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HARMONIC ANALYSIS ON THE LORENTZ GROUP AS DYNAMICAL RELATIVISTIC FOURIER ANALYSIS

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Abstract: A dynamical relativistic Fourier analysis connecting both the relative momentum and the relativistic coordinate has been constructed in the framework of the theory of Galilei and Poincare group unitary representations and the concept of »geometrization« of the relativistic two-body problem. The usual three-dimensional quantum-mechanical language of the formalism enables one to obtain easily some consequences for relativistic quantum mechanics.

1. *Introduction*

The aim of this paper is to formulate a three dimensional dynamical relativistic Fourier analysis that connects both the relative relativistic momentum and the coordinate. Such a formalism is expected to be effective in describing high-energy phenomena.

The possibility of stating such a problem is based on the correspondence principle, the theory of unitary representations of kinematical space-time groups (Galilei and Poincaré groups) and the concept of »geometrization« of the relativistic two-body problem.

The principle of relativity means that a unitary representation of the kinematical space-time group (Galilei or Poincaré) is realized in the space of wave functions of a quantum system[1,2].

Reduction[3] of the representation of a subgroup of translations to irreducible representations leads to a harmonic (Fourier) analysis with a four-dimensional

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plane wave $e^{i(Et-\vec{P}\cdot\vec{X})}$ both in the relativistic and in the »nonrelativistic« case, but with a different connection between the energy and momentum of the system.

Nonrelativistic quantum mechanics is a welldeveloped scheme for description of atomic systems. This is due to the fact that the Schrödinger equation exists and the way of introducing the interaction is known.

Analyzing a two-particle stationary system on the basis of the Schrödinger equation, one can find (after separating the variables into the centre-of mass variables and the relative ones) that the dynamical information is contained in the wave function of relative motion.

Thus, the two-particle problem is reduced to the problem of motion of one »effective« particle in an (effective) potential field.

A unit representation of a Galilei group (if one is not interested in rotations) is realized in the space of »relative« wave functions.

A group of translations in the space of relative momentum is isomorphic to the Galilei boost group, but it is not a group of covariance for the effective Schrödinger equation.

Reduction of unitary representations of such a group into irreducible ones leads to a harmonic analysis with a three-dimensional plane wave $e^{i\vec{q}\cdot\vec{r}}$. This harmonic analysis will be called the dynamical nonrelativistic Fourier analysis; it connects both the relative momentum and coordinate.

Consequently, if the geometry of the relative momentum space for some two-particle problem is known, the above group-theoretical construction can be used to define the relative coordinate.

In the framework of the quasipotential approach[4)], a relativistic analog of the effective wave function was introduced. However, the development of the quasipotential approach[5)] has shown that the effective wave function cannot be defined uniquely because of the nonuniqueness[6)] of the offmass-shell continuation.

In the quasipotential approach[7)], proposed by Kadishevsky on the basis of a special type of Hamiltonian formulation of quantum field theory, relativistic analogs of the Schrödinger and the Lippman-Schwinger equations were obtained[8)]. The only difference between the nonrelativistic and relativistic cases is that the three-dimenstional Euclidean momentum space is used in the former case, while the three-dimensional Lobachevsky momentum space is employed in the latter.

According to the correspondence principle and in agreement with the approach[7 8)] one can postulate that the relativistic effective wave function is defined in the Lobachevsky momentum space[9)].

The group of motion in this space is isomorphic to the kinematical Lorentz group but is not a group of covariance. The harmonic analysis on this Lorentz group leads to a dynamical relativistic Fourier analysis that connects both the relativistic relative momentum and coordinate.

The relativistic Fourier analysis, formulated in a form suitable for the quasipotential approach[10)], has been successfully applied for parametrization of the nucleon form factor[11)].

In Section 1, the formulas of the nonrelativistic dynamical Fourier analysis are discussed from the group-theoretical point of view. The role of the plane wave for deriving the uncertainty relation between the momentum and the coordinate is also noted.

The formulas of the relativistic dynamical Fourier analysis are given in Section 2 (see Refs.[3]).

In Section 3, the angle dependence of the wave function is studied as a methodological example of application of such a technique. The Hamiltonian of a »free effective« particle with arbitrary spin is constructed. The relativistic generalization (in the above sense) of the Heinsenberg relation, namely the uncertainty relation for rapidity and the relativistic coordinate is derived. It should be pointed out that a similar example of uncertainty relation follows from the solution of the potential-well problem[8]. The experimentally observed transition from the exponential to power behaviour of the elastic scattering amplitude with increasing momentum transfer is interpreted as a transition from Euclidean to Lobachevsky geometry.

