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EVOLUTION OF STATE FUNCTION DURING MEASUREMENT PROCESS

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An evolution law of the state of physical system during measurement process is proposed and some consequences as well as the possibility of experimental verification of the law are investigated.

1. Introduction

In its present form quantum mechanics describes the state of physical system by a normed function of the separable Hilbert space. There are two different ways in which this function can change. In the first case, we have a continuous and deterministic change of the state according to Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = H \Psi. \tag{1}$$

If the energy operator H does not depend explicitly on time, this change is, essentially, a continuous transition of the identity operator I into a unitary operator U. Symbolically,

$$I \rightarrow U(t) = e^{-\frac{i}{\hbar} t H}. \tag{2}$$

In the second case, we have a discontinuous and undeterministic change of the state brought about by the measurement of a quantity (we shall restrict ourselves on energy measurements only). If $\varphi_1, \varphi_2, \ldots\ldots$ are the eigenstates, then the state Ψ will be changed into the state φ_j with probability $|(\varphi_j, \Psi)|^2$. In this case we can write

$$\Psi \rightarrow \frac{P_{[\varphi_j]} \Psi}{\| P_{[\varphi_j]} \Psi \|} \tag{3}$$

where $P_{[\varphi_j]}$ is the projection operator on the eigenspace spanned by the eigenfunction φ_j. This is, essentially, a instantaneous transition of the identity operator I into the projection $P_{[\varphi_j]}$. Simbolically,

$$I \rightarrow P_{[\varphi_j]} \,. \tag{4}$$

This instantaneous reduction of the state function has been recognized very early as the chief weakness of quantum mechanics[1]. In this paper we shall try to take into account the natural assumption of minimum duration of the measurement and propose an evolution law which will lead from the identity operator to the corresponding projection.

2. *Some mathematical facts*

Let n be an element of the set of all integers and $\tau > 0$ an arbitrary real number. Let $S = \{U(n\tau)\}$ be a set of unitary operators defined on the additive abelian group $G = \{t_n = n\tau\}$, the elements of which satisfy the following conditions

$$U(n\tau)\, U(m\tau) = U((n+m)\,\tau) \tag{5}$$

$$U(0) = I. \tag{6}$$

If φ is an arbitrary element of the separable Hilbert space, then Herglotz theorem[2] assures that there exists a unique distribution function $F(s)$ with the property $F(0) = 0$, such that

$$(\varphi, U(k\tau)\,\varphi) = \int_0^{\frac{2\pi}{\tau}} e^{ik\tau s}\, dF(s). \tag{7}$$

Starting from Herglotz theorem and following the same reasoning which leads from Bochner's to Stone's theorem, we get easily this result.

There exists a bounded selfadjoint operator A such that

$$U(k\tau) = e^{ik\tau A} \tag{8}$$

$$A = \int_0^{\frac{2\pi}{\tau}} s \, \mathrm{d} P(s) \qquad P(s) = \begin{cases} 0 & s < 0 \\ I & s \geqslant 2\pi/\tau \end{cases} \tag{9}$$

$$|A| \leqslant \frac{2\pi}{\tau} \tag{10}$$

where by $|A|$ we denoted the norm of the operator A. This result is analogous to Stone's theorem, the only difference being that the operator A must be bounded.

3. *Operator of evolution*

As we mentioned in the introduction, quantum mechanics gives the set of possible results of a measurement and, generally, never the result itself. Now, suppose that in a energy measurement the choice of energy E_j is assured by a selfadjoint operator A_j for which we shall assume the following properties:

I A_j is bounded.

II A_j has pure discrete spectrum with positive eigenvalues.

III One of the eigenvalues of A_j, say, a_j^j is equal to the norm $|A_j|$.

IV A_j represents a quantity of the dimension of frequency.

V All A_j and H form a set of mutually commuting operators. Furthermore, we shall denote by φ_i the common eigenfunction of all A_j and H and by a_i^j and E_i the corresponding eigenvalues. Thus, we have

$$A_j \varphi_i = a_i^j \varphi_i \tag{11}$$

$$H \varphi_i = E_i \varphi_i. \tag{12}$$

Finally, by

$$\tau_j = \frac{2\pi}{|A_j|} \tag{13}$$

we define a time interval which we shall assign to the operator A_j.

