i(\vec{k}\vec{x} + k_0 t)} \right\} \doteq$$

$$\doteq \frac{1}{\sqrt{2}} \frac{1}{L^{3/2}} \sum_{\vec{k}} (\varkappa + k_0) \begin{bmatrix} a^1_{\vec{k}2} \\ a^2_{\vec{k}2} \\ 0 \\ 0 \end{bmatrix} e^{i(\vec{k}\vec{x} + k_0 t)}. \tag{103}$$

Let us notice that Ψ_{I} contains only the negative frequency terms and Ψ_{II} only the positive frequency terms. Due to this reason the nonrelativistic limit of the canonical equations (25—28) one gets writing

$$\Psi_{\mathrm{I}} = e^{-i\varkappa t} \chi_{\mathrm{I}} \equiv e^{-i\varkappa t} \begin{pmatrix} \chi_{\mathrm{I}1} \\ \chi_{\mathrm{I}2} \end{pmatrix}, \tag{104}$$

$$\Psi_{\mathrm{II}} = e^{i\varkappa t} x_{\mathrm{II}} \equiv e^{i\varkappa t} \begin{pmatrix} \chi_{\mathrm{II}1} \\ \chi_{\mathrm{II}2} \end{pmatrix}. \tag{105}$$

The substitution in (25—28) gives

$$
\begin{aligned}
& i\,\partial_0\,\chi_{I1} + i\,\partial_j\,\sigma^j\,\chi_{I2} = 0, \\
& -2\,\varkappa\,\chi_{I2} - i\,\partial_0\,\chi_{I2} - i\,\partial_j\,\sigma^j\,\chi_{I1} = 0,
\end{aligned}
\tag{106}
$$

$$
\begin{aligned}
& i\,\partial_0\,\chi_{II1} + i\,\partial_j\,\sigma^j\,\chi_{II2} = 0, \\
& 2\,\varkappa\,\chi_{II2} - i\,\partial_0\,\chi_{II2} - i\,\partial_j\,\sigma^j\,\chi_{II1} = 0.
\end{aligned}
\tag{107}
$$

Assuming

$$
\begin{aligned}
& |\varkappa\,\chi_{I2}| \gg |i\,\partial_0\,\chi_{I2}|, \\
& |\varkappa\,\chi_{II2}| \gg |i\,\partial_0\,\chi_{II2}|
\end{aligned}
\tag{108}
$$

it becomes

$$
\begin{aligned}
& i\,\partial_0\,\chi_{I1} + \frac{1}{2\varkappa}\Delta\,\chi_{I1} = 0, \\
& \chi_{I2} = -\frac{i}{2\varkappa}\,\partial_j\,\sigma^j\,\chi_{I1},
\end{aligned}
\tag{109}
$$

$$
\begin{aligned}
& -i\,\partial_0\,\chi_{II1} + \frac{1}{2\varkappa}\Delta\,\chi_{II1} = 0, \\
& \chi_{II2} = \frac{i}{2\varkappa}\,\partial_j\,\sigma^j\,\chi_{II1}.
\end{aligned}
\tag{110}
$$

We see that the large components, χ_{I1}, χ_{II1}, satisfy the Schrödinger's equations with the positive and negative time derivative, respectively. These equations are the nonrelativistic approximation of the canonical equations (25—28).

Let us write also the Lagrange's equation for Φ in the nonrelativistic approximation. By making use of (39), we have

$$
\Phi = \frac{1}{\varkappa\sqrt{2}}\left(e^{-i\varkappa t}\,\chi_I + e^{i\varkappa t}\,\chi_{II}\right)
\tag{111}
$$

and after substitution in Eq. (19)

$$
(-2\,i\,\varkappa\,\partial_0\,\chi + \partial_0^2\,\chi - \Delta\,\chi_I)\,e^{-i\varkappa t} + (2\,i\,\varkappa\,\partial_0\,\chi_{II} + \partial_0^2\,\chi - \Delta\,\chi_{II})\,e^{i\varkappa t} = 0.
$$