1. Nonrelativistic Fourier analysis

Plancherel's theorem and its group-theoretical sense. Plancherel's theorem[1 2] reads: Let a function $\Psi(\vec{p})$ be square integrable

$$\int d^3 \vec{p}\, |\Psi(\vec{p})|^2 < \infty.$$

Then there exists a function $\Psi(\vec{r})$ such that

$$\int d^3 \vec{r}\, |\Psi(\vec{r})| < \infty$$

and

$$\Psi(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3 p\, e^{i\vec{p}\cdot\vec{r}}\, \Psi(\vec{p}). \tag{1}$$

Moreover,

$$\int d^3 r\, |\Psi(\vec{r})|^2 = \int d^3 \vec{p}\, |\Psi(\vec{p})|^2$$

and we have

$$\Psi(\vec{p}) = \frac{1}{(2\pi)^{3/2}} \int d^3 \vec{r}\, e^{-i\vec{p}\cdot\vec{r}}\, \Psi(\vec{r}).$$

We shall emphasize some group-theoretical aspects of this theorem.

Let a representation of the Galilei boosts be given in p-representation

$$T_q \Psi(\vec{p}) = \Psi(\vec{p} - \vec{q}). \tag{3}$$

With the help of formula (2) and the »addition theorem« for the plane wave

$$e^{-i\vec{p}\cdot\vec{r}}\,e^{i\vec{q}\cdot\vec{r}} = e^{-i(\vec{p}-\vec{q})\cdot\vec{r}}, \tag{4}$$

one can obtain that the reducible representation T_q is decomposed in terms of irreducible representations $T_q^{[r]}$ which act on a function in coordinate space in the following way

$$T_q^{[r]}\,\Psi(\vec{r}) = e^{-i\vec{q}\cdot\vec{r}}\,\Psi(\vec{r}). \tag{5}$$

The orthogonality and completeness conditions for plane waves follow from Equs. (1) and (2)

$$\begin{aligned} \frac{1}{(2\pi)^3}\int d^3\vec{r}\, e^{i(\vec{p}-\vec{q})\cdot\vec{r}} &= \delta(\vec{p}-\vec{q}),\\ \frac{1}{(2\pi)^3}\int d^3 p\, e^{i\vec{p}(r-r')} &= \delta(\vec{r}-\vec{r}) \end{aligned} \tag{6}$$

As it is well known in quantum mechanics, formulas (6) can be proved using the expansion

$$e^{i\vec{p}\cdot\vec{r}} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \mathcal{J}_l(\mathrm{pr})\,\overline{Y}_{lm}(\vec{p})\,Y_{lm}(\vec{r}) \tag{7}$$

and orthogonality and completeness conditions for Bessel and spherical functions.

It is interesting to point out that the usual three-dimensional exponent is a generating function for the boost matrix element (Bessel functions) if the basis functions are chosen to be spherical functions.

Parseval's theorem and its group-theoretical sense. Parseval's theorem[12]: If the functions $\Psi(\vec{p})$, $\Phi(\vec{p})$ and their convolution $\int d^3\vec{p}\,\Psi(\vec{p})\Phi(\vec{p}-\vec{q})$ are square integrable, then

$$\int d^3\vec{r}\,\overline{\Psi}(r)\,\Phi(\vec{r})\,e^{i\vec{q}\cdot\vec{r}} = \int d^3\vec{p}\,\overline{\Psi}(\vec{p})\,\Phi(\vec{p}-\vec{q}).$$

This formula shows that Parseval's theorem is an »isometric« statement for both p- and r-spaces with respect to the boost

$$\int d^3\vec{r}\,\overline{\Phi}(\vec{r})\,T_q^{[r]}\,\Phi(\vec{r}) = \int d^3\vec{p}\,\overline{\Psi}(p)\,T_q\,\Phi(\vec{p}).$$

The uncertainty relation. Formally, the uncertainty relation is a consequence of the fact that the commutator of the momentum and coordinate operators is a c-number

$$[\vec{p}, \vec{r}] = -i, \tag{9}$$

and that the integral

$$\int d^3 \vec{r} \, |(a\hat{r} + \hat{p}) \Psi(\vec{r})|^2,$$

where a is an arbitrary positive number, is positive definite.

