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ON APPROXIMATE LORENTZ TRANSFORMATIONS OF THE FIRST AND SECOND ORDERS

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Abstract: Approximate Lorentz transformations of the first $((c^{-2})^0)$ (Galilean transformations) and second (c^{-2}) $(x = (x' + \beta ct')(1 + \beta^2/2),\ \ t = t'$ $(1 + \beta^2/2) + \beta x'/C)$ orders are considered. It is emphasized that in a first approximation the transformation formula for energy takes the form $E = E'$. The corresponding transformations for covariant coordinates (covariant transformations) are considered as well; they coincide with space-like ones which follow from the Lorentz transformations if $ct \approx \beta x$. It is shown that only based on the 1st order covariant transformations one can obtain the known transformation formulae for probability density ($\varrho = \varrho'$) and position operator ($X = X'$). The group properties of the 1st order covariant transformations (Galilean covariant group) are discussed. A particular role of the 2nd order covariant transformations is emphasized when the questions on the invariance of nonrelativistic equations in physics are under consideration.

1. *First approximation*

Galilean transformations (approximate) Lorentz transformations of the first order $(c^{-2})^0$). Let us consider the Galilean transformations

$$x = x' + v\,t', \quad t = t'. \tag{1}$$

Further we introduce, as usually, the momentum and energy of the particle with mass m by means of the expressions

$$p = p^1 = m\frac{\mathrm{d}x}{\mathrm{d}s},\ E = p^0 = m\frac{\mathrm{d}t}{\mathrm{d}s}, \tag{2a, b}$$

where s is a proper time, $c = 1$.

As a result, according to (1), we have

$$p = p' + v\,E', \quad E = E', \tag{3}$$

for the transformation formulae of momentum and energy in a Galilean approximation, i. e., with a precision of up to the terms of the order of $(\beta^2)^0$ where $\beta = v$.

Here, perhaps, it should be noted that in Galilean approximations time is not transformed when we transit from one reference frame to another, i. e., it is practically a scalar value. Therefore, according to (2b), both p^0 and E are actual scalars. But the latter means that in this case the energy E should be expressed only through a scalar value (which is not transformed when we transit from one to another reference frame), i. e., it represents a rest energy. Just from the fact indicated, the first of formulae (3) can be written as

$$p = p' + m\,v. \tag{3a}$$

The application of the formula for energy $E = m$ in a Galilean approximation can be born out by an additional consideration. Only in this case the relativistic expression for the center mass coordinate $X = \Sigma\,Ex/\Sigma\,E$ transits to the conventional classical expression $X = \Sigma\,m\,x/\Sigma\,m$. One more argument in favour of the specified formula can be obtained, e. g., by considering the transition from the relativistic expression for probability density to the nonrelativistic* limit by means of the known substitution

$$\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = -\,E\psi. \tag{4}$$

As a result, we have

$$\varrho = |\,\psi\,|^2\,\frac{E}{m}. \tag{5}$$

Whence it follows with necessity that the known expression

$$\varrho = |\,\psi\,|^2 \tag{6}$$

can be obtained from (5) only on the assumption that in a Galilean approximation the energy E is a rest energy.

If we now introduce, as usually, momentum and energy as partial derivatives of some scalar action function S

* Below the term »nonrelativistic« is practically used in the sense of »approtimate«.

$$p_1 = \frac{\partial S}{\partial x}, \quad p_4 = -\frac{\partial S}{\partial t}, \tag{7}$$

then, according to (1), we obtain that the transformation formulae p_1 and p_0 will be different from the required ones for energy and momentum (3).

This fact is due to that in the framework of the discussed approach the covariant components of a vector are transformed by the formulae which differ from the transformation formulae for contravariant components.