Assuming

$$|\varkappa\,\partial_0\,\chi_{\rm I}| \geqslant |\partial_0^2\,\chi_{\rm I}|,$$
$$|\varkappa\,\partial_0\,\chi_{\rm II}| \geqslant |\partial_0^2\,\chi_{\rm II}| \tag{112}$$

it leads to

$$(-\,2i\,\varkappa\,\partial_0\,\chi_{\rm I} - \Delta\,\chi_{\rm I})\,e^{-i\varkappa t} + (2\,i\,\varkappa\,\partial_0\,\chi_{\rm II} - \Delta\,\chi_{\rm II})\,e^{i\varkappa t} = 0. \tag{113}$$

From here follows

$$2\,i\,\varkappa\,\partial_0\,\chi_{11} + \Delta\,\chi_{11} = 0, \tag{114}$$

$$-2\,i\,\varkappa\,\partial_0\,\chi_{111} + \Delta\,\chi_{111} = 0, \tag{115}$$

in accordance to (109) and (110).

One can get formally Eq. (114) from the energy equation

$$E = \frac{p^2}{2\varkappa} \tag{116}$$

and the substitution

$$E \to {\rm i}\,\partial_0, \qquad \vec{p} \to -\,i\,\nabla. \tag{117}$$

Similarly, one can get formally Eq. (115) from Eq. (116) and the substitution

$$E \to -\,i\,\partial_0, \qquad \vec{p} \to i\,\nabla. \tag{118}$$

In terms of the Poisson brackets it means in the first case

$$\{\,,\} \to \frac{1}{i}\,[\,,] \tag{119}$$

and in the second case

$$\{\,,\} \to -\,\frac{1}{i}\,[\,,]. \tag{120}$$

One can easily see that these »quantizations« give the dynamical equations.

The nonrelativistic approximation of the Lagrange's equation (19) one gets, therefore, by the decomposition

$$\Phi = {\rm e}^{-i\varkappa t}\,\Phi_1 + {\rm e}^{i\varkappa t}\,\Phi_2, \tag{121}$$

where Φ_1 satisfies Eq. (114) and Φ_2 Eq. (115).

Having the nonrelativistic approximation of the field equations, we now evaluate the constants of motion in this approximation.

First, we write these constants in terms of Ψ_{I} and Ψ_{II}. According to (39), (60), (64), (65), (70) and taking $K = \varkappa$, we have

$$Q = \text{const}_Q \int (\Psi_{\text{I}}^{\dagger} \Psi_{\text{I}} - \Psi_{\text{II}}^{\dagger} \Psi_{\text{II}})\, \mathrm{d}^3x, \tag{122}$$

$$P_0 = \text{const}_P \frac{i}{2} \int \{(\Psi_{\text{I}}^{\dagger} \partial_0 \Psi_{\text{I}} - \partial_0 \Psi_{\text{I}}^{\dagger} \Psi_{\text{I}}) - (\Psi_{\text{II}}^{\dagger} \partial_0 \Psi_{\text{II}} - \partial_0 \Psi_{\text{II}}^{\dagger} \Psi_{\text{II}})\}\, \mathrm{d}^3x, \tag{123}$$

$$P_l = \text{const}_P \frac{i}{2} \int \{(\Psi_{\text{I}}^{\dagger} \partial_l \Psi_{\text{I}} - \partial_l \Psi_{\text{I}}^{\dagger} \Psi_{\text{I}}) - (\Psi_{\text{II}}^{\dagger} \partial_l \Psi_{\text{II}} - \partial_l \Psi_{\text{II}}^{\dagger} \Psi_{\text{II}})\}\, \mathrm{d}^3x, \tag{124}$$

$$M^{\beta\gamma} = \text{const}_M \{ \int (x^{\gamma} T^{\beta 0} - x^{\beta} T^{\gamma 0})\, \mathrm{d}^3x +$$

$$+ \frac{1}{2} \int (\Psi_{\text{I}}^{\dagger} \sigma^{\beta\gamma} \Psi_{\text{I}} - \Psi_{\text{II}}^{\dagger} \sigma^{\beta\gamma} \Psi_{\text{II}})\, \mathrm{d}^3x\}. \tag{125}$$

