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MOTION OF PARTICLES RELATIVE TO A SINGLE OBSERVER

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Abstract: Method of introduction of a special frame of reference — the single observer approach — is applied to problems in test particles machanics in general relativity. The equations of the first and the second deviations of a geodesic relative to an arbitrarily moving observer are obtained in the case of a »scalar« (non-rotating) particle. Similar approach is developed for description of rotating particles (generalization of the Papapetrou equation). Corresponding Lagrangians are considered.

1. Introduction

There is a number of definitions of FR (frame of reference) and each of them can be connected with certain set of problems. Considering mechanics of mass points we use here the FR of single observer (SO). The notion of FRSO was introduced earlier by a study of kinematics and observables such as gravitational energy etc[1,2]. The FRSO implies that on the Riemannian manifold there are given:

- the world line (WL) of SO;
- the transport paths of physical quantities (from outside onto WLSO);
- the transport procedure for physical quantities along these paths (in particular the parallel transport is here adequate);
- the definition of physical time and basis spatial directions of SO.

We consider here close vicinities of WLSO what corresponds to the case of weak gravitational field (with respect to the SO). Such approximation is sufficient in the Solar system scale, e. g. for study of the Mercury perihelion advance[3]. It is worth mentioning that this approximation was used by Bazansky to find the energy of a particle in gravitational fields up to the second order in deviation[4].

2. *The case of scalar test particle*

We use Lagrangian in order to describe the behaviour of particles. The action integral for a test particle is usually written as $S = -\int m_0 \, ds$ where m_0 is the rest mass of the particle and s is its proper time. However the use of the proper time of particle as a parameter leads to difficulties e. g. by a transition to mechanics of continua. It is more convenient to introduce an independent parameter which simplifies the variational technics and yields reasonable expression for the energy-momentum tensor. If the independent parameter becomes equal to proper time, the equations of motion have to coincide. This goal is achieved by difierent ways, e. g. Bartrum[5] demands the invariance of the Lagrangian under the parameter transformations. We shall avoid this limitation by applying the conditional extremum approach. In order to describe the free scalar test particle we use the action integral[6]

$$S = \int \frac{1}{2} m_0 \, u^\mu u^\nu g_{\mu\nu} \, \mathrm{d}t = \int u \, \mathrm{d}t, \tag{1}$$

where u^μ is the parametrical 4-velocity of the particle, $u^\mu u^\nu \pm 1$, t is an independent parameter along WL of the particle, $g_{\mu\nu}$ is the metric (sign $g_{\mu\nu} = -2$), Greek indices run the numbers 0, 1, 2, 3; Latin indices run the numbers: 1, 2, 3. The condition

$$u^\mu u_\mu = 1 \tag{2}$$

would lead to the variational principle for the action integral

$$\overline{S} = \int \frac{1}{2} (m_0 + \lambda) \, u^\mu u^\nu g_{\mu\nu} \, \mathrm{d}t,$$

where λ is a Lagrange multiplier found from the condition (2) $\frac{\mathrm{d}}{\mathrm{d}t} (m_0 + \lambda) = 0$. If $m_0 = \text{const.}$ then $\lambda = \text{const}$, i. e. we can put λ equal to zero and ignore the condition (2); then in transition to the proper time the parameter t should be replaced by s in the equations of motion.

Consider FRSO. Let $x^\mu = x^\mu (v^{(i)})$ be the transport trajectories of physical quantities to WLSO, $v^{(i)}$ being three parameters along the WL of transport,

$v^{(l)} = 0$ being the position of the observer. Suppose also that the condition $\frac{\mathrm{d}}{\mathrm{d}t}$ $v^{(l)} = 0$ is satisfied along the WL of the particle. Define the first deviation of the WL of the particle relative to WLSO as $l^\mu = \frac{\partial x^\mu}{\partial v^{(l)}} v^{(l)}$ and n-th deviation as

$$(n)\, l^\mu = \left(\frac{D^{(n-1)}}{\mathrm{d}\, v^{(l)\,(n-1)}}\, l^\mu\right) v_r^{(l)\,(n-1)}.$$

Find now the rule of transport. Let $\widetilde{A}_B(x)$ be a quantity obtained by transport onto WLSO. It satisfies the equation

$$\frac{D}{\mathrm{d}v} \widetilde{A}_B(x) \equiv \frac{\mathrm{d}}{\mathrm{d}v} \widetilde{A}_B + A_B \int\limits_\mu^\nu \Gamma^\mu_{\nu\lambda} \frac{\mathrm{d}x^\lambda}{\mathrm{d}v} = 0; \tag{3}$$

here $A_B \Big|_\mu^\nu = \frac{\partial}{\partial \xi^\mu_\nu}, (A'_B(x') - A_B(x))$ under infinitesimal transformations of co-ordinates $x'^\mu = x^\mu \pm \xi^\mu$ (see[7]). Formal solution of this equation has the form

$$\widetilde{A}_B(x) = \exp\left(l^\mu \frac{D}{\mathrm{d}x^\mu}\right) A_B = A_B + A_B;\, {}_\mu l^\mu + \frac{1}{2} (A_B;\, {}_\mu l^\mu)\, {}_\nu l^\nu + \ldots \tag{4}$$