First order covariant transformations. Using the definition of covariant vector

$$A_i = \frac{\partial x^{\varkappa'}}{\partial x^i} A'_\varkappa \tag{8}$$

and taking into account (1), the first order transformation formulae for the covariant components of a »coordinate vector« read as

$$x_1 = x'_1, \; x_0 = x'_0 + v\,x', \tag{9}$$

(we have also changed the sign of a time coordinate).

It is evident that the expressions obtained in such a manner coincide with the approximate space-like transformations of the first order resulting from the Lorentz transformations if $t \approx \beta x$ due to the rejection of the terms of the order of β^2 and higher.

Now it is possible to agree to the definition of momentum and energy as partial derivatives of the action function and the above definition (2). With this aim we should use covariant coordinate derivatives, i. e., contravariant derivatives[1]. Replacing (7) by the expressions

$$p^1 = \frac{\partial S}{\partial x_1}, \quad p^0 = -\frac{\partial S}{\partial x_0} \tag{10}$$

we find according to (9) that the values, introduced by means of (10), actually obey the transformation law for momentum and energy (3).

The following condition

$$\left|\frac{\partial S}{\partial x}\right| \ll \left|\frac{\partial S}{\partial t}\right| \tag{11}$$

is an evident consequence of the definition (10) taking into account (3) in the framework of the approximation considered.

It is precisely this condition that is an actual base for transition from the Hamilton-Jacobi classical relativistic equation

$$\left(\frac{\partial S}{\partial t}\right)^2 = \left(\frac{\partial S}{\partial x}\right)^2 + m^2 \tag{12}$$

to the nonrelativistic limit.

As a matter of fact, in order to make the indicated limiting transition, let us extract a square root from the both parts of (12) and then expand the right--hand side into powers of $[(\partial S/\partial x)/m]^2$. Limiting ourselves to the second order terms and changing to the function $S_1 = S + m\,t$, we obtain

$$\frac{\partial S_1}{\partial t} = \frac{1}{2m}\left(\frac{\partial S_1}{\partial x}\right)^2, \tag{13—II}$$

(note that the Hamilton-Jacobi nonrelativistic equation of the first order, i. e., for a given approximation, takes the trivial form $\partial S_1/\partial t = 0$ (13)).

That we made use of smallness of the value $(\partial S/\partial x)/m$ or, so far as $\partial S/\partial t \approx m$, of the condition (11) is very essential. And this, in its turn, means that the Hamilton-Jacobi equation under consideration should represent a partial differential equation in covariant coordinates.

Similarly to (10), we introduce a wave vector with components

$$k^1 = \frac{\partial n}{\partial x_1}, \quad k^0 = -\frac{\partial n}{\partial x_0}, \tag{14}$$

where n is a scalar value.

Using further the de Broglie and Einstein-Planck relations

$$p^1 = \hbar k^1, \; E = \hbar k^0, \tag{15}$$

we get that the momentum and energy defined by formulae (15) will be transformed according to the required law (3).

Thus the fact (an indication of this fact is ascribed to Lande by Bunge[2)] and in this connection, see also Ref.[3)]) that the wave length λ_1, which must be subject to transformations (9), is actually invariant for the considered approximation whereas the momentum is tranformed by the Galilean formula (3 a), finds its explanation.

In addition, it should be noted that in the framework of this approach, the questions concerning the transformation formulae for probability density and position operator in nonrelativistic quantum mechanics are sequently solved.

In fact, in order to provide the wanted transformation formula for probability density in a Galilean approximation

$$\varrho = \varrho', \tag{16}$$

(which coincides, say, with the transformation formula for electric charge density), we must use contravariant derivatives in the relativistic expression for ϱ, i. e., we must believe that

$$\varrho = \frac{i\hbar}{2m}\left(\psi^{*}\frac{\partial \psi}{\partial x_0} - \psi\frac{\partial \psi^{*}}{\partial x_0}\right). \tag{17}$$