On the other hand, the commutator (9) can be derived making use of the diagonality of the momentum and coordinate operators in p- and r-representations, respectively, and that the wave functions in $\vec{p}$- and $\vec{r}$-representations are connected by the Fourier transformation (1). Thus, the commutator (9) is a consequence of the type of the plane wave. Consequently, the uncertainty relation also results from the type of the nonrelativistic plane wave

$$\varepsilon_{NR}^{(o)}(\vec{p}, \vec{r}) = e^{i\vec{p}\cdot\vec{r}}.$$

2. *Relativistic Fourier analysis*

Fourier transform connecting the momentum space and the relativistic coordinate space. Let the function $\Psi_\mu{}^{(s)}(\vec{p})$ be a relativistic wave function of an (effective) particle with mass m, spin s, spin projection μ, and momentum $\vec{p}$, belonging to the Lobachevsky space $p_0^2 - \vec{p}^2 = m^2$, $p_0 > 0$.

The representation of the Lorentz group is

$$T_q \Psi_\mu{}^{(s)}(\vec{p}) = \sum_{\mu'=-s}^{s} D^{(s)}_{\mu\mu'}[V(q, p)) \Psi^{(s)}_{\mu'}(\vec{p}(-)\vec{q}), \tag{10}$$

where the Wigner rotation $V(q, p)$ is

$$V(q, p) = B_p^{-1} B_q B_{p(-)q} = V^{-1}(p, q) \tag{11}$$

and

$$B_p = \frac{m + p}{\sqrt{2m(m + p_0)}}, \quad p = p_0 - \vec{p}\cdot\vec{\sigma}$$

$$\vec{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\frac{p_0 q_0 - \vec{p}\cdot\vec{q}}{m_q}$$

$$p(-)q \equiv \Lambda_q^{-1} p = \left(p - \frac{\vec{q}}{m_q}\,\frac{m_q p_0 + p_0 q_0 - \vec{p}\cdot\vec{q}}{m_q + q_0}\right), \tag{12}$$

and the $D^{(s)}$ function is a matrix element of the unitary representation of the SU(2) group with weight s. The representation (10) is unitary with respect to the scalar product

$$\sum_{\mu=-s}^{s} \int \frac{d^3\vec{p}}{2p_0} \overline{\Psi_\mu^{(s)}(\vec{p})}\, \Phi_\mu^{(s)}(\vec{p}). \tag{13}$$

In spherical coordinate one has

$$p = \begin{pmatrix} m & \operatorname{ch}\chi_p \\ m & \operatorname{sh}\chi_p \vec{n}_p \end{pmatrix},$$

where $\chi_p = \ln((p_0 + p)/m)$. It is clear from Equ. (12) that the transformation of rapidity

$$\chi_p \to \chi_{p(-)q} = \chi_{q(-)q}$$

corresponds to the Lorentz boost of momentum p

$$p \to p(-)q.$$

The reduction of the representation (10) into irreducible $[\nu,r]$ representations $(\nu = -s, \ldots, s,\ 0 \leq r \leq \infty)$ is given by the formulas[10]

$$\Psi_\mu^{(s)}(\vec{p}) = \frac{1}{(2\pi)^{3/2}} \sum_{\nu=-s}^{s} \int_0^\infty (\nu + r^2)\,dr \int d^2\vec{r}\, \zeta_{\mu\nu}^{\dagger(s)}(\vec{p},\vec{r})\, \Psi_\nu^{(s)}(r), \tag{14a}$$

$$\Psi_\mu^{(s)}(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \sum_{\nu=-s}^{s} \int \frac{d^3\vec{p}}{2p_0}\, \zeta_{\mu\nu}^{(s)}(\vec{p},\vec{r})\, \Psi_\nu^{(s)}(\vec{p}), \tag{14b}$$

where

$$d^2\vec{r} = \sin\vartheta\, d\vartheta\, d\varphi, \quad \vec{r} = r\vec{n}, \quad \vec{n} = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix},$$

and

$$\xi^{(s)}_{\mu\nu}(\vec{p},\vec{r}) = \xi^{(0)}(\vec{p},\vec{r})\, D^{(s)}_{\mu\nu}(\vec{r}(-)\vec{p}), \quad \xi^{(0)}(\vec{p},\vec{r}) = (p_0 - \vec{p}\cdot\vec{n})^{-1-ir}, \tag{15}$$

where

$$\vec{r}(-)\vec{p} = r\vec{n}(-)\vec{p}, \quad \vec{n}(-)\vec{p} = \frac{1}{p_0 - \vec{p}\cdot\vec{n}}\left(\vec{n} - \vec{p}\,\frac{m + p_0 - \vec{p}\cdot\vec{n}}{m + p_0}\right).$$

Obviously, (compare with Equs. 1 and 2), the functions $\xi^{(s)}_{\mu\nu}(\vec{p},\vec{r})$ play the role of »relativistic plane waves«.

