YU ISSN 0015—3206
FZKAAA 14 (2) 133 (1982)

SOME GENERAL PROPERTIES OF THE SCHWINGER-DYSON EQUATION FOR THE FINITE ELECTRON PROPAGATOR

B. DRAGOVIĆ and B. SAZDOVIĆ

Institute of Physics, P. O. Box 57, 11001 *Beograd, Yugoslavia*

Received 5 April 1982

UDC 539.12

Original scientific paper

On the basis of the charge conjugation invariance of *QED* some general properties of the Schwinger-Dyson equation for the finite electron propagator are derived.

One of the necessary conditions for a finiteness (without the ultraviolet divergences before applying the renormalisation procedure) of quantum electrodynamics (*QED*) in the Johnson-Baker-Willey[1] (*JBW*) programme, is an equality of the bare electron mass to zero ($m_0 = 0$). Namely, taking $m_0 = 0$ and choosing appropriate gauge parameter G, in the *JBW* perturbation expansion of the Schwinger--Dyson (*SD*) equation of the electron propagator, one can achieve a finiteness of the electron propagator. Hence, the *JBW* approach to *QED* leads to an attractive possibility of studying dynamical (electromagnetic) electron mass generation. The problem of electromagnetic mass generation is studied in many papers[2] and it is shown[3] that there is an absence of electron mass generation in the first approximation of the *SD* equation in the *JBW* perturbation expansion. Investigating how to include some higher-order corrections to the *SD* equation for the electron propagator we found out some symmetrical properties applicable to any higher-order contribution (which give finite solution) as well as to the first *JBW* approximation. As a consequence of $m_0 = 0$, a property called the dilatation invariance of *QED* is earlier studied[4].

The property which we present in this paper is a consequence of $m_0 = 0$ and the charge conjugation invariance.

Now, let us start with the unrenormalized SD equation for the electron propagator[5)]

$$S^{-1}(p) = -\hat{p} - \frac{i e_0^2}{(2\pi)^4} \int \mathrm{d}^4 q \, D_{\mu\nu}(p-q) \, \gamma^\mu \, S(q) \, \Gamma^\nu(p, q) \tag{1}$$

where $S(p)$ is the complete electron propagator, $D_{\mu\nu}(k)$ is the complete photon propagator, $\Gamma^\nu(p, q)$ is the complete vertex function and $m_0 = 0$. Taking (the Bjorken-Drell metric)

$$S^{-1}(p) = \alpha(-p^2) - \hat{p}\,\beta(-p^2), \quad (\hat{p} = p_\mu \gamma^\mu) \tag{2}$$

and performing relevant mathematical operations (traces techniques, the Wick rotation) we obtain

$$\begin{aligned} \alpha(x) &= \int_0^\infty M\left(\frac{y}{x}, g\right) \frac{\alpha(y)\,\mathrm{d}y}{\alpha^2(y) + y\,\beta^2(y)}, \\ \beta(x) &= 1 + \int_0^\infty N\left(\frac{y}{x}, g\right) \frac{\beta(y)\,\mathrm{d}y}{\alpha^2(y) + y\,\beta^2(y)}, \end{aligned} \tag{3}$$

where $x = p_0^2 + \vec{p}^{\,2}$, $y = q_0^2 + \vec{q}^{\,2}$, $g = (e_0/4\pi)^2 = \alpha_0/4\pi$,

$$\begin{aligned} M\left(\frac{y}{x}, g\right) &= \frac{g\,d(g)\,y}{8\pi^2} \int \frac{\mathrm{d}^3 \Omega_q}{(p-q)^2} P_{\mu\nu}(p-q)\, Tr\{\gamma^\mu \Gamma^\nu(p, q)\}, \\ N\left(\frac{y}{x}, g\right) &= -g\,\frac{y\,d(g)}{x\,8\pi^2} \int \frac{\mathrm{d}^3 \Omega_q}{(p-q)^2} P_{\mu\nu}(p-q)\, Tr\{\gamma^\mu \hat{q}\, \Gamma^\nu(p, q)\, \hat{p}\}, \\ P_{\mu\nu}(k) &= g^{\mu\nu} - [1 - G(g)]\,\frac{k_\mu k_\nu}{k^2}. \end{aligned} \tag{4}$$