The substitution Ψ_{I} and Ψ_{II} from (104 —105) gives

$$Q = \text{const}_Q \int (\chi_{\text{I}}^{\dagger} \chi_{\text{I}} - \chi_{\text{II}}^{\dagger} \chi_{\text{II}})\, \mathrm{d}^3x, \tag{126}$$

$$P_0 = \text{const}_P\, \varkappa \int (\chi_{\text{I}}^{\dagger} \chi_{\text{I}} + \chi_{\text{II}}^{\dagger} \chi_{\text{II}})\, \mathrm{d}^3x + \tag{127}$$

$$+ \text{const}_P \frac{i}{2} \int \{(\chi_{\text{I}}^{\dagger} \partial_0 \chi_{\text{I}} - \partial_0 \chi_{\text{I}}^{\dagger} \chi_{\text{I}}) - (\chi_{\text{II}}^{\dagger} \partial_0 \chi_{\text{II}} - \partial_0 \chi_{\text{II}}^{\dagger} \chi_{\text{II}})\}\, \mathrm{d}^3x,$$

$$P_l = \text{const}_P \frac{i}{2} \int \{(\chi_{\text{I}}^{\dagger} \partial_l \chi_{\text{I}} - \partial_l \chi_{\text{I}}^{\dagger} \chi_{\text{I}}) - (\chi_{\text{II}}^{\dagger} \partial_l \chi_{\text{II}} - \partial_l \chi_{\text{II}}^{\dagger} \chi_{\text{II}})\}\, \mathrm{d}^3x, \tag{128}$$

$$M^{\beta\gamma} = \text{const}_M \{ \int (x^{\beta} T^{\gamma 0} - x^{\gamma} T^{\beta 0})\, \mathrm{d}^3x + \tag{129}$$

$$+ \frac{1}{2} \int (\chi_{\text{I}}^{\dagger} \sigma^{\beta\gamma} \chi_{\text{I}} - \chi_{\text{II}}^{\dagger} \sigma^{\beta\gamma} \chi_{\text{II}})\, \mathrm{d}^3x\}. \tag{130}$$

In the nonrelativistic limit $\chi_{\text{I}} \to \chi_{\text{II}}$, $\chi_{\text{II}} \to \chi_{\text{III}}$. The functions $\chi_{\text{II}}, \chi_{\text{III}}$ are independent and they are given by Eqs. (114—115). Therefore, we may them normalize to unit (when different from zero);

$$\int \chi_{\text{II}}^{\dagger} \chi_{\text{II}}\, \mathrm{d}^3 x = 1, \tag{131}$$

$$\int \chi_{\text{III}}^{\dagger} \chi_{\text{III}}\, \mathrm{d}^3x = 1. \tag{132}$$

Selecting const_Q, const_P and const_M according to (78) the constants of motion (126—128) then become

$$Q_{nonrel} = \begin{cases} q, & \chi_{\mathrm{I}1} \neq 0, \quad \chi_{\mathrm{II}1} = 0, \\ -q, & \chi_{\mathrm{I}1} = 0, \quad \chi_{\mathrm{II}1} \neq 0, \\ 0, & \chi_{\mathrm{I}1} \neq 0, \quad \chi_{\mathrm{II}1} \neq 0, \end{cases} \tag{133}$$

