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CANONICAL FORMALISM FOR THE DIRAC FIELD*

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Abstract: The canonical formalism for the Dirac field based on the Lagrangian formalism is developed.

1. *Introduction*

The standard theory of the Dirac field[1] uses the field components as the Lagrangian variables in the Lagrangian formalism. Since the field equation is of the first order differential equation there is no corresponding canonical equations. Indeed, taking the Lagrangian density in the form

$$\mathcal{L} = \frac{i}{2} (\bar{\Psi} \partial_\mu \gamma^\mu \Psi - \partial_\mu \bar{\Psi} \gamma^\mu \Psi + 2i\kappa\bar{\Psi}\Psi) \tag{1.1}$$

the Lagrangian equations are

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$$(i\partial_\mu \gamma^\mu - \kappa)\Psi = 0 ,$$
$$-i\partial_\mu \bar{\Psi}\gamma^\mu - \kappa\bar{\Psi} = 0 , \tag{1.2}$$

where Ψ is the Dirac field and γ^μ Dirac's matrices

$$\gamma^0 = \beta , \quad \gamma^k = \beta\,\alpha^k \quad k=1,2,3 , \quad \gamma^0\gamma^{\mu+}\gamma^0 = \gamma^\mu ,$$
$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \qquad x^\mu = (t,\vec{x}) , \tag{1.3}$$
$$\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu = -2i\sigma^{\mu\nu} \qquad x_\mu = (t,-\vec{x}) , \ \partial_\mu \equiv \frac{\partial}{\partial x^\mu} ,$$

(we use units $c=\hbar=1$).

The canonical conjugate momentum to Ψ is

$$\Pi_\Psi = \frac{\partial\mathcal{L}}{\partial\dot{\Psi}} = \frac{i}{2}\Psi^+ . \tag{1.4}$$

We see that Π_Ψ doesn't depend of the time derivative of the field components and consequently it is not correct canonical momentum. Because of this there is no the Hamiltonian density which comes from the Lagrangian procedure and there are no canonical equations.

We understand this situation as a consequence of the unadequate starting point. The Lagrangian (and canonical) formalism is developed in the classical mechanics for second order differential equations for field components of a continuous system (or coordinates of particles). The Maxwell equations for the electromagnetic field can illustrate this situation. The Maxwell equations are the first order differential equations. The corresponding Lagrangian formalism doesn't use the field components as the Lagrangian variables but the components of the electromagnetic's field potential and the differential equations for the potentials are of the second order. Relying on this fact, if we want to have a correct Lagrangian and canonical formalism for the Dirac field, it seems correct conclusion that the Dirac equation has to be considered as a canonical equation.

There are two possibilities in the interpretation of the Dirac equation on this line:

(1) that Ψ is the momentum of a canonical conjugate pair and
(2) that Ψ contains complete canonical pair in its internal structure.

The first possibility is much closer to the present situation of the theory of the Dirac field. Because of this we consider the first possibility in this paper and the second possibility we shall consider at another place.

We show that correct Lagrangian and canonical formalism for the Dirac field can be developed with essentially the same results in the case of free Dirac field as it the standard theory gives.

In Section 2 the canonical conjugate pair of the Dirac field is introduced. Section 3 contains solutions of the canonical and Lagrangian equations. The constants of motion are evaluated in Section 4. At the end, Section 5, some conclusions are given.

2. *Canonical pair of the Dirac field*

We expect that the basic properties of the Dirac field are contained in the Ψ. We ask, therefore, the Lagrangian density in the form which is completely expressed by the Ψ. The simplest scalar of that type is $\bar{\Psi}\Psi$. Consequently, we take

$$\mathcal{L} = \frac{k}{2}\,\overline{\Psi}\Psi \quad , \quad (\text{k is a constant}) \quad . \tag{2.1}$$

Now, we look for $\Psi = \Psi(\phi)$ where ϕ satisfies the second order differential Lagrange's equation and in the corresponding canonical equations Ψ satisfies the Dirac equation.

Due to linearity of the Lagrange's equation for the most general functional dependance Ψ of ϕ is

$$\begin{aligned} \Psi &= a_{i\lambda}\Gamma^{i}\partial^{\lambda}\phi + b_{i}\Gamma^{i}\phi \quad , \\ \overline{\Psi} &= \partial^{\lambda}\overline{\phi}\, a^{*}_{i\lambda}\Gamma^{i} + \overline{\phi}\, b^{*}_{i}\Gamma^{i} \quad , \end{aligned} \tag{2.2}$$

where are

$$\Gamma^{i} = \{1, \gamma^{\mu}, \sigma^{\mu\nu}, i\gamma^{5}\gamma^{\mu}, \gamma^{5} = \gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}\} \quad , \tag{2.3}$$

with the property $\gamma^{0}\Gamma^{i+}\gamma^{0} = \Gamma^{i}$ and $a_{i\lambda}$, b_{i} are constants.