The orthogonality and completeness conditions for the functions $\xi^{(s)}(\vec{p},\vec{r})$ have the form

$$\frac{1}{(2\pi)^3}\sum_{\nu=-s}^{s}\int_0^\infty (\nu^2 + r^2)\, dr \int d^2\vec{r}\, \xi^{\dagger(s)}_{\mu\nu}(\vec{p},\vec{r})\, \xi^{(s)}_{\nu\sigma}(\vec{p},\vec{r}) = 2\delta_{\mu\sigma}\,\delta(\vec{p}(-)\vec{q}), \tag{16}$$

$$\frac{1}{(2\pi)^3}\sum_{\nu=-s}^{s}\int \frac{d^3\vec{p}}{2p_0}\, \xi^{(s)}_{\mu\nu}(\vec{p},\vec{r})\, \xi^{\dagger(s)}_{\nu\sigma}(\vec{p},\vec{r}') = \delta_{\mu\sigma}\,\frac{r^2}{\mu^2 + r^2}\,\delta(r - r').$$

As in the nonrelativistic case, these relations can be proved on the basis of the expansion

$$\xi^{(s)}_{\mu\nu}(p, r) = \sum_{J=s}^{\infty}\sum_{M=-J}^{J} d^{[\nu,r]}_{Js\mu}(\chi_p)\, D^{\dagger(J)}_{\mu M}(\vec{p})\, D^{(J)}_{M\nu}(\vec{r}), \tag{17}$$

using the orthogonality and completeness relations for both the matrix elements of the rotation group $D^{(J)}$ and the Lorentz boosts $d^{[\nu,r]}_{Js\mu}$.

Unlike the addition theorem for $\xi^{(0)}_{NR}(\vec{p},\vec{r})$, the addition theorem for the relativistic plane wave takes place under the sign of integration

$$\int d^2\vec{r}\, \xi^{\dagger(s)}(\vec{p},\vec{r})\, \xi^{(s)}(\vec{q},\vec{r}) = D^{(s)}[V(q,p)] \int d^2\vec{r}\, \xi^{\dagger(s)}(\vec{p}(-)\vec{q},\vec{r})\, D^{(s)}(\vec{r}). \tag{18}$$

The scalar product in coordinate space has the form

$$\sum_{\nu=-s}^{s}\int (\nu^2 + r^2)\, dr\, d^2\vec{r}\, \overline{\Psi_\nu^{(s)}(r)}\, \Phi_\nu^{(s)}(r).$$

The nonrelativistic limits of formulas (14) are (up to rotations in r space) formulas (1) and (2) for every spin-projection component. The limits of formulas (16) and (17) are exactly formulas (6) and (7).

Plancherel's theorem. In the relativistic Fourier analysis, Plancherel's theorem is the equality

$$\sum_{\nu=-s}^{s} \int_0^{\infty} (\nu^2 + r^2)\,\mathrm{d}r \int \mathrm{d}^2 r \,|\Psi_\nu^{(s)}(\vec{r})|^2 = \sum_{\nu=-s}^{s} \int \frac{\mathrm{d}^3\vec{p}}{2p_0}\,|\Psi_\nu^{(s)}(\vec{p})|^2.$$

This can be proved explicitly using Equs. (14) and (16).