$$P^0{}_{nonrel} = \begin{cases} \varkappa + \int \chi^\dagger_{\mathrm{I}1} \left(-\frac{1}{2\varkappa}\Delta\right) \chi_{\mathrm{I}1}\, d^3x, & \chi_{\mathrm{I}1} \neq 0, \quad \chi_{\mathrm{II}1} = 0, \\ \varkappa + \int \chi^\dagger_{\mathrm{II}1} \left(-\frac{1}{2\varkappa}\Delta\right) \chi_{\mathrm{II}1}\, d^3x, & \chi_{\mathrm{I}1} = 0, \quad \chi_{\mathrm{II}1} \neq 0, \\ 2\varkappa + \int \chi^\dagger_{\mathrm{I}1} \left(-\frac{1}{2\varkappa}\Delta\right) \chi_{\mathrm{I}1}\, d^3x + & \\ \quad + \int \chi^\dagger_{\mathrm{II}1} \left(-\frac{1}{2\varkappa}\Delta\right) \chi_{\mathrm{II}1}\, d^3x, & \chi_{\mathrm{I}1} \neq 0, \quad \chi_{\mathrm{II}1} \neq 0, \end{cases} \tag{134}$$

$$P^j{}_{nonrel} = \begin{cases} \int \chi^\dagger_{\mathrm{I}1} (-i\,\partial_j)\, \chi_{\mathrm{I}1}\, d^3x, & \chi_{\mathrm{I}1} \neq 0, \quad \chi_{\mathrm{II}1} = 0, \\ \int \chi^\dagger_{\mathrm{II}1} (i\,\partial_j)\, \chi_{\mathrm{II}1}\, d^3x, & \chi_{\mathrm{I}1} = 0, \quad \chi_{\mathrm{II}1} \neq 0. \\ \int \chi^\dagger_{\mathrm{I}1} (-i\,\partial_j)\, \chi_{\mathrm{I}1}\, d^3x + & \\ + \int \chi^\dagger_{\mathrm{II}1} (i\,\partial_j)\, \chi_{\mathrm{II}1}\, d^3x, & \chi_{\mathrm{I}1} \neq 0, \quad \chi_{\mathrm{II}1} \neq 0. \end{cases} \tag{135}$$

The spin pseudovector, according to (72), is

$$S_{r\, nonrel} = \frac{1}{2} \begin{cases} \int \chi^\dagger_{\mathrm{I}1}\, \sigma_r\, \chi_{\mathrm{I}1}\, d^3x, & \chi_{\mathrm{I}1} \neq 0, \quad \chi_{\mathrm{II}1} = 0, \\ -\int \chi^\dagger_{\mathrm{II}1}\, \sigma_r\, \chi_{\mathrm{II}1}\, d^3x, & \chi_{\mathrm{I}1} = 0, \quad \chi_{\mathrm{II}1} \neq 0, \\ \int \chi^\dagger_{\mathrm{I}1}\, \sigma_r\, \chi_{\mathrm{I}1}\, d^3x - \int \chi^\dagger_{\mathrm{II}1}\, \sigma_r\, \chi_{\mathrm{II}1}\, d^3x. & \chi_{\mathrm{I}1} \neq 0, \quad \chi_{\mathrm{II}1} \neq 0. \end{cases} \tag{136}$$

From here we may say that $\chi_{\mathrm{I}1}$ is the wave function (the probability amplitude) of a particle with the charge q, the energy operator $\varkappa + (-1/2\,\varkappa)\,\Delta$, the momentum operator $-i\,\partial_j$ and with spin 1/2. Similarly, the function $\chi_{\mathrm{II}1}$ is the wave function (the probability amplitude) of a particle with the charge -q, the energy operator $\varkappa + (-1/2\,\varkappa)\,\Delta$, the momentum operator $i\,\partial_j$ and spin 1/2.

Thus, in the nonrelativistic limit we come to the conclusion that the field $(\Phi, \Phi\dagger)$ describes a statistical behaviour of two particles; a particle and its antiparticle. This conclusion is consistent with the standard nonrelativictis quantum mechanics.