After substitution Ψ from (2.2) into (2.1) we get the Lagrangian density for ϕ . The conjugate momenta to ϕ and $\overline{\phi}$ are

$$\begin{aligned} \Pi_{\phi} &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{k}{2}\,\overline{\Psi}(a_{io}\Gamma^{i}) \quad , \\ \Pi_{\overline{\phi}} &= \frac{\partial\mathcal{L}}{\partial\dot{\overline{\phi}}} = \frac{k}{2}\,(a^{*}_{io}\Gamma^{i})\Psi \quad , \end{aligned} \tag{2.4}$$

where $\frac{k}{2}(a^{*}_{io}\Gamma^{i})$ is constant matrix later denoted by A.

The corresponding Hamiltonian density is

$$\begin{aligned} \mathcal{H} = \frac{2}{k}\,&\Pi_{\phi}\,(a_{jo}\Gamma^{j})^{-1}(a^{*}_{io}\Gamma^{i})^{-1}\Pi_{\overline{\phi}} - \Pi_{\phi}\,(a_{jo}\Gamma^{j})^{-1}\cdot \\ &(a_{ik}\Gamma^{i}\partial^{k}\phi + b_{j}\Gamma^{j}\phi) - (\partial^{k}\overline{\phi}\, a^{*}_{ik}\Gamma^{i} + \overline{\phi}\, b^{*}_{i}\Gamma^{i})\cdot \\ &(a^{*}_{io}\Gamma^{i})^{-1}\,\Pi_{\overline{\phi}}\,. \end{aligned} \tag{2.5}$$

The canonical equations are then

$$\dot{\Pi}_\phi = \Pi_\phi (a_{jo}\Gamma^j)^{-1}(b_j\Gamma^j) - \partial^l \Pi_\phi (a_{jo}\Gamma^j)^{-1}(a_{il}\Gamma^i) \quad ,$$

$$\dot{\Pi}_{\bar\phi} = (b_i^*\Gamma^i)(a_{jo}^*\Gamma^j)^{-1}\Pi_{\bar\phi} - (a_{il}^*\Gamma^i)(a_{jo}^*\Gamma^j)^{-1}\partial^l \Pi_{\bar\phi} \quad ,$$

$$\dot\phi = \frac{2}{k}(a_{jo}\Gamma^j)^{-1}(a_{io}^*\Gamma^i)^{-1}\Pi_{\bar\phi} - (a_{jo}\Gamma^j)^{-1}(a_{il}\Gamma^i\partial^l\phi + b_j\Gamma^j\phi) \quad , \tag{2.6}$$

$$\dot{\bar\phi} = \frac{2}{k}\Pi_\phi (a_{jo}\Gamma^j)^{-1}(a_{io}^*\Gamma^i)^{-1} - (\partial^l\bar\phi\, a_{il}^*\Gamma^i + \bar\phi\, b_j^*\Gamma^j)(a_{io}^*\Gamma^i)^{-1} \; .$$

The requirement that $\Pi_{\bar\phi} \equiv \Psi' = A\Psi$, $\Pi_\phi \equiv \bar\Psi' = \bar\Psi\gamma^0 A^+\gamma^0$ and that the corresponding equations are the Dirac equations gives

$$i\partial_\mu\gamma^\mu\Psi - \kappa\Psi = 0 \quad ,$$
$$-i\partial_\mu\bar\Psi\gamma^\mu - \kappa\bar\Psi = 0 \quad , \tag{2.7}$$

and

$$a_{j\lambda}^*\Gamma^j = iB\gamma_\lambda \quad , \qquad b_j^*\Gamma^j = \kappa B \quad ,$$
$$a_{j\lambda}\Gamma^j = -i\gamma_\lambda B \quad , \qquad b_j\Gamma^j = \kappa B \quad , \tag{2.8}$$

where B is a constant nonsingular matrix. Without loosing generality we take B = - 1 .