(if the quantity A_B is given along its transport line). This is the definition of the Taylor expansion in the Riemannian space-time. Applying the formula (4) to the velocity of the particle we obtain

$$\widetilde{U}^\mu = \exp\left(l^\nu \frac{D}{\mathrm{d}x^\nu}\right) V^\mu = V^\mu + \dot{l}^\mu + \frac{1^{(a)}}{2}\, \dot{l}^\mu + \frac{1}{2} V^\alpha l^\beta l^\gamma R^\mu_{\gamma\rho\alpha} + \ldots, \tag{5}$$

where $V^\mu = \frac{\mathrm{d}y^\mu}{\mathrm{d}t}$ is the velocity of the observer.

Considering a scalar test particle we set the problem to find its equation of motion relative to SO up to the second deviation. In order to achieve this goal we expand the action integral (1) up to the third deviation with the help of (4) and (5)

$$L = \frac{m_0}{2} \Big((V^2 + 2\, \dot{l}_\mu V^\mu + {}^{(2)} \dot{l}_\mu V^\mu + \dot{l}_\mu \dot{l}^\mu + R_{\alpha\beta\gamma\delta} V^\alpha l^\beta l^\gamma V^\delta +$$

$$+ \frac{1}{3} {}^{(3)} \dot{l}_\mu V^\mu + {}^{(2)} \dot{l}_\mu \dot{l}^\mu + R_{\alpha\beta\gamma\delta} V^{\alpha\,(2)} l^\beta l^\gamma V^\delta + \frac{4}{3} \overline{R}_{\alpha\beta\gamma\delta} V^\alpha l^\beta l^\gamma \dot{l}^\delta +$$

$$+ \frac{1}{3} R_{\alpha\beta\gamma\delta;\,\sigma} V^\alpha l^\beta l^\gamma l^\delta V^\delta l^\sigma + \ldots \Big), \tag{6}$$

where $R_{\mu\nu\lambda\varrho}$ is the curvature tensor. Since WLSO is not subject to variation, $\delta x^\mu = 0$; the quantities δl^μ, $D_l \delta l^\mu$, $D_l^q \delta l^\mu$ are independent variations. It is also convenient to introduce the notion of »variation at a point«

$$\delta A_B(t) = \bar{\delta} A_B(t) + A_B |^\nu_\mu (t) \Gamma^\mu_{\nu\lambda} \delta x^\lambda,$$

where $\bar{\delta} A_B(t)$ is an ordinary variation. *The properties* should be pointed out

$$\left[\delta, \frac{D}{dt}\right] A_B = A_B |^\nu_\mu R^\mu_\nu \alpha\beta \; \delta x^\alpha V^B;$$

$$[\delta, D_l] A_B = A_B |^\nu_\mu R^\mu_{\nu\alpha\beta} \delta x^\alpha l^B,$$

where $[a, b] = ab - ba$. Putting the coefficient at the independent variation δl_μ equal to zero, we obtain the equation

$$-\frac{D}{dt}\left[m_0\left(V^\mu + \dot{l}^\mu + \frac{1}{2}\overset{(2)}{\cdot}\; \dot{l}^\mu + \frac{1}{2} R^\mu \cdot \alpha\beta\gamma\, l^\alpha l^B V^\gamma\right)\right] +$$

$$+ m_0 R^\mu \cdot \alpha\beta\gamma\, V^\alpha V^\beta l^\gamma + \frac{1}{2} m_0 R^\mu \cdot \alpha\beta\gamma\, V^\alpha V^{B(2)} l^\gamma + \tag{7}$$

$$+ \frac{1}{2} m_0 R^\mu \cdot \alpha\beta\gamma\, V^\alpha \dot{l}^\beta l^\gamma + m_0 R^\mu \cdot \alpha\beta\gamma\, l^\alpha V^\beta l^\gamma +$$

$$+ \frac{1}{2} m_0 R^\mu \alpha\beta\gamma;\; \delta\, V^\alpha V^\beta l^\gamma l^\delta = 0,$$

the coefficients at $D_l \, \delta \, l_\mu$ and $D^2 \;\; \delta \, l_\mu$ give only lower order approximations of the same equation, for they are infinitesimals of higher orders. These equations are written relative to time of the particle; the transition to the time of the observer meets no difficulty. Each following equation in the given form naturally differs from the preceding one only by addition of the next order in deviation because of the difference between orders of variations.