Having this conclusion we now go over to the relativistic case.

(*b*) *Relativistic case*

Essential point in the nonrelativistic approximation is the existence of two particles, a particle and its antiparticle. It has to be preserved also in the relativistic case.

The characteristic of the nonrelativistic approximation which led to the two particles was the separation of the field components into positive and negative frequency terms. We have to follow this line also in the relativistic case.

Therefore, we write

$$\Phi = \xi + \zeta, \tag{137}$$

where ξ contains only the negative frequency terms and ζ contains only the positive frequency terms. The corresponding momenta and Ψ_{I}, Ψ_{II} are:

$$\Pi_{\Phi^\dagger} = \Pi_{\xi^\dagger} + \Pi_{\zeta^\dagger}, \tag{138}$$

$$\Psi_{I} = \Psi_{I\xi} + \Psi_{I\zeta}, \tag{139}$$

$$\Psi_{II} = \Psi_{II\xi} + \Psi_{II\zeta} \tag{140}$$

(for $|\vec{k}| \ll \varkappa$, $\Psi_{I\zeta} \to 0$, $\Psi_{II\xi} \to 0$).

Collecting the terms with the same frequencies the constants of motion become

$$Q = Q_\xi + Q_\zeta, \tag{141}$$

$$P^a = P^a_\xi + P^a_\zeta, \tag{142}$$

$$M^{\beta\gamma} = M^{\beta\gamma}_\xi + M^{\beta\gamma}_\zeta, \tag{143}$$

where

$$Q_\xi = \mathrm{const}_Q \int (\Psi^\dagger_{I\xi} \Psi_{I\xi} - \Psi^\dagger_{II\xi} \Psi_{II\xi})\, \mathrm{d}^3x, \tag{144}$$

$$Q_\zeta = \mathrm{const}_Q \int (\Psi^\dagger_{I\zeta} \Psi_{I\zeta} - \Psi^\dagger_{II\zeta} \Psi_{II\zeta})\, \mathrm{d}^3x, \tag{145}$$

$$\begin{aligned} P^a_\xi = \mathrm{const}_P \frac{i}{2} \int \{(\Psi^\dagger_{I\xi}\, \partial^a \Psi_{I\xi} - \partial^a \Psi^\dagger_{I\xi} \Psi_{I\xi}) - \\ - (\Psi^\dagger_{II\xi}\, \partial^a \Psi_{II\xi} - \partial^a \Psi^\dagger_{II\xi} \Psi_{II\xi})\}\, \mathrm{d}^3x, \end{aligned} \tag{146}$$

$$\begin{aligned} P^a_\zeta = \mathrm{const}_P \frac{i}{2} \int \{(\Psi^\dagger_{I\zeta}\, \partial^a \Psi_{I\zeta} - \partial^a \Psi^\dagger_{I\zeta} \Psi_{I\zeta}) - \\ - (\Psi^\dagger_{II\zeta}\, \partial^a \Psi_{II\zeta} - \partial^a \Psi^\dagger_{II\zeta} \Psi_{II\zeta})\}\, \mathrm{d}^3x, \end{aligned} \tag{147}$$

$$M^{\beta\gamma}_\xi = \mathrm{const}_M \{ \int (x^\beta\, T^{\gamma 0}_\xi - x^\gamma\, T^{\beta 0}_\xi)\, \mathrm{d}^3x + \tag{148}$$

$$+\frac{1}{2}\int(\Psi^{\dagger}_{\mathrm{I}\xi}\sigma^{\beta\gamma}\Psi_{\mathrm{I}\xi}-\Psi^{\dagger}_{\mathrm{II}\xi}\sigma^{\beta\gamma}\Psi_{\mathrm{II}\xi})\,\mathrm{d}^3x\}$$