Eqs. (2.8) determine completely the equations (2.2) which now read

$$\Psi = i\partial_\mu\gamma^\mu\phi - \kappa\phi \; ,$$
$$\bar\Psi = -i\partial_\mu\bar\phi\gamma^\mu - \kappa\bar\phi \; . \tag{2.9}$$

The second pair of the canonical equations (2.6) are

$$\dot{\phi} = -i\gamma^0\Psi - \gamma^0\gamma^k\partial_k\phi - i\kappa\gamma^0\phi \quad , \qquad (2.11)$$
$$\dot{\bar{\phi}} = i\bar{\Psi}\gamma^0 - \partial_k\bar{\phi}\gamma^k\gamma^0 + i\kappa\bar{\phi}\gamma^0 \quad .$$

They are in accordance to (2.9). Thus we have solution of the formulated problem.

The Hamiltonian of the system is now

$$\mathcal{H} = \frac{2}{k}\,\Pi_\phi\Pi_{\bar{\phi}} - \Pi_\phi(\,\alpha^k\partial_k\phi + i\kappa\beta\phi\,) + (\partial_k\bar{\phi}\alpha^k + i\kappa\bar{\phi}\beta\,)\,\Pi_{\bar{\phi}} \quad . \qquad (2.11)$$

The substitution Ψ from (2.9) into (2.7); or explicitly evaluating the Lagrangian equation, gives the Lagrangian equation for ϕ :

$$-\,\partial_\mu\partial^\mu\phi + \kappa^2\phi - 2i\kappa\partial_\mu\gamma^\mu\phi = 0 \quad , \qquad (2.12a)$$

or

$$(i\partial_\mu\gamma^\mu - \kappa\,)(\,i\partial_\nu\gamma^\nu - \kappa\,)\,\phi = 0 \quad . \qquad (2.12b)$$

It is second order differential equation. Let's mention that this equation is not separated* in the bispinor components of ϕ .

3. Solutions of canonical and Lagrangian equations

We start with the Lagrangian equation

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* The separated equations in the bispinor components of ϕ gives the pseudoscalar's Lagrangian density: const $\bar{\Psi}\gamma^5\Psi$. It can be easily proved.

$$(i\partial_\mu\gamma^\mu - \kappa)(i\partial_\nu\gamma^\nu - \kappa) \phi = 0 \quad . \tag{3.1}$$

In order to get solution of Eq. (3.1) it is useful to introduce a new function f for which the corresponding system of equations is separated in its components. One obtains it by

$$\phi = (i\partial_\mu\gamma^\mu + \kappa)(i\partial_\nu\gamma^\nu + \kappa) f \quad . \tag{3.2}$$

The equation for f is

$$(\Box - \kappa^2)(\Box - \kappa^2) f = 0 \quad . \tag{3.3}$$

The operator on the left side is diagonal and therefore it is separated system in components of f.

Writing

$$(\Box - \kappa^2) f = \chi \quad . \tag{3.4}$$

Eq. (3.3) becomes

$$(\Box - \kappa^2) \chi = 0 \quad . \tag{3.5}$$

The solution of Eq. (3.5) we write in the form

$$\chi = \sum_{\vec{k}_\mu} (a_{k_\mu} e^{-ik_\mu x^\mu} + b^*_{k_\mu} e^{+ik_\mu x^\mu}) \quad ,$$

$$k^{o^2} - \vec{k}^2 = \kappa^2 \quad . \tag{3.6}$$

Then we have

$$(\Box - \kappa^2) f = \sum_{\vec{k}_\mu} (a_{k_\mu} e^{-ik_\mu x^\mu} + b^*_{k_\mu} e^{ik_\mu x^\mu}) . \tag{3.7}$$

The general solution of this equation is

$$f = \alpha\, \tilde{\chi} + \frac{1}{2\kappa^2}\, x^{\nu}\partial_{\nu}\, \chi \quad , \tag{3.8}$$

where $\tilde{\chi}$ is general solution of homogeneous equation (i.e. of Eq. (3.5)) and the second term is a particular solution.

After substitution f from (3.8) into (3.2) we obtain

$$\phi = \left(2\alpha\kappa\tilde{\Psi} + \frac{1}{\kappa}\,\Psi \right) + \frac{1}{\kappa}\, x^{\nu}\partial_{\nu}\Psi \quad , \tag{3.9}$$

where $\tilde{\Psi}$ and Ψ are solutions of the Dirac equations[3] and α is a constant.

Eg. (3.9) can be written in the form

$$\phi = \phi_h + a\Psi + \frac{1}{\kappa}\, x^{\nu}\partial_{\nu}\Psi \quad , \tag{3.1o}$$

where ϕ_h is also a solution of the Dirac equation and a is arbitrary constant. By this ϕ is determined with solutions of the Dirac equation.

