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MAGNETOHYDRODYNAMIC FLOW BETWEEN TWO ROTATING CYLINDERS

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Abstract: The steady MHD flow of an incompressible viscous, electrically conducting fluid is studied between two infinite concentric rotating cylinders subject to a radial magnetic field, taking a uniform suction velocity at the wall of the outer cylinder and a uniform injection velocity at the wall of the inner cylinder. Induced magnetic field is taken into account and using suitable boundary conditions exact solution is obtained.

1. *Introduction*

Ramoorthy[1] has considered the steady magnetohydrodynamic (MHD) flow between two concentric cylinders without taking the induced magnetic field into account. Here we consider the MHD flow of an incompressible, viscous, electrically conducting fluid between two infinite concentric cylinders of radii a and b ($a < b$), subject to a radial magnetic field $\left(= \frac{A}{r};\ A = \text{constant}\right)$, taking a uniform suction velocity v_b at the wall of the outer cylinder and a uniform injection velocity $v_a \left(= \frac{a v_b}{b}\right)$ at the wall of the inner cylinder. We take induced magnetic field into account. Muhuri[2] has investigated a similar problem. But the boundary condition on the azimuthal component of the magnetic field which he has used is corrected in our investigation.

Because of the axisymmetry the velocity and the magnetic field will be of the form

$$\vec{V}\,[v_r(r),\ v_\Theta(r),\ 0],$$

$$\vec{H}\,[H_r(r),\ H_\Theta(r),\ 0],$$

where $v_r(r)\,H_r(r)$ can be written by the divergence relation as $v_r(r) = \frac{b\,v_b}{r}$, $H_r(r) = = \frac{A}{r}$. Therefore the induced magnetic field has its non-zero component only along the azimuthal direction. Following Muhuri[2] we get the equation of motion and Ohm's law as

$$\frac{d^2 v_\Theta}{dr^2} - (R-1)\frac{1}{r}\frac{d v_\Theta}{dr} - (R+1)\frac{v_\Theta}{r^2} + \frac{A\,\mu\,H_\Theta}{\varrho\nu\;r^2} + \frac{A\,\mu}{\varrho\;\nu}\frac{1}{r}\frac{d\,H_\Theta}{d\,dr} - 0, \tag{1}$$

and

$$v_\Theta = \frac{\nu}{A\,\varkappa}\left[(\varkappa R - 1)\,H_\Theta - r\frac{d\,H_\Theta}{dr}\right], \tag{2}$$

where $R = \frac{b\,v_b}{\nu}$ is the suction Reynold's number and $\varkappa = \mu\,\sigma\,\nu$ the magnetic Prandtl number.

2. Boundary conditions

We suppose the inner and the outer cylinders are rotating with uniform angular velocities ω_1 and ω_2 respectively. Then from the no-slip conditions we have

$$v_\Theta = a\,\omega_1, \text{ at } r = a \tag{3}$$

$$v_\Theta = b\,\omega_2, \text{ at } r = b. \tag{4}$$

The electric current is given by

$$I_z = (0,\ 0,\ J_z),$$

$$I_z = \mu\,\sigma\,(v_r H_\Theta - v_\Theta H_r).$$

This implies that I_z will give rise to an azimuthal component of magnetic field in the annulus and outside it. Therefore H_Θ needs not necessarily be zero at $r = b$. This, in turn, implies that Muhuri's boundary condition $H_\Theta = 0$ at $r = b$ is not correct. However, if we assume the existence of no current inside the cylinder $r = a$, we can write down the boundary condition on H_Θ at $r = a$ as

$$H_\Theta = 0, \text{ at } r = a. \tag{5}$$

3. Solutions of the equations

Eliminating v_Θ from (1) and (2) we get

$$r^3 \frac{d^3 H_\Theta}{d r^3} + (4 - R - \varkappa R)\, r^2 \frac{d^2 H_\Theta}{d r^2} + [(R - 1)(\varkappa R - 2) - (R + 1) - \\ - M^2]\, r \frac{d H_\Theta}{d r} + [(R + 1)(\varkappa R - 1) - M^2]\, H_\Theta = 0, \tag{6}$$

where $M^2 = \dfrac{A^2 \mu^2 \sigma}{\varrho \nu}$, M = appropriate Hartmann number.