As far as the transformation formula for position operator X is concerned, using (9) we naturally obtain

$$X = X'. \tag{18}$$

At the same time one can hardly agree with the usual derivation of formula (18), (see, for example, Ref.[3]) from instantaneous Galilean transformations. The point is that the transformations for coordinates (though it were Lorentz transformations or their particular case-Galilean ones) practically describe event pairs. The instantaneous Galilean transformation must describe the pair of events for which

$$\Delta x = x \quad \text{and} \quad \Delta t = 0. \tag{19}$$

However, as is known, the transition from the Lorentz transformations to Galilean ones takes place if

$$x \ll t. \tag{20}$$

That is why the introduction of the instantaneous Galilean transformations cannot be admitted as a consistent step.

*Covariant Galilean group**. As the above first order transformations (9) for covariant coordinates play a definite role in nonrelativistic physics and, in particular, in nonrelativistic quantum mechanics, we will especially touch upon the group properties of these transformations.

If the translations are taken into account, transformations (9) are written as

$$\begin{aligned} x &= x' \qquad\qquad + a, \\ t &= v\,x' + t' + b. \end{aligned} \tag{21}$$

* In this connection see also paper by Lewy-Leblond[4].

The group of covariant transformations (21), covariant Galilean group C, is generated by three one-dimensional subgroups, with their canonical parametrization:

— space translation (a),

— time translation (b), and

— pure covariant transformations (v).

From the space-time action of the group (21), one can derive a three-dimensional matrix representation δ

$$\delta(a, b, v) = \begin{pmatrix} 1 & 0 & a \\ v & 1 & b \\ 0 & 0 & 1 \end{pmatrix}. \tag{21'}$$

Hence, denoting the element of the group C by (a, b, v), for the law of group multiplication, we find

$$(a', b', v')(a, b, v) = (a' + a, b' + b + a v', v' + v). \tag{22}$$

The Lie algebra of the group C can be built by means of the representation δ. Thus we find the infinitesimal generators to have the matrix representation

$$P_\delta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for space translations,}$$

$$H_\delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for time translations,} \tag{23}$$

$$K_\delta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for pure covariant transformations.}$$

By exponentiation, easy here due to the nilpotent nature of these generators, we obtain for δ

$$\delta(a, b, v) = \exp[a P_\delta)] \exp[b H_\delta] \exp[v K_\delta].$$

The commutation relations of the generators, which define the Lie algebra of the covariant group C, take the form

$$[H, P] = 0, \tag{24 a}$$

$$[K, P] = H, \tag{24 b}$$

$$[K, H] = 0. \tag{24 c}$$

Here it is (24 b) which expresses the non-Abelian character of the group C.

It should be stressed that the time translation generator obviously belongs to the centre of the Lie algebra and thus it should be invariant in the framework of any irreducible representation.

Taking further the Baker-Campbell-Hausdorff formula

$$e^{-B} A e^{B} = A + [A, B] + \frac{1}{2!} [[A, B], B] + \dots \tag{25}$$

in the particular case ($B = -vK$) of pure transformations (9) for the transformation formulae P and H, we find

$$P = p, \tag{26 a}$$

$$H = m,$$

$$K = i m \frac{\partial}{\partial p}. \tag{26 b}$$

Taking into account (24 b) and (24 c), we conclude that P and H are transformed by the formulae for momentum and energy (3) in a Galilean approximation.

In this case, for example, in the momentum-energy representation for the (Hermitian) generators P, H and K, we have

$$P = p, \tag{27 a}$$

$$H = m, \tag{27 b}$$

$$K = i m \frac{\partial}{\partial p}. \tag{27 c}$$

Note that, according to (27 b), in the right-hand side of (24 b) in this case there appears the rest mass. However this does not lead to difficulties similar to those which arise when for the considered approximation in the formula for energy the next order term is taken into account (this is groundless, however).