The solution (3.1o) is, naturally, also a solution of the canonical equations (2.7) and (2.9). Indeed, from (2.9) follows

$$\phi = \phi_h + \phi_p \quad , \tag{3.11}$$

where ϕ_h satisfies the Dirac equation $(i\partial_{\nu}\gamma^{\nu} - \kappa)\phi_h = 0$ and ϕ_p is a particular solution. Taking ϕ_p in the form $\phi_p = a\,\Psi + \frac{1}{\kappa}\, x^{\nu}\partial_{\nu}\Psi$ one can easily prove that it is a particular solution of Eq. (2.9) if Ψ satisfies the canonical equation (2.7), i.e. Ψ is a solution of the Dirac equation.

The solution ϕ_h is completely arbitrary. Our concentration is on Ψ . Because of this we eliminate this solution by the initial condition: $\phi_h(x^k, x^0=0) = 0$. Then we have

$$\phi = a\Psi + \frac{1}{\kappa} x^\nu \partial_\nu \Psi \quad . \tag{3.12}$$

This is, now, inversion of Eq. (2.9). From (3.12) follows that each quantity expressed by ϕ can be also expressed by the Ψ .

4. *Constants of motion*

The Lagrangian density (2.1) is invariant to the transformation

$$\phi \to e^{i\alpha}\phi \quad , \qquad \bar{\phi} \to e^{-i\alpha}\bar{\phi} \quad . \tag{4.1}$$

Due to the Neter's theorem we have

$$\partial_\mu j^\mu = 0 \quad ,$$

where

$$j^\mu = i\left(\bar{\phi} \frac{\partial \mathcal{L}}{\partial \bar{\phi}_{|\mu}} - \frac{\partial \mathcal{L}}{\partial \phi_{|\mu}} \phi \right) \quad .$$

Explicitly for the Lagrangian (2.1) with

$$\frac{\partial \mathcal{L}}{\partial \bar{\phi}_{|\mu}} = \frac{k}{2} (-i\gamma^\mu)\Psi \quad , \qquad \frac{\partial \mathcal{L}}{\partial \phi_{|\mu}} = \frac{k}{2} (i\bar{\Psi})\gamma^\mu$$

$$j^\mu = \frac{k}{2} (\bar{\phi}\gamma^\mu\Psi + \bar{\Psi}\gamma^\mu\phi) \quad . \tag{4.2}$$

From here follows the constant of motion

$$Q = C \int j^0 d^3x = C \frac{k}{2} \int (\phi^+ \Psi + \Psi^+ \phi) d^3x \quad . \qquad (4.3)$$

The substitution ϕ from (3.12) into (4.3) gives*

$$Q = C \frac{k}{2} \int \left(2a \Psi^+ \Psi + \frac{1}{\kappa} x^\mu \partial_\mu \Psi^+ \Psi + \frac{1}{\kappa} \Psi^+ x^\mu \partial_\mu \Psi \right) d^3x =$$

$$= C\ k\ \left(a - \frac{3}{2\kappa} \right) \int \Psi^+ \Psi\ d^3x \quad . \qquad (4.4)$$

It is known constant of motion of the Dirac field connected with the charge.

The energy-momentum tensor

$$T_\alpha{}^\beta = \bar{\phi}_{|\alpha} \frac{\partial \mathcal{L}}{\partial \bar{\phi}_{|\beta}} + \frac{\partial \mathcal{L}}{\partial \phi_{|\beta}} \phi_{|\alpha} - \delta_\alpha{}^\beta \mathcal{L} \quad , \qquad (4.5)$$

for the Lagrangian (2.1) is

$$T_\alpha{}^\beta = \frac{k}{2} \left(-i \bar{\phi}_{|\alpha} \gamma^\beta \Psi + i \bar{\Psi} \gamma^\beta \phi_{|\alpha} - \delta_\alpha{}^\beta \bar{\Psi} \Psi \right) \quad . \qquad (4.6)$$

Due to

$$\partial_\beta T_\alpha{}^\beta = 0 \quad , \qquad (4.7)$$

the energy-momentum vector

$$P_\alpha = C \frac{k}{2} \int d^3x \left(i \Psi^+ \phi_{|\alpha} - i \phi_{|\alpha}^+ \Psi - \delta_\alpha{}^0 \bar{\Psi} \Psi \right) \quad , \qquad (4.8)$$

is a constant of motion. By making use of Eq. (3.12) it is*

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* Assuming that surface integrals are zero at partial integrations. This means that Ψ vanishes in the infinity as r^{-s} , $s > 2$.