Using the boundary conditions (3) and (4) in (2), we get the derived boundary conditions on H_Θ as

$$\frac{d H_\Theta}{d r} = - \frac{A \varkappa}{\nu}\, \omega_1, \text{ at } r = a \tag{7}$$

and

$$\left[(\varkappa R - 1) H_\Theta - b \frac{d H_\Theta}{d r}\right] = \frac{A \varkappa}{\nu}\, b\, \omega_2, \text{ at } r = b. \tag{8}$$

Solution of (6) subject to (5), (7) and (8) is given by

$$H_\Theta = P_1 \left(\frac{1}{r}\right) + P_2\, r^{\lambda_1} + P_3\, r^{\lambda_2}, \tag{9}$$

where

$$\lambda_1, \lambda_2 = \frac{1}{2}\, [R(1 + \varkappa) \pm \{R^2 (1 + \varkappa)^2 - 4(R + 1)(\varkappa R - 1) + \\ + 4 M^2\}^{1/2}], \tag{10}$$

$$P_1 = \frac{A \varkappa}{\nu c} [(\varkappa R - 1 - \lambda_1)\, b^{1+\lambda_1} a^{3+\lambda_2} \omega_1 - (\varkappa R - 1 - \lambda_2)\, b^{1+\lambda_2} a^{3+\lambda_1} \omega_1 - \\ - a^{\lambda_1+\lambda_2+2} (\lambda_2 - \lambda_1)\, b^2 \omega_2],$$

$$P_2 = \frac{A \varkappa}{\nu c} [(\varkappa R - 1 - \lambda_2)\, a^2 b^{1+\lambda_2} \omega_1 - \varkappa R\, a^{3+\lambda_2} \omega_1 + (\lambda_2 + 1)\, a^{1+\lambda_2} b^2 \omega_2],$$

$$P_3 = \frac{A\varkappa}{\nu c}\left[-(\varkappa R - 1 - \lambda_1)\, a^2\, b^{1+\lambda_1}\, \omega_1 + \varkappa R\, a^{3+\lambda_1}\, \omega_1 - (\lambda_1 + 1)\, a^{1+\lambda_1}\, b_2\, \omega_2\right],$$

and

$$C = \varkappa R\, a^{\lambda_1+\lambda_2+2}\,(\lambda_1 - \lambda_2) + (\lambda_2 + 1)(\varkappa R - 1 - \lambda_1)\, b^{1+\lambda_1}\, a^{1+\lambda_2} - \\ - (1 + \lambda_1)(\varkappa R - 1 - \lambda_2)\, b^{1+\lambda_2}\, a^{1+\lambda_1}. \quad (11)$$

Using (9) in (2) we get

$$\frac{A\varkappa}{\nu} v_\Theta = P_1\, \varkappa R \left(\frac{1}{r}\right) + P_2\,(\varkappa R - 1 - \lambda_1)\, r^{\lambda_1} + P_3\,(\varkappa R - 1 - \lambda_2)\, r^{\lambda_2}. \quad (12)$$

Moment (about the axis) of the shearing stress per unit length on the outer cylinder is given by

$$T = 2\pi \varrho \nu\, b^3 \left\{\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{v_\Theta}{r}\right)\right\}_{r = b_z},$$

where v_Θ is given by (12). Thus

$$T = \frac{2\pi \varrho\, \nu^2\, b^3}{A\varkappa}\left[-2\varkappa R\left(\frac{1}{b^3}\right) P^1 + (\lambda_1 - 1)(\varkappa R - 1 - \\ - \lambda_1)\, b^{\lambda_1 - 2}\, P_2 + (\lambda_2 - 1)(\varkappa R - 1 - \lambda_2)\, b^{\lambda_2 - 2}\, P_3]\right]. \quad (13)$$