In the coordinate-time representation the indicated generators read as

$$P = -i \frac{\partial}{\partial x_1}, \tag{28 a}$$

$$H = i \frac{\partial}{\partial x_0}, \tag{28 b}$$

$$K = i x_1 \frac{\partial}{\partial x_0}. \tag{28 c}$$

Let us note that the position operator X is correctly transformed under both a space translation $U(a)$

$$U(a) X U^{-1}(a) = X + a I \tag{29}$$

and pure transformations (9)

$$U(v) X U^{-1}(v) = X. \tag{30}$$

The latter, in particular, is connected with the validity of the relation

$$[X, K] = 0. \tag{31}$$

However, as is indicated earlier (see, e. g.,[5,6]), when we consider important questions on the invariance of the nonrelativistic equations of physics, which are in fact the second order approximate equations (so far as the transition to them — from the corresponding relativistic equations — assumes that, together with maximal terms, the addens, which are small with respect to them — of the order of β^2 — are taken into account as well) the first order transformations are already lacking. We must lean upon the transformation formulae for coordinates in which the second order terms are also taken into account.

2. *Second aproximation*

Approximate Lorentz transformations of the second order c^{-2}. The second order transformations are an extension of the Galilean transformations as a result of taking into account the terms of the order of β^2 (note that the indicated transformations were earlier considered e. g., by Fichtenholz[7a)], Fock [7b)] and Chandrasekhar & Contopolous[7c)]). They take the form

$$x = (x' + vt')\left(1 + \frac{1}{2} v^2\right), \quad t = t'\left(1 + \frac{1}{2} v^2\right) + v x'. \tag{1-II*}$$

* We simply add figure II to the formulae which are a second aproximation to the above expressions.

Using (1-II) and (2), we have for the transformation formulae of momentum and energy in the considered approximation

$$p = (p' + v E')\left(1 + \frac{1}{2} v^2\right), \quad E = E'\left(1 + \frac{1}{2} v^2\right) + v p'. \tag{3-II}$$

If now the framework of this approximation we pass to a range of problems discussed in the Section 1, we must again turn to covariant coordinates. It should be emphasized that just these second order covariant transformations should serve as the basis for considering the questions on the invariance of the nonrelativistic Hamilton-Jacobi equation and the Schrödinger one.

Second order covariant transformations. The outlined transformations represent a corresponding extension of transformations (9) due to the fact, that the terms of the order of β^2 are taken into account and are obtained directly from formulae (8) and (1-II). They take the form

$$x_1 = x_1'\left(1 + \frac{1}{2} v^2\right) + v x_4', \quad x_0 = (x_0' + v x_1'\left(1 + \frac{1}{2} v^2\right). \tag{9-II}$$

In order to establish commutation relations for the infinitesimal generator of pure covariant transformations of the second order (9-II), let us use formula (25). For values P and H in the approximation under consideration we have

$$P = P' + i a [P', K] + \frac{(i a)^2}{2!} [P, K], K] + \frac{(i a)^3}{3!} [[[P', K], K], K] + O(c^{-4}), \tag{26 a-II}$$

$$H = H' + i a [H', K] + \frac{(i a)^2}{2!} [[H', K], K] + O(c^{-4}), \tag{26 b-II}$$

where in the given approximation $a = v\left(1 + \frac{1}{3} v^2\right)$, $a^2 = v^2$, $a^3 = v^3$.

The underlined terms are the terms of the first order corresponding to the Galilean approximation.

A comparison of equations (26-II) with transformation formulae (3-II)' when

$$[K, P] = i H \tag{14b'}$$

is taken into account, results in the following commutation relation

$$[K, H] = i P, \tag{24c'-II}$$

which coincides with the corresponding one for the Lorentz group generators (in the general case we have a complete coincidence of the commutation relations with the corresponding ones for the Poincare group generators).

Let us now deal with several problems concerning the application of transformations (9-II) to nonrelativistic equations of physics.