From (13) and III we see that

$$\tau_j a_i^j \begin{cases} = 2\pi & i = j \\ < 2\pi & i \neq j\,. \end{cases} \tag{14}$$

Now, we are in position to postulate the evolution law by the following time average

$$V(t) = \frac{1}{n} \sum_{k=0}^{n-1} U(k\tau_j), \qquad (n-1)\tau_j \leqslant t < n\tau_j,\ n = 1, 2, 3, \ldots \tag{15}$$

where

$$U(k\tau_j) = e^{ik\tau_j A_j}. \tag{16}$$

From (15) we see that the evolution law represents a series of discrete changes of the state which occur at the end of time intervals of length τ_j. Evidently,

$$V(0) = I. \tag{17}$$

Of the other hand, taking into account the group property of the set of operators U, we can write for the evolution operator

$$V(t) = \frac{1}{n} \sum_{k=0}^{n-1} U^k(\tau_j), \qquad (n-1)\tau_j \leqslant t < n\tau_j, \quad n = 1, 2, 3, \dots \tag{18}$$

Now, we can use a result well known from the ergodic theorem[3] and say that after passing to the limit $n \to \infty$ (or $t \to \infty$) the evolution operator converges to the projection on the eigenspace of unit eigenvalue of operator $U(\tau_j)$. Thus, taking into account (13) we have

$$U(\tau_j)\varphi_j = e^{i\tau_j A_j}\varphi_j = e^{i\tau_j a_j^j}\varphi_j = \varphi_j. \tag{19}$$

Therefore,

$$V(+\infty) = P_{[\varphi_j]} \tag{20}$$

and we can conclude that the operator $V(t)$ has the desired property (4).

The evolution operator (15) represents an infinitely long evolution which may be physically unacceptable as well as an instantaneous change of the state function. But, the situation is not so bad as it seems. For, if we calculate the unnormed function of state for the time $t = n\tau_j$, we get

$$V(n\tau_j)\Psi = \frac{1}{n}\sum_{k=0}^{n-1} e^{ik\tau_j A_j}\Psi = \sum_{m=1}^{\infty} f_m \left(\frac{1}{n}\sum_{k=0}^{n-1} e^{ik\tau_j a_m^j}\right)\varphi_m \tag{21}$$

where $\Psi = \sum_{m=1}^{\infty} f_m \varphi_m$. Taking the norms of both sides of (21), we get

$$\| V(n\tau_j)\Psi \|^2 = |f_j|^2 + \sum_{\substack{m=1 \\ m\neq j}}^{\infty} |f_m|^2 \frac{\sin^2 n\alpha_m}{n^2 \sin^2 \alpha_m}, \quad \alpha_m = \frac{\tau_j a_m^j}{2}. \tag{22}$$

From the last equation we see that the evolution operator does not change the Fourier coefficient f_j while the other Fourier coefficients converge very rapidly to zero as $n \to \infty$ (or $t \to \infty$). If we accept a very natural assumption that the time interval τ_j must be very short, we see that the initial state Ψ changes very rapidly into the state which is practically the eigenstate φ_j. In other words, the operator A_j reduces the state Ψ very rapidly to the eigenstate φ_j. On account of this property we shall call the operator A_j the reduction operator.

The evolution law (15) enables us to classify all measurements in two groups. We shall call a measurement complete if the interaction between physical system and measuring instrument lasts infinitely long. In all other cases the measurement will be incomplete. Obviously, only after performing a complete measurement the physical system will be found in a eigenstate. After an incomplete measurement the state of physical system will be left in a state more or less far from an eigenstate. However, in the light of the property of the evolution operator to change the initial state very rapidly, the duration of the interaction in a complete measurement need be only sufficiently long.

4. Experimental verification

In order to describe the evolution of the state function during a measurement process we have introduced further parameters beyond the state function. These extraparameters, together with the initial state function, are the initial conditions in the calculation of the final state. If we have all initial parameters, we can predict the final state of physical system after a measurement. In the case of a complete measurement this state will be an eigenstate. This fact seems very appealing, although it should not be overestimated. For, we predicted only one and already known result. If we want to increase the confidence in the theory we must predict a result unsuspected before the formulation of the theory that may be, at least in principle, experimentally verified.