If the inner cylinder is at rest and the outer cylinder is rotating with ω_2, then T becomes

$$T = \frac{2\pi\varrho\nu\, b^5\, \omega_2}{c}\left[-\frac{2\varkappa R}{b^3}(\lambda_1 - \lambda_2)\, a^{\lambda_1+\lambda_2+2} + \\ + (\lambda_1 - 1)(\lambda_2 + 1)(\varkappa R - 1 - \lambda_1) \cdot b^{\lambda_1 - 2}\, a^{\lambda_2+1} - \\ - (\lambda_2 - 1)(\lambda_1 + 1)(\varkappa R - 1 - \lambda_2)\, b^{\lambda_2 - 2}\, a^{\lambda_1+1}\right]. \quad (14)$$

When the suction Reynold's number $R = 0$, we see that our results (12) and (14) coincide with the results (5) and (12) of Rammoorthy[1] and H_Θ becomes

$$H_\Theta = \frac{-\frac{A\varkappa}{\nu}[(\lambda - 1)(\omega_2\, b^{\lambda+1} - \omega_1\, a^{\lambda+1})(r^{\lambda+1} - a^{\lambda+1}) - \\ - (\lambda + 1)(a\,b)^{\lambda+1}(\omega_1\, b^{\lambda-1} - \omega_2\, a^{\lambda-1})(r^{1-\lambda} - a^{1-\lambda})]}{(\lambda_2 - 1)(b^{2\lambda} - a^{2\lambda})\, r}, \quad (15)$$

where $\lambda = (1 + M^2)^{1/2}$.

The solution for v_Θ for the non-magnetic case can not be obtained from (12) directly by putting $A = M = \varkappa = 0$, because in our case the velocity field is obtained from (2) only when a non-zero magnetic field is present and the fluid is conducting. However non-magnetic result may be obtained directly from the equation (1) by putting $M = 0$ and is given by

$$v_\Theta = A_1\, r^{R+1} + B_1\, r^{-1} \tag{16}$$

— using boundary conditions (3) and (4) —,

where $A_1 = \dfrac{a^2\,\omega_1 - b^2\,\omega_2}{a^{R+2} - b^{R+2}}$, $B_1 = \dfrac{a^2 b^2\,(a^R\,\omega_2 - b^R\,\omega_1)}{a^{R+2} - b^{R+2}}$.

For $R = 0$, the equation (16) reduces to the Coutte's hydrodynamics result[3].

4. *Numerical results and discussion*

We now introduce the following dimensionless quantities and parameters

$$\begin{aligned} v_\Theta &= \omega_1\, a\,\bar{v}_\Theta, & H_\Theta &= \frac{a\,\omega_1\,A\,\varkappa}{\nu}\,\bar{H}_\Theta, \\ T &= \varrho\,\nu\,a^2\,\omega_1\,\bar{T}, & r &= a\,\bar{r}, \\ \bar{\lambda} &= \frac{b}{a}, & \bar{\Omega} &= \frac{\omega_2}{\omega_1}. \end{aligned} \tag{17}$$

Using them in the equations (9), (12) and (13), we have

$$\bar{v}_\Theta = \bar{P}_1\,\varkappa\,R\left(\frac{1}{\bar{r}}\right) + \bar{P}_2\,(\varkappa\,R - 1 - \lambda_1)\,\bar{r}^{\lambda_1} + \bar{P}_3\,(\varkappa\,R - 1 - \lambda_2)\,\bar{r}^{\lambda_2}, \tag{18}$$

$$\bar{H}_\Theta = \bar{P}_1\left(\frac{1}{\bar{r}}\right) + \bar{P}_2\,\bar{r}^{\lambda_1} + \bar{P}^3\,\bar{r}^{\lambda_2}, \tag{19}$$