The Fourier transformation of the »Lorentz boost«. In the nonrelativistic case, the irreducible representation of the boost group was derived by calculating the Fourier transformation of the function

$$T_q \Psi(\vec{p}) = \Psi(\vec{p} - \vec{q}).$$

Analogously, in the relativistic case, the irreducible representation in r-space can be derived by solving one of the conditions

$$T_q \Psi_\mu^{(s)}(\vec{p}) = \frac{1}{(2\pi)^{3/2}} \sum_{\nu=-s}^{s} \int (\nu^2 + r^2)\,\mathrm{d}r\,\mathrm{d}^2\vec{r}\,\xi_{\mu\nu}^{+(s)}(\vec{p},\vec{r})\,T_q^{[\nu,r]}\,\Psi_\nu^{(s)}(r),$$

$$T_q^{[\nu,r]}\,\Psi_\mu^{(s)}(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \sum_{\nu=-s}^{s} \int \frac{\mathrm{d}^3\vec{p}}{2p_0}\,\xi_{\mu\nu}^{(s)}(\vec{p},\vec{r})\,T_q \Psi_\nu^{(s)}(\vec{p}).$$

The »invariance« of the measures

$$\frac{\mathrm{d}^3\vec{p}}{2p_0} = \frac{\mathrm{d}^3\vec{p}(-)\vec{q}}{2(p(-)q)_0},\quad \mathrm{d}^2\vec{r} = (q_0 - \vec{q}\cdot\vec{n})^2\,\mathrm{d}^2\vec{r}(-)\vec{q} \tag{19}$$

and the properties of the Wigner rotation (11) permits one to obtain

$$T_q^{[\nu,r]}\,\Psi_\nu^{(s)}(\vec{r}) = \xi^{(0)}(\vec{p},\vec{r})\,\Psi_\nu^{(s)}(\vec{r}(-)\vec{q}). \tag{20}$$

Thus, the Lorentz boosts change the direction of the relative coordinate but do not change its spin projection and modulus.

Parseval's theorem. It follows from formulas (14), (16) and (18) that*)

$$\int (\nu^2 + r^2)\, dr\, d^2\vec{r}\, \overline{\Psi_\nu^{(s)}}(\vec{r})\, T_q^{[\nu,r]}\, \Phi_\nu^{(s)}(r) \equiv$$

$$\equiv \int (\nu^2 + r^2)\, dr\, d^2\vec{r}\, \overline{\Psi_\nu^{(s)}}(\vec{r})\, \Phi_\nu^{(s)}(\vec{r}(-)\vec{q})\, \xi^{(0)}(\vec{q}, \vec{r}) =$$

$$= \int \frac{d^3\vec{p}}{2p_0}\, \overline{\Psi_\nu^{(s)}}(\vec{p})\, D_{\nu\sigma}^{(s)}[V(q,p)]\, \Phi_\gamma^{(s)}(\vec{p}(-)\vec{q}) \equiv$$

$$\equiv \int \frac{d^3\vec{q}}{2p_0}\, \overline{\Psi_\nu^{(s)}}(\vec{p})\, T_q\, \Phi_\nu^{(s)}(\vec{p}),$$

in full analogy with the nonrelativistic case in the Section 1.

However, one can note that in the relativistic case the Fourier transformation of two wave functions in r-space is not the convolution of the wave functions in p-space. This is connected with the facts that the Lorentz boosts change the direction of the coordinate and the wave function is multiplied by the spin-zero plane wave $\xi^{(0)}(\vec{p}, \vec{r})$ but not by the spin-s plane wave $\xi^{(s)}(\vec{p}, \vec{r})$.

Finally, it is interesting to note that the integral over r-space of more than two plane waves is not a δ-function, in contrast to the nonrelativistic case.

3. *Some applications*

Angle dependence of the relativistic wave function for a particle with arbitrary spin. Let us examine formulas (14) under the assumption that the wave functions on the right-hand side have no angle dependence. Then, taking into account formulas (11), (19) and the expansion (17) and integrating over angles, one can obtain that the left-hand sides of Equs. (16) are equal to zero when the spin projection is non-zero. This contradiction is an illustration of the intuitively obvious physical fact that the wave function of relative motion always has an angle dependence when the spin is a half-integer or an integer, but the spin-projection is non-zero.

Hamiltonian of a »free effective« particle with arbitrary spin[18]. In the case $s = 0$ it is easy to check that[8]

$$H_0^{(0)}\, \xi^{(0)}(\vec{p}, \vec{r}) = 2p_0\, \xi^{(0)}(\vec{p}, \vec{r}),$$

where

$$H_0^{(0)} = 2\,\mathrm{ch}\, i\partial_r + \frac{2i}{r}\, \mathrm{sh}\, i\partial_r + \frac{1}{r^2}\, \vec{L}^2 \exp i\partial_r,$$

*) The sign for summation over spin-projection indices is here omitted.