$$P_\alpha = C'k\left(a - \frac{1}{\kappa}\right)\frac{i}{2}\int d^3x\,(\psi^+\psi_{,\alpha} - \psi^+_{,\alpha}\psi) \tag{4.9}$$

Similarly, the angular momentum tensor

$$M^{\alpha,\beta\gamma} = x^\beta T^{\gamma\alpha} - x^\gamma T^{\beta\alpha} - \frac{\partial \mathcal{L}}{\partial \phi_{,\alpha}}\left(\frac{i}{2}\sigma^{\beta\gamma}\right)\phi + \bar{\phi}\left(\frac{i}{2}\sigma^{\beta\gamma}\right)\frac{\partial \mathcal{L}}{\partial \bar{\phi}_{,\alpha}}, \tag{4.10}$$

for the Lagrangian (2.1) is

$$M^{\alpha,\beta\gamma} = x^\beta T^{\gamma\alpha} - x^\gamma T^{\beta\alpha} + \frac{k}{4}\bar{\psi}\gamma^\alpha\sigma^{\beta\gamma}\phi + \frac{k}{4}\bar{\phi}\sigma^{\beta\gamma}\gamma^\alpha\psi \quad . \tag{4.11}$$

Due to

$$\partial_\alpha M^{\alpha,\beta\gamma} = 0 \quad , \tag{4.12}$$

constant of motion is

$$M^{\beta\gamma} = C''\int d^3x\,\left(x^\beta T^{\gamma 0} - x^\gamma T^{\beta 0} + \frac{k}{4}\psi^+\sigma^{\beta\gamma}\phi + \frac{k}{4}\bar{\phi}\sigma^{\beta\gamma}\gamma^0\psi\right), \tag{4.13}$$

or when using Eq. (3.12)

$$M^{\beta\gamma} = C''\frac{k}{4}\left(2a - \frac{3}{2\kappa}\right)\int d^3x\,\{x^\beta[\partial^\gamma]\psi^+\psi - x^\gamma[\partial^\beta]\psi^+\psi +$$

$$+ \frac{i}{2}\bar{\psi}\sigma^{\beta\gamma}\gamma^0\psi + \frac{i}{2}\psi^+\sigma^{\beta\gamma}\psi\} \quad , \tag{4.14}$$

where is $[\partial^\gamma]\psi^+\psi = \partial^\gamma\psi^+\psi - \psi^+\partial^\gamma\psi$.

The constants C, C', C'' in equations (4.4),(4.9) and (4.14) can be selected so that Q, P_α and $M^{\beta\gamma}$ are the standard charge, energy-momentum vector and the angular momentum tensor of the Dirac field, respectively.

5. *Conclusions*

First, what we conclude is that there is a correct canonical and Lagrangian formalism for the Dirac field. This formalism gives canonical pair where the standard Dirac field is the canonical momentum. By the initial condition whole physical contest can be reduced to the canonical momentum which is then equivalent to the standard Dirac field. In this case one obtains the known properties of the free Dirac field.

Second, we conclude that correct Lagrangian and canonical formalism of the Dirac field doesn't lead generally to the conception of the potential as in the theory of electromagnetic field in present form. However, by selection of initial condition the canonical variable may be interpreted as potential of the Dirac field with the difference that it doesn't separate the Dirac equation*.

The presented analysis is restricted to the free Dirac field. Interaction with the electromagnetic field and its consequencies we consider at another place. The quantum aspects of this problem we also do not consider here. We only make remark that the presented formalism gives correct basis for the quantum procedure.

R e f e r e n c e s

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2) *Landau L.D., Lifšic E.M., Teorija polja, Moskva 1973.*
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* Another way of introduction of the potential in the Dirac field theory is presented in the paper[3] and its extensions.

KANONSKI FORMALIZAM ZA DIRACOVO POLJE

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Sadržaj

Razvijen je ispravan kanonski formalizam za Diracovo slobodno polje sa masom mirovanja različitom od nule. Lagrangeove varijable zadovoljavaju diferencijalne jednadžbe drugog reda, a kanonski konjugirani impulsi Diracove jednadžbe. Nadjena su rješenja kanonskog sistema jednadžbi. Izbor početnog uvjeta omogućava interpretaciju Lagrangeovih varijabli kao potencijala. Izračunate su konstante kretanja i one se podudaraju sa izrazima standardne teorije slobodnog Diracovog polja.