$$\begin{aligned} \bar{T} = 2\,\pi\,\bar{\lambda}^3 \Big[&- 2\,\varkappa\,R\left(\frac{1}{\bar{\lambda}^3}\right)\bar{P}_1 + (\lambda_1 - 1)\,(\varkappa\,R - 1 - \lambda_1)\,\bar{\lambda}^{\lambda_1 - 2}\,\bar{P}_2 + \\ &+ (\lambda_2 - 1)\,(\varkappa\,R - 1 - \lambda_2)\,\bar{\lambda}^{\lambda_2 - 2}\,\bar{P}_3\Big], \end{aligned} \tag{20}$$

where

$$\bar{P}_1 = \frac{1}{\bar{c}}\,[(\varkappa\,R - x - \lambda_1)\,\bar{\lambda}^{1+\lambda_1} - (\varkappa\,R - 1 - \lambda_2)\,\bar{\lambda}^{1+\lambda_2} - (\lambda_2 - \lambda_1)\,\bar{\lambda}^2\,\bar{\Omega}],$$

$$\overline{P}_2 = \frac{1}{\overline{c}} [(\varkappa R - 1 - \lambda_2)\overline{\lambda}^{1+\lambda_2} - \varkappa R + (\lambda_2 + 1)\lambda_2 \overline{\Omega}],$$

$$\overline{P}_3 = \frac{1}{\overline{c}} [-(\varkappa R - 1 - \lambda_1)\overline{\lambda}^{1+\lambda_1} + \varkappa R - (\lambda_1 + 1)\lambda^2 \overline{\Omega}], \tag{21}$$

$$\overline{C} = \varkappa R(\lambda_1 - \lambda_2) + (1 + \lambda_2)(\varkappa R - 1 - \lambda_1)\overline{\lambda}^{1+\lambda_1} -$$

$$- (1 + \lambda_1)(\varkappa R - 1 - \lambda_2)\overline{\lambda}^{1+\lambda_2},$$

and bar sign indicates that the quantity is dimensionless.

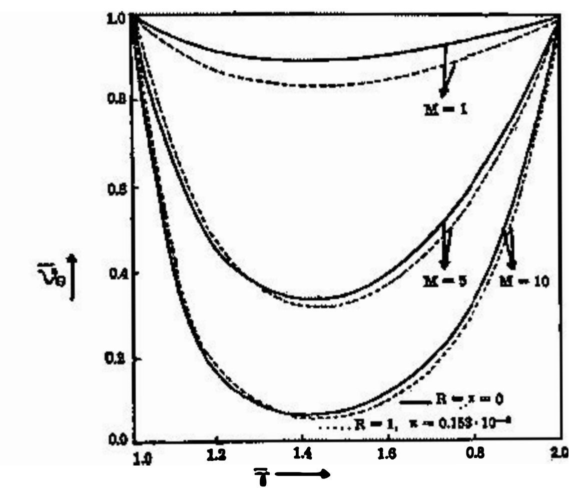

Fig. 1. Velocity distribution for various values of M, when $\overline{\Omega} = 0.5$.

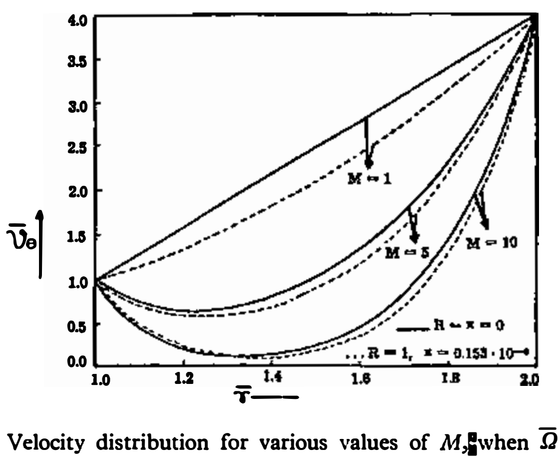

Fig 2. Velocity distribution for various values of M, when $\overline{\Omega} = 2$.