$$M^{\beta\gamma}_{\zeta}=\mathrm{const}_M\{\int(x^{\beta}T^{\gamma0}_{\zeta}-x^{\gamma}T^{\beta0}_{\zeta})\,\mathrm{d}^3x+ \tag{149}$$

$$+\frac{1}{2}\int(\Psi^{\dagger}_{\mathrm{I}\zeta}\sigma^{\beta\gamma}\Psi_{\mathrm{I}\zeta}-\Psi^{\dagger}_{\mathrm{II}\zeta}\sigma^{\beta\gamma}\chi_{\mathrm{II}\zeta})\,\mathrm{d}^3x\}$$

The integrals with mixed terms with respect to ξ and ζ are zero (see Eq. (86)). Here we have taken also $K=\varkappa$.

Due to the fact that in the nonrelativistic limit $\Psi_{\mathrm{I}\zeta}\to 0$, $\Psi_{\mathrm{II}\xi}\to 0$ and $Q_{\xi}+ + Q_{\zeta}\to Q$ given by (133) we take

$$\int(\Psi^{\dagger}_{\mathrm{I}\xi}\Psi_{\mathrm{I}\xi}-\Psi^{\dagger}_{\mathrm{II}\xi}\Psi_{\mathrm{II}\xi})\,\mathrm{d}^3x=1, \tag{150}$$

$$\int(\Psi^{\dagger}_{\mathrm{II}\zeta}\Psi_{\mathrm{II}\zeta}-\Psi^{\dagger}_{\mathrm{I}\zeta}\Psi_{\mathrm{I}\zeta})\,\mathrm{d}^3x=1. \tag{151}$$

Introducing the notation

$$\Psi_+=\begin{pmatrix}\Psi_{\mathrm{I}\xi}\\ \Psi_{\mathrm{II}\xi}\end{pmatrix},\qquad \Psi_-=\begin{pmatrix}\Psi_{\mathrm{I}\zeta}\\ \Psi_{\mathrm{II}\zeta}\end{pmatrix}, \tag{152}$$

$$\tau_+=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad \tau_-=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \tag{153}$$

Eqs. (150—151) become

$$\int\Psi^{\dagger}_{+}\tau_+\Psi_+\,\mathrm{d}^3x=1, \tag{154}$$

$$\int\Psi^{\dagger}_{-}\tau_-\Psi_-\,\mathrm{d}^3x=1. \tag{155}$$

The expressions under these integrals are positive definite, what one can see from Eqs. (85—87), because of

$$|\varkappa-k_0|<\varkappa+k_0$$

and since $\Psi_{\mathrm{II}\xi}$ and $\Psi_{\mathrm{I}\zeta}$ contain the factors $(\varkappa-k_0)$ while $\Psi_{\mathrm{I}\xi}$ and $\Psi_{\mathrm{II}\zeta}$ contain the factors $(\varkappa+k_0)$.

Selecting const_Q, const_P and const_M according to (78), the constants of motion instead of (133—135) then become

$$Q=\begin{cases} q, & \Psi_+\neq 0,\ \Psi_-\neq 0,\\ -q, & \Psi_+=0,\ \Psi_-\neq 0,\\ 0, & \Psi_+\neq 0,\ \Psi_-\neq 0,\end{cases} \tag{156}$$

$$P^{a} = \begin{cases} \int \Psi_{+}^{\dagger} \tau_{+} (\mathrm{i}\, \partial^{a}) \Psi_{+} \, \mathrm{d}^{3} x, & \Psi_{+} \neq 0,\ \Psi_{-} = 0, \\ \int \Psi_{-}^{\dagger} \tau_{-} (-i\, \partial^{a}) \Psi_{-} \, \mathrm{d}^{3} x, & \Psi_{+} = 0,\ \Psi_{-} \neq 0, \\ \int \{\Psi_{+}^{\dagger} \tau_{+} (i\, \partial^{a}) \Psi_{+} + \Psi_{-}^{\dagger} \tau_{-} (-i\, \partial^{a}) \Psi_{-}\} \, \mathrm{d}^{3} x, & \Psi_{+} \neq 0,\ \Psi_{-} \neq 0. \end{cases} \tag{157}$$