As was shown earlier[5], the conventional nonrelativistic Hamilton-Jacobi equation (13-II) is invariant under the indicated transformations both in the free case and in the presence of electromagnetic field. At the same time the application of transformations (9-II) to the Schrödinger equation (see, e. g.,[6]) leads to certain difficulties.

In particular, the appearance of four additional terms ($\hbar = 1$)

$$\varDelta - \frac{\beta^2}{2} \frac{\partial \psi}{\partial \psi x_0'} - \beta \frac{\partial \psi}{\partial x_1'} + \frac{i}{2m} \left(2p \frac{\partial^2 \psi}{\partial x_1' x_0'} - \beta^2 \frac{\partial^2 \psi}{\partial x_0'^2} \right) \tag{32}$$

is a result of the transition to another reference frame using (9-II) for the Schrödinger equation (the mass term being taken into account)

$$i \frac{\partial \psi}{\partial x_0} = m\psi - \frac{1}{2m} \frac{\partial^2 \psi}{\partial x_1^2}. \tag{33}$$

Hence, in order that the equation under consideration may keep its form after this transition, it is evident we should require to fulfil the equation $\varDelta = 0$ which can be also written as

$$\frac{\beta^2}{2} \frac{\partial \Phi}{\partial x_0'} = \beta \frac{\partial \Phi}{\partial x_1'}, \tag{34}$$

where

$$\Phi = \psi - \frac{i}{m} \frac{\partial \psi}{\partial x_0'}. \tag{35}$$

It should be noted that recently Gaida[8] has also applied the second order transformations to the Schrödinger equation in the form of (33). He showed that Equ. (33) was invariant under the indicated transformations if

$$\Phi = 0. \tag{36}$$

However, it is necessary to stress that although, e. g., in the case of free particle, the condition (36) is sure to be fulfilled, the application of similar conditions asks for caution. Indeed, in the framework of (36) the left-hand part of (33) can be written as $-i\,(\psi\, \partial^2 \psi / x_0^2)^{1/2}$. It is the last expression that is a direct result of the transition* from the Klein-Gordon equation to the nonrelativistic limit[9].

* An analog of the above transition from Equ. (12) to (13-II).

As far as the above-mentioned transformation formula for probability density is concerned, in the considered approximation it is of the form

$$\varrho = \varrho' \left(1 + \frac{1}{2} v^2\right) + v j'. \tag{16-II}$$

As in this case $E = m + p^2/2m$, the above transition to the known expression (6) (even for free particle) is already impossible. Consequently, in a second approximation we must use the relativisic expression (17) for probability density (as it is so, we cannot avoid difficulties when deriving the continuity equation using the conventional Schrödinger equation).

Using the above results, in conlusion we would like to touch upon the question concerning the transformation of the spinless particle wave function under Galilean transformations (1).

As usualby, when one considers the indicated question, it is assumed that under the Galilean transformations the wave function transformation formula, that is, say, subject to the Schrödinger equation, is determined by the expression

$$\psi'_{Sch} = \exp(if)\, \psi_{Sch}, \tag{37}$$

where ψ' is the ψ-function which depends on the primed coordinates.

At first glance, one could agree with this statement as, to exclude the mass term in the wave equation, the known transformation must take place

$$\psi = \psi_{Sch} \exp(-i m t). \tag{38}$$

Using this and taking into account the equation

$$\psi' = \psi \tag{39}$$

we are actually led to (37). For phase f we have

$$f = i m (t' - t). \tag{40}$$

However, as it follows from the Galilean transformation formula for time, $f = 0$. In other words, this means that, as time is invariant under the Galilean transformations, the function ψ_{Sch} (as well as ψ), obtained due to the transformation inverse to (38), is in fact a scalar value. That is why the above statement concerning Equ. (37) cannot be called valid (in this connection see also Ref.[6] where, in particular, it is shown that for the correct transformation of the plane wave ψ-function there actually appears no additional phase in the nonrelativistic case for ψ).

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