If the observer moves along a geodesic, and only the first deviation is taken into account, the equation (8) acquires the well known form of geodesic deviation $\frac{D^2}{dt^2} l^\mu = R^\mu \alpha\beta\gamma\, V^\alpha V^\beta l^\gamma$ (see Ref.[1].) As one of the present authors showed[7], the Lagrangian for this equation is $L = l_\mu V^\mu$, which coincides with the second term in (7). The expression (8) shows that the second deviation satisfies the equation

$$\frac{D^2}{dt^2} {}^{(2)} l^\mu + R^\mu \alpha\beta\gamma\, V^{\alpha\,(2)} l^\beta V^\gamma = (R^\mu \cdot \beta\gamma\delta;\; \varepsilon - R^\mu \cdot \delta\varepsilon\beta;\; \mu) \cdot$$

$$\cdot V^\beta V^\gamma l^\delta l^\varepsilon + 4 R^\mu \beta\gamma\delta\, l^\beta V^\gamma l^\delta, \tag{8}$$

which was first obtained by Bazansky[4].

3. Motion of a rotating test particle

The case of a scalar particle excluded its proper rotation, but the motion of a particle of nonzero proper angular momentum represents special interest. Equations of motion for this kind of particle were written by Papapetrou[8)]

$$\begin{cases} \frac{D}{\mathrm{d}t} P_\mu + \frac{1}{2} S^{\alpha\beta} U^\gamma R_{\alpha\beta\gamma\mu} = 0; \\ \frac{D}{\mathrm{d}t} S^{\mu\nu} - P^\mu U^\nu + P^\nu U^\mu = 0. \end{cases} \tag{9}$$

U^μ is here 4-velocity of the particle, $S^{\mu\nu}$ is its proper angular momentum, P_μ is its linear momentum found from the second equation (9) as

$$P_\mu = m\, U_\mu + \dot{S}_{\mu\nu}\, U^\nu / U^2, \quad m = \frac{P_\mu U^\mu}{U^2}. \tag{10}$$

It is important that the system (9) is non-complete and needs an auxiliary condition interpreted as a definition of the centre of mass of the particle. In a number of papers (e. g., by Dixon[9)]) it was shown that the centre of mass supplementary condition in the comoving system, in the sense of velocity: $S^{\alpha\beta} U_\beta = 0$, does not single out a unique solution, but the centre of mass condition in the comoving system in the sense of momentum

$$S^{\alpha\beta} P_\beta = 0 \tag{11}$$

solves the problem of uniqueness. It is shown in Ref. 10) that the »particle's WL« described with the help of the system (9) together with the condition (11) remains inside the convex body of the particle world tube when natural assumptions are taken.

Consider now the motion of the rotating particle relative to SO. The dynamic quantities have the form

$$P_\mu = P_\mu (U^\alpha, \omega^{\sigma\tau}, m); \quad S^{\mu\nu} = S^{\mu\nu} (U^\alpha \omega^{\sigma\tau} m), \tag{12}$$

where $\omega^{\mu\nu} = \dot{g}_{(a)}{}^{\mu} g^{(a)\nu}$ is the angular velocity of the particle, $g_{(a)}\,\mu$ is the tetrad rigidly connected with the particle and rotating with it. Since the tetrad is transported parallelly onto WLSO, then

$$D_t\, g_{(a)}{}^{\mu} = 0 \tag{13}$$

and the observed angular velocity up to the first deviation equals

$${}^{(1)}\omega^{\mu\nu} = \omega^{\mu\nu} + R^{\mu\nu} \cdot \alpha\, \beta\, V^\alpha l^\beta. \tag{14}$$

From the expressions (5), (12)—(14), follows that the dynamic observables have the form

$$ {}^{(1)}P_\mu = P_\mu + \frac{\partial P_\mu}{\partial U^\alpha} \dot{l}^\alpha + \frac{\partial P_\mu}{\partial \omega_{\alpha\beta}} R_{\alpha\beta\gamma\delta} V^\gamma l^\delta, $$

$$ {}^{(1)}S^{\mu\nu} = S^{\mu\nu} + \frac{\partial S^{\mu\nu}}{\partial U^\alpha} \dot{l}^\alpha + \frac{\partial S^{\mu\nu}}{\partial \omega_{\alpha\alpha}} R_{\alpha\beta\gamma\delta} V^\gamma l^\delta. \tag{15} $$