The spin pseudovector is

$$S_{r} = \frac{1}{2} \begin{cases} \int \Psi_{+} \tau_{+} \tilde{\sigma}_{r} \Psi_{+} \, \mathrm{d}^{3}x, & \Psi_{+} \neq 0,\ \Psi_{-} = 0, \\ \int \Psi_{-}^{\dagger} \tau_{-} \tilde{\sigma}_{r} \Psi_{-} \, \mathrm{d}^{3}x, & \Psi_{+} = 0,\ \Psi_{-} \neq 0, \\ \int \{\Psi_{+}^{\dagger} \tau_{+} \tilde{\sigma}_{r} \Psi_{+} + \Psi_{-}^{\dagger} \tau_{-} \tilde{\sigma}_{r} \Psi_{-}\} \, \mathrm{d}^{3}x, & \Psi_{+} \neq 0,\ \Psi_{-} \neq 0. \end{cases} \tag{158}$$

The constant P^{0} can be written in the form, taking into account (12),

$$P^{0} = \begin{cases} \int \Psi_{+}^{\dagger} \tau_{+} \begin{pmatrix} -i\, \partial_{j}\, \alpha^{j} + \varkappa\, \beta & 0 \\ 0 & -i\, \partial_{j}\, \alpha^{j} - \varkappa\, \beta \end{pmatrix} \Psi_{+} \, \mathrm{d}^{3} x, \quad \Psi_{+} \neq 0,\ \ \Psi_{-} = 0, \\ \int \Psi_{-}^{\dagger} \tau_{-} \begin{pmatrix} i\, \partial_{j}\, \alpha^{j} - \varkappa\, \beta & 0 \\ 0 & i\, \partial_{j}\, \alpha^{j} + \varkappa\, \beta \end{pmatrix} \Psi_{-} \, \mathrm{d}^{3}x, \quad \Psi_{+} = 0,\ \Psi_{-} \neq 0, \\ \int \{\Psi_{+}^{\dagger} \tau_{+} \begin{pmatrix} -i\, \partial_{j}\, \alpha^{j} + \varkappa\, \beta & 0 \\ 0 & -i\, \partial_{j}\, \alpha^{j} - \varkappa\, \beta \end{pmatrix} \Psi_{+} + \\ + \Psi_{-}^{\dagger} \tau_{-} \begin{pmatrix} i\, \partial_{j}\, \alpha^{j} - \varkappa\, \beta & 0 \\ 0 & i\, \partial_{j}\, \alpha^{j} + \varkappa\, \beta \end{pmatrix} \Psi_{-}\} \, \mathrm{d}^{3}x, \quad \Psi_{+} \neq 0,\ \Psi_{-} \neq 0. \end{cases} \tag{159}$$

The physical meaning of these constants of motion is the same as in the nonrelativistic limit. These are physical quantities for two particles: a particle and its antiparticle of the spin 1/2. The operators in the average values of the energies play the role of the relativistic Hamiltonians of these particles. In the nonrelativistic limit they go over into already derived nonrelativistic expressions.

9. *Some historical remarks*

The Dirac equations with positive and negative mass terms were investigated by several authors in the history of the quantum theory. In order to relate our approach to these investigations we make remarks to some of them.