The graphs of $\overline{v_\theta}$ against $\bar{r}$ for different values of M are shown in Figs. 1 and 2 for $\bar{\Omega} = 0.5$ and $\bar{\Omega} = 2$ respectively, taking $\bar{\lambda} = 2$. Also variation of $\overline{H_\theta}$ against $\bar{r}$ for different values of M are shown in Figs. 3 and 4 for $R = 0$ and $R = 1$

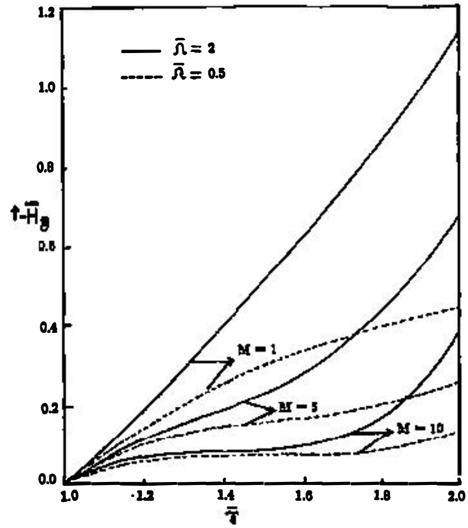

Fig. 3. Distribution of induced magnetic field for various values of M, when R = $\varkappa$ = 0.

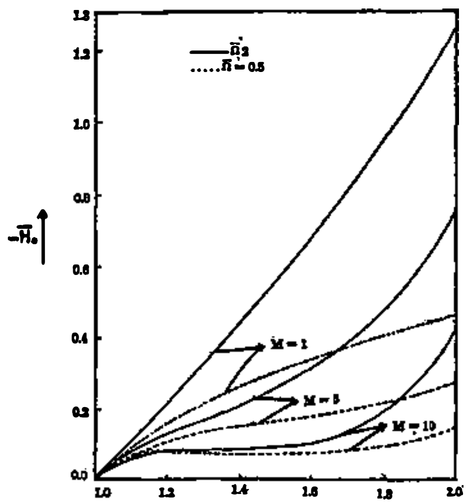

Fig. 4. Distribution of induced magnetic field for various values of M, when R = 1, $\varkappa = 0.153 \cdot 10^{-6}$.

respectively, taking $\bar{\lambda} = 2$. It may be noted from Figs. 1 and 2 that the effect of the suction and injection is to sweep the velocity profiles towards the outer cylinder. Effects of change in rotational velocities of the cylinders on the induced magnetic field as is evident from the Figs. 3 and 4 are significant. With the decrease in the velocity ratio $\overline{\Omega}$ the induced magnetic field decreases rapidly.

TABLE

M	$\overline{T}$ for $R = 0$, $\overline{\Omega} = 0.5$	$\overline{T}$ for $R = 0$, $\overline{\Omega} = 2$	$\overline{T}$ for $R = 1$, $\overline{\Omega} = 0.5$	$\overline{T}$ for $R = 1$, $\overline{\Omega} = 2$
1	4.492438	28.594744	0.417035	61.696811
5	47.876799	202.73414	54.428672	232.65811
10	113.48630	454.66002	120.46205	482.82259

Numerical values of $\overline{T}$ for $\overline{\lambda} = 2$ and $\varkappa = 0.153 \cdot 10^{-6}$ are given in the Table. It may be observed trom the Table that $\overline{T}$ increases with the increase of Hartmann number. This may be attributed to the reason that an extra amount of work is being done to bend the magnetic lines of force. Also $\overline{T}$ for $R = 1$ is larger than that for $R = 0$ and $\overline{T}$ increases as $\overline{\Omega}$ increases.

References

1) P. Rammoorthy, Phys. Fluid **4** (1961) 1444;
2) P. K. Muhuri, A I A A. **2** (1964) 1328;
3) S. I. Pai, Viscous Flow Theory, I, Laminar Flow (New Jersy, 1956).