Applying the operation (4), (5) to the system of equations (9) we obtain the equations of motion of the rotating particle relative to SO where the first deviation is taken into account

$$ -\frac{D}{dt} {}^{(1)}P_\mu - R^\alpha{}_{.\mu\beta\gamma} P_\alpha V^\beta l^\gamma + \frac{1}{2} {}^{(1)}S^{\alpha\beta} V^\gamma R_{\alpha\beta\gamma\mu} + $$

$$ + \frac{1}{2} S'^{\alpha\beta} \dot{l}^\gamma R_{\alpha\beta\gamma\mu} + \frac{1}{2} S^{\alpha\beta} V^\gamma l^\delta R_{\alpha\beta\gamma\mu,\delta} = 0; $$

$$ \frac{D}{dt} {}^{(1)}S^{\mu\nu} + S^{\alpha\nu} R^\mu{}_{.\alpha\beta\gamma} l^\beta V^\gamma + S^{\mu\alpha} R^\nu{}_{.\alpha\beta\gamma} l^\beta V^\gamma - $$

$$ - {}^{(1)}P^\mu V^\nu + {}^{(1)}P^\nu V^\mu - P^\mu \dot{l}^\nu + P^\nu \dot{l}^\mu = 0; $$

$$ \frac{D}{dt} S^{\mu\nu} + P^\mu V^\nu - P^\nu V^\mu = 0. \tag{16} $$

The same result can be obtained, as in the case of a scalar particle, from the Lagrangian approach if the Lagrangian for the system (9) found by Bartrum[5] and analysed in the papers[11,12], is used

$$ L = L(V, g_{\alpha\mu}, \omega_{\mu\nu}, m_A, \dot{m}_A). \tag{17} $$

Here m_A are moments of mass in the proper tetrad of the particle. Expanding (17) up to the second deviation we obtain the Lagrangian for the system (16) in the form

$$ {}^{(1)}L = L + \frac{\partial L}{\partial V^\alpha} \dot{l}^\alpha + \frac{\partial L}{\partial \omega_{\mu\nu}} R_{\mu\nu\alpha\beta} V^\alpha l^\beta + \frac{1}{2} \frac{\partial L}{\partial V^\alpha} \cdot ({}^{(2)}\dot{l}^\alpha + R^\alpha{}_{.\beta\gamma\delta} l^\beta l^\gamma V^\delta) + $$

$$ + \frac{1}{2} \frac{\partial L}{\partial \omega_{\mu\nu}} (R_{\mu\nu\alpha\beta;\gamma} V^\alpha l^\beta l^\gamma + R_{\mu\nu\alpha\beta} \dot{l}^\alpha l^\beta + R_{\mu\nu\alpha\beta} V^{\alpha(2)} l^\beta) + \frac{1}{2} \frac{\partial^2 L}{\partial V^\alpha \partial V^\beta} \dot{l}^\alpha \dot{l}^\beta + $$

$$ + \frac{1}{2} \frac{\partial^2 L}{\partial \omega_{\mu\nu} \partial \omega_{\lambda\varrho}} R_{\mu\nu\alpha\beta} V^\alpha l^\beta R_{\lambda\varrho\gamma\delta} V^\gamma l^\delta + \frac{\partial^2 L}{\partial V^\alpha \partial \omega_{\mu\nu}} R_{\mu\nu\gamma\beta} V^\gamma l^\beta \dot{l}^\alpha. \tag{18} $$

Note that the momentum defined from (18) differs to some extent from (15) ${}^{(1)}\overline{P} = {}^{(1)}P_\mu - \frac{1}{4} S^{\alpha\beta} l^\gamma R_{\alpha\beta\gamma\mu}$, but nevertheless the equations obtained from (18) and from (16) are equivalent.

The system (16) can be simplified if we notice that up to the first deviation

$$- \frac{D}{dt}{}^{(1)}P_\mu - R^\alpha{}_{.\mu\beta\gamma}{}^{(1)}P_\alpha V^\beta l^\gamma + \frac{1}{2}{}^{(1)}S^{\alpha\beta} V^\gamma R_{\alpha\beta\gamma\mu} +$$

$$+ \frac{1}{2}{}^{(1)}S^{\alpha\beta} \dot{l}^\gamma R_{\alpha\beta\gamma\mu} + \frac{1}{2}{}^{(1)}S^{\alpha\beta} V^\gamma l^\delta R_{\alpha\beta\gamma\mu;\,\delta} = 0;$$

$$\frac{D}{dt}{}^{(1)}S^{\mu\nu} + {}^{(1)}S'^{\alpha\nu} R^\mu{}_{.\alpha\beta\gamma} l^\beta V^\gamma + {}^{(1)}S^{\mu\alpha} R^\nu{}_{\alpha\beta\gamma} l^\beta V^\gamma -$$

$$- {}^{(1)}P^\mu V^\nu + {}^{(1)}P^\nu V^\mu - {}^{(1)}\dot{P}^\mu l^\nu + {}^{(1)}P^\nu \dot{l}^\mu = 0$$

(we excluded »non-observable« quantities).

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