An extensive analysis of the two Dirac equations one finds in papers due to Markov[12,13]. Markov considered the problem of two neutrinos and in connection with it the two types of the Dirac field. He wrote the Dirac equations[12]

$$(i\, \hat{p} + m)\, \Psi_{e} = 0, \qquad \hat{p} = -i\, \partial_{a}\, \gamma^{a},$$
$$(i\, \hat{p} - m)\, \Psi_{\mu} = 0 \tag{160}$$

in the form

$$(i\hat{P} + \Gamma m)\Psi = 0, \quad \Psi = \begin{pmatrix}\Psi_e \\ \Psi_\mu\end{pmatrix}, \quad \hat{P} = \Gamma_a P^a, \tag{161}$$

$$\Gamma = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}_{8\times 8}, \quad \Gamma_a = \begin{pmatrix}\gamma_\mu & 0 \\ 0 & \gamma_\mu\end{pmatrix},$$

and used for the field Ψ the Lagrangian

$$\mathscr{L} = -\frac{1}{2}(\bar{\Psi}\Gamma_\nu \partial^\nu \Psi - \partial^\nu \bar{\Psi}\Gamma_\nu \Psi) - m\bar{\Psi}\Gamma\Psi. \tag{162}$$

From the scalar constant of motion of this field Markov interpreted the first Dirac equation as a equation for electron and the second for μ-meson.

Various other aspects of Markov's works can be found in our work.

Belinfante and Pauli considered the two Dirac equations in connection with the statistical behaviour of elementary particles[14,15]. Using the Lagrangian

$$\mathscr{L} = -K_+ \Psi_+^\dagger \beta(i\hat{p} + \varkappa)\Psi_+ - K_- \Psi_-^\dagger \beta(i\hat{p} - \varkappa)\Psi_- \tag{163}$$

for the equations (160), where $\Psi_e \to \Psi_+$ and $\Psi_\mu \to \Psi_-$, they established the quantum rules.

Essential difference between these works and our work is in the correct canonical description of the field and the elimination of the »dual« solutions what is present in our work. However, these works are of remarkable value in the analysis of this subject.

10. Conclusions

We have shown in this paper that the Dirac equations with positive and negative mass terms are equivalent to the canonical equations of a bispinor field whose Lagrangian is bispinor scalar with the first derivatives and a mass term. The theory is developed in correct Lagrange's and canonical form. Elimination of the »dual« solutions gives the positive definite energy of the field. The canonical quantization of the field leads to the particle of the spin 1/2 and the Bose statistics. The nonquantized field describes a statistical behaviour of two particles: a particle and its antiparticle of the spin 1/2. The nonrelativistic approximation of the nonquantized field equations are the Schrödiger's equations for these particles, where the equation for antiparticle has the time derivative with negative sign. The quantum version of the field has one infinity term in the scalar constant of motion. The statistical version is without difficulties and is completely consistent.

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DUALNA SIMETRIJA I TEORIJA DIRACOVOG POLJA

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Proučavanjem simetrije Diracovog polja koja je analogna dualnoj simetriji elektromagnetskog polja formulirana je nova teorija Diracovog polja u kojoj obe Diracove jednadžbe, sa pozitivnim i negativnim članom mase, opisuju fizikalni objekt. Nađeno je da su ove jednadžbe ekvivalentne kanonskim jednadžbama bispinorskog polja čija je Lagrangeova gustoća kvadratični bispinorski skalar sa prvim derivacijama polja i članom sa masom. Teorija je razvijena u ispravnoj Lagrangeovoj i kanonskoj formi. Izbacivanje »dualnih« rješenja daje pozitivno definitnu energiju polja. Kanonsko kvantiziranje polja vodi na čestice sa spinom 1/2 i Bose statistikom. Nekvantizirano polje opisuje statističko vladanje dvije čestice: jedne čestice i njoj antičestice. Nerelativistička aproksimacija jednadžbi nekvantiziranog polja su Schrödingerove jednadžbe za ove čestice, pri čemu Schrödingerova jednadžba za antičesticu ima negativni predznak uz vremensku derivaciju. Kvantna verzija polja sadrži jedan beskonačni član u skalarnoj konstanti kretanja. Statistička verzija ne sadrži poteškoće i potpuno je u sebi konzistentna.