and $\vec{L}$ is the generator of the SU(2) group. In Ref.[19], explicit expressions for the operators $\hat{\vec{p}}^{(0)}$

$$\hat{\vec{p}}^{(0)} \xi^{(0)} (\vec{p}, \vec{r}) = \vec{p}\, \xi^{(0)} (\vec{p}, \vec{r})$$

were obtained in spherical coordinates

$$p_1^{(0)} = \sin\vartheta \cos\varphi \frac{H_0^{(0)}}{2} - [\sin\vartheta \cos\varphi + \frac{i}{r} (\cos\vartheta \cos\varphi\, \partial_\Theta - \frac{\sin\varphi}{\sin\vartheta} \partial_\varphi)] \exp i\partial_\sigma,$$

$$p_2^{(0)} = \sin\vartheta \sin\varphi \frac{H_0^{(0)}}{2} - [\sin\vartheta \sin\varphi - \frac{i}{r} (\cos\vartheta \sin\varphi\, \partial_\Theta - \frac{\cos\varphi}{\sin\vartheta} \partial_\varnothing)] \exp i\partial_\sigma,$$

$$p_3^{(0)} = \cos\vartheta \frac{H_3^{(0)}}{2} - [\cos\vartheta - \frac{i}{r} \sin\vartheta\, \partial_\varphi] \exp i\partial_\sigma.$$

In the case of arbitrary spin, the plane wave of a free particle has the form

$$\xi^{s,\nu} (\vec{p}, \vec{r}) = \xi^{(s)} (\vec{p}, \vec{r})\, \chi^{s,\nu}.$$

The spinor $\chi^{s,\nu}$ is defined by the equations

$$S_\pm\, \chi_s^{s,\nu} = [(s \mp \nu)(s \pm \nu + 1)]^{1/2} \chi^{s,\nu\pm 1}, \qquad S_3\, \chi^{s,\nu} = \nu\, \chi^{s,\nu}.$$

The vector matrix operator $\vec{S}$ is defined from the equality

$$\vec{S} = D^{(s)} (\vec{r})\, \vec{L}\, D^{\dagger(s)} (r),$$

where the generator $\vec{L}$ operates on the $D^{(s)}$ function only.

Let us consider the operators

$$\vec{J}^{(s)} = D^{(s)} (\vec{r}(-)\vec{p})\, \vec{L}\, D^{\dagger(s)} (\vec{r}(-)\vec{p}),$$

$$H_0^{(s)} = D^{(s)} (\vec{r}(-)\vec{p} H_0^{(0)} D^{\dagger(s)} (\vec{r}(-)\vec{p}), \tag{21}$$

$$\vec{P}^{(s)} = D^{(s)} (\vec{r}(-)\vec{p})\, \vec{P}^{(0)} D^{\dagger(s)} (\vec{r}(-)\vec{p}).$$

They satisfy the commutation relations of Poincaré algebra for both the angular momentum generator and for the four-dimensional momentum generator

$$[\vec{J}^{(s)}, \vec{J}^{(s)}] = i\vec{J}^{(s)}, \quad [\vec{J}^{(s)}, H_0^{(s)}] = 0,$$

$$[\vec{J}^{(s)}, \vec{P}^{(s)}] = i\vec{P}^{(s)}, \quad [H_0^{(s)}, \vec{P}^{(s)}] = 0, \quad [\vec{P}^{(s)}, \vec{P}^{(s)}] = 0.$$

From the explicit expressions it follows

$$H_0^{(s)} \xi^{s,\nu}(\vec{p}, \vec{r}) = 2p_0 \xi^{s,\nu}(\vec{p}, \vec{r}), \quad \vec{P}^{(s)} \xi^{s,\nu}(\vec{p}, \vec{r}) = \vec{p}\, \xi^{s,\nu}(\vec{p}, \vec{r}).$$

The nonrelativistic limit of the first equation is

$$[-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2}(\vec{L} + \vec{S})^2]\, e^{i\vec{p}\cdot\vec{r}} D^{(s)}(\vec{r})\, \chi^{s,\nu} = P^2 e^{i\vec{p}\cdot\vec{r}} D^{(s)}(\vec{r})\, \chi^{s,\nu}.$$

After using the operator equality

$$D^{(s)}(\vec{r})\, \vec{L}\, \vec{D}^{\dagger(s)}(\vec{r}) = \vec{L} + \vec{S},$$

one has

$$\vec{P}^2 e^{i\vec{p}\cdot\vec{r}} \chi^{s,\nu} = \vec{p}^2 e^{i\vec{p}\cdot\vec{r}} \chi^{s,\nu},$$

where $\vec{P}^2$ is the usual three-dimensional Laplacean.