Let us begin with a simple example. Suppose, the physical system is in the time $t = 0$ in the state

$$\Psi(0) = f_1\varphi_1 + f_2\varphi_2 \tag{23}$$

where φ_1 and φ_2 are the eigenfunctions of the energy which belong to the eigenvalues E_1 and E_2, respectively. Let A_1 and A_2 be the reduction operators that determine the transitions from the state $\Psi(0)$ to the states φ_1 and φ_2, respectively. The transitions to other states are, obviously, impossible because the other reduction operators would lead to the zero state which we rule out as a unacceptable state. Furthermore, let be

$$\frac{2\pi}{|A_1|} \equiv \tau_1 < \tau_2 \equiv \frac{2\pi}{|A_2|}. \tag{24}$$

Suppose, an incomplete measurement was performed on the system. Let the time duration of the interaction be t such that

$$\tau_1 < t < \begin{cases} 2\tau_1 \\ \tau_2 \end{cases}. \tag{25}$$

Furthermore, suppose that we ignore in the initial conditions the reduction operator. If the initial parameters were $\Psi(0)$ and A_1, then the final state would be according to the evolution law (15)

$$\Psi_1(t) = \frac{f_1\varphi_1 + (f_2 e^{i\beta}\cos\beta)\varphi_2}{\sqrt{|f_1|^2 + |f_2|^2\cos^2\beta}}, \qquad \beta = \frac{\tau_1 a_2^1}{2} \tag{26}$$

and if they were $\Psi(0)$ and A_2, the final state would be

$$\Psi_2(t) = f_1\varphi_1 + f_2\varphi_2. \tag{27}$$

Thus, we cannot predict the final state because we ignore one of two initial parameters of the process, namely, the reduction operator. However, we can, in accord with quantum mechanics, assert that the probability that the system will be in the state Ψ_1 or Ψ_2 is equal to the probability of getting the energies E_1 or E_2 if the measurement were complete. In other words, we assume that the initial parameters A_1 and A_2 are distributed and the measurement process is strictly deterministic. So, probability aspects of the outcomes of the measurement arise as a result of our ignorance of the initial parameters. Denoting these probabilities by $P(\Psi_1)$ and $P(\Psi_2)$, we have, according to quantum mechanics,

$$P(\Psi_1) = |f_1|^2 \tag{28}$$

$$P(\Psi_2) = |f_2|^2. \tag{29}$$

If we, immediately after the first measurement, perform on the system another and, now, complete measurement, then the final result of the second measurement will be one of two possible energies E_1 and E_2. If we denote the conditional probability by $P(E/\Psi)$, we have the following probabilities

$$P(E_1/\Psi_1) = \frac{|f_1|^2}{|f_1|^2 + |f_2|^2 \cos^2\beta} \tag{30}$$

$$P(E_1/\Psi_2) = |f_1|^2 \tag{31}$$

$$P(E_2/\Psi_1) = \frac{|f_2|^2 \cos^2\beta}{|f_1|^2 + |f_2|^2 \cos^2\beta} \tag{32}$$

$$P(E_2/\Psi_2) = |f_2|^2. \tag{33}$$

Therefore, the probability of getting one of two values E_1 or E_2 after the second measurement is

$$P(E_1) = P(\Psi_1)\,P(E_1/\Psi_1) + P(\Psi_2)\,P(E_1/\Psi_2) =$$

$$= |f_1|^2 \left(|f_2|^2 + \frac{|f_1|^2}{|f_1|^2 + |f_2|^2 \cos^2\beta}\right) \tag{34}$$

$$P(E_2) = P(\Psi_1)\,P(E_2/\Psi_1) + P(\Psi_2)\,P(E_2/\Psi_2) =$$

$$= |f_2|^2 \left(|f_2|^2 + \frac{|f_1|^2 \cos^2\beta}{|f_1|^2 + |f_2|^2 \cos^2\beta}\right). \tag{35}$$

Let us take the special case

$$f_1 = f_2 = \frac{1}{\sqrt{2}}. \tag{36}$$

In this case we have

$$P(E_1) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{1+\cos^2\beta}\right) \tag{37}$$

$$P(E_2) = \frac{1}{2}\left(\frac{1}{2} + \frac{\cos^2\beta}{1+\cos^2\beta}\right). \tag{38}$$

From the equations (37) and (38) we see that two probabilities are unequal although we started with equally distributed eigenstates. At this point the theory could be compared with experiment.

5. *Conclusion*

The theory, based on the proposed evolution law, is a purely deterministic theory which, in the case of complete measurement, gives predictions in accord with quantum mechanics. It constitutes a framework in which we can investigate the evolution of the state function in the measurement process. It introduces further parameters which describe the process of measurement but it does not supply a prescription to say what is the correct mathematical form of these extraparameters. Nevertheless, it was possible to find a simple example in which these extraparameters may manifest themselves.

References

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EVOLUCIJA FUNKCIJE STANJA U TOKU MERNOG PROCESA

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Originalni naučni rad

U radu je predložen zakon evolucije stanja fizičkog sistema u toku mernog procesa a zatim su razmatrane neke posledice predloženog zakona i mogućnost njegove eksperimentalne provere.