The uncertainty relation for rapidity and relativistic coordinates. Let $s = 0$. Then, $\psi(\vec{r}) = \psi(r)$ and one obtains

$$\psi(p) = \frac{2}{\sqrt{2\pi}} \frac{1}{p} \int_0^\infty r\, dr \sin \chi_p r\, \psi(r),$$

where $\chi_p = \ln \frac{p_0 + p}{m_p}$.

Let us further propose that (as in the nonrelativistic case) the rapidity operator is diagonal in P-representation and that the coordinate operator is diagonal in r-representation. Proceeding in the same way as in Sec. 1, we obtain the operator of rapidity in r-representation

$$\hat{\chi} = -i\left(\partial_r + \frac{1}{r}\right).$$

Therefore, the commutation relations between the rapidity and relativistic coordinates are the same as between the nonrelativistic momentum and coordinate

$$[\hat{\chi}, \hat{r}] = -i.$$

Hence it follows, in complete analogy with quantum mechanics, that the uncertainty relation for the rapidity and relativistic coordinates is of the form

$$\Delta \chi \Delta r \geq \frac{1}{2} \frac{\hbar}{2m}.$$

Geometrical interpretation of the exponential-power behaviour of the elastic scattering amplitude. In the case of the Schrödinger equation, the Born elastic-scattering amplitude can be derived as a Fourier image of the potential

$$T(p, q) = -\frac{m}{2\pi} \int e^{-i(\vec{p}-\vec{q})\vec{r}} V(r) \, d^3r.$$

In the relativistic case we know neither the equation nor the interaction potential.

Let us suppose that a quasipotential exists in the relative coordinate space and

$$\int V(s, r) \, d^3 r < \infty,$$

where s is the invariant energy squared of the two-particle system.

For simplicity, we neglect the influence of the spin in the relativistic case and obtain

$$T(\vec{p}, \vec{q}) = -\frac{m}{2\pi} \int \xi^{\dagger}(\vec{p}(-)\vec{q}, \vec{r}) \, V(s, r) \, d^3 r.$$

Because of the equalities

$$\frac{(\vec{p}(-)\vec{q})_0}{m_p} = \frac{m_p{}^2 + m_q{}^2 - t}{2m_p m_q},$$

$$\frac{\vec{p}(-)\vec{q}}{m_p} = \frac{(t^2 + m_q^2 + m_q^2 - 2t(m_q^2 + m_q^2) - 2m_q^2 m_q^2)^{1/2}}{2m_p m_q},$$

where t is the momentum transfer squared, the limit $|t| \ll m^2$ is equivalent to the limit $(\vec{p}(-)\vec{q})/m \ll 1$. Consequently, for $|t| \ll m^2$, the amplitude will be a nonrelativistic Fourier image of the quasipotential.

So, we have the following mnemonic rule: the correspondence between the nonrelativistic and relativistic cases can be derived automatically by substituting the transferred momentum by the transferred rapidity

$$\chi_t = \ln \frac{m_p^2 + m_q^2 - t + (t^2 + m_p^4 + m_q^2 - 2t(m_p^2 + m_q^2) - 2m_p^2 m_q^2)^{1/2}}{2m_p m_q}$$

Thus, if the analytic structure of the quasipotential is such that the amplitude has the power behaviour with increasing momentum transfer, the power behaviour for $|t| \ll m^2$ will be changed into the exponential one. Geometrically, this fact can be explained by the well-known statement: the local geometry of the Lobachevsky space is the Euclidean geometry, in full correspondence with the case of the real space-time.

4. *Conclusion*

A dynamical relativistic Fourier analysis has been constructed on the basis of the correspondence principle, the theory of Galilei and Poincaré group unitary representations and the concept of »geometrization« of the relativistic two-body problem. This Fourier analysis connects both the relative momentum and the relativistic coordinate.

The usual three-dimensional quantum-mechanical language of the formalism enables one to obtain some physical consequences:

— a relativistic generalization of the Heisenberg relation — the uncertainty relation between both the relative rapidity and the relativistic coordinate;

— the exponential-power behaviour of the elastic scattering amplitude can be interpreted as a geometrical effect.

The deveploped formalism can be employed for a heuristic description of high-energy phenomena[20].

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