Equations (3) and (4) are written in Euclidean momentum space. The kernels of the integral equations M and N depend of x and y through the ratio y/x because they are dimensionless. This is valid only for a four space-time dimensions since the coupling constant g is then dimensionless. Writing Eqs. (4) we have in mind that the photon propagator, in any finite order approximation of the usual perturbation theory, can be presented as follows

$$D_{\mu\nu}(k) = -\frac{d(g)}{k^2}\left\{g_{\mu\nu} - [1 - G(g)]\,\frac{k_\mu k_\nu}{k^2}\right\} \tag{5}$$

where $d(g)$ is the only function of g (because $m_0 = 0$). Hence, any massless higher-order contribution to the photon propagator can be incorporated in the coupling constant g by its redefinition. It means that, in practical calculations, the photon propagator can be taken as the free one. Making the usual perturbation expansion of the vertex function $\Gamma^\nu(p, q)$ we can present the kernels M and N as follows:

$$
\begin{aligned}
M\left(\frac{y}{x}, g\right) &= g\, M_1\left(\frac{y}{x}\right) + g^2 M_2\left(\frac{y}{x}\right) + g^3 M_3\left(\frac{y}{x}\right) + \dots, \\
N\left(\frac{y}{x}, g\right) &= g\, N_1\left(\frac{y}{x}\right) + g^2 N_2\left(\frac{y}{x}\right) + g^3 N_3\left(\frac{y}{x}\right) + \dots.
\end{aligned} \tag{6}
$$

M_1 and N_1 have been obtained many years ago while the kernels M_2 and N_2 we calculated recently[6]. Note that our programme of an investigation of the SD equation (1) is the modified JBW perturbation expansion, but with the same finiteness properties.

The following properties

$$
\begin{aligned}
\frac{1}{y} M\left(\frac{y}{x}, g\right) &= \frac{1}{x} M\left(\frac{x}{y}, g\right), \\
\frac{x}{y} N\left(\frac{y}{x}, g\right) &= \frac{y}{x} N\left(\frac{x}{y}, g\right)
\end{aligned} \tag{7}
$$

should be of great importance for further analyses of the SD equation (1). We shall only mention the main points of the derivation of properties (7) on the basis of the charge conjugation invariance of QED. Let $C = C^{-1} = \gamma^2 \gamma^0$ be the charge conjugation operator[5] then it can be shown that (using transformation properties of the fields and the propagators as well as the definition of the vertex function with the help of the inverse electron propagator)

$$\Gamma^\nu_c(p, q) = -\, C\, \Gamma^{\nu T}(q, p)\, C \tag{8}$$

where T denotes a matrix transposition. It can be also shown

$$
\begin{aligned}
Tr\{\gamma^\mu \Gamma^\nu_c(p, q)\} &= Tr\{\gamma^\mu \Gamma^\nu(q, p)\}, \\
Tr\{\gamma^\mu \hat{p}\, \Gamma^\nu_c(p, q)\, \hat{q}\} &= Tr\{\gamma^\mu \hat{q}\, \Gamma^\nu(q, p)\, \hat{p}\}.
\end{aligned} \tag{10}
$$

Rewriting Eqs. (4) so that on their left-hand side are placed

$$\frac{1}{y} M\left(\frac{y}{x}, g\right), \qquad \frac{x}{y} N\left(\frac{y}{x}, g\right), \tag{11}$$

then on the basis of the charge conjugation invariance of such Eqs. (4) it follows that we can interchange the places of p and q on the corresponding right-hand side what just gives equalities (7).

Since the kernels M and N can be written in the following form

$$M\left(\frac{y}{x}, g\right) = A\left(\frac{y}{x}, g\right)\Theta(x-y) + A_0\left(\frac{y}{x}, g\right)\Theta(y-x),$$
$$N\left(\frac{y}{x}, g\right) = B\left(\frac{y}{x}, g\right)\Theta(x-y) + B_0\left(\frac{y}{x}, g\right)\Theta(y-x), \tag{12}$$

with the help of Eqs. (7) we obtain

$$\alpha(x) = \left[\int_0^x A\left(\frac{y}{x}, g\right) + \int_x^\infty \frac{y}{x} A\left(\frac{x}{y}, g\right)\right] \frac{\alpha(y)\,\mathrm{d}y}{\alpha^2(y) + y\,\beta^2(y)},$$
$$\beta(x) = 1 + \left[\int_0^x B\left(\frac{y}{x}, g\right) + \int_x^\infty \left(\frac{y}{x}\right)^2 B\left(\frac{x}{y}, g\right)\right] \frac{\beta(y)\,\mathrm{d}y}{\alpha^2(y) + y\,\beta^2(y)}. \tag{13}$$

From these Eqs. (13) it follows that (for a fixed x or y) an infra-red behaviour of the kernels determines their ultra-violet behaviour and vice versa. Eqs. (13) maintain their validity for any approximation of the kernels given like (6).

Note that we can rewrite Eqs. (13) in the following form

$$\alpha(x) = a(x, g) + \int_0^x \left[A\left(\frac{y}{x}, g\right) - \frac{y}{x} A\left(\frac{x}{y}, g\right)\right] \frac{\alpha(y)\,\mathrm{d}y}{\alpha^2(y) + y\,\beta^2(y)}$$
$$\beta(x) = b(x, g) + \int_0^x \left[B\left(\frac{y}{x}, g\right) - \left(\frac{y}{x}\right)^2 B\left(\frac{x}{y}, g\right)\right] \frac{\beta(y)\,\mathrm{d}y}{\alpha^2(y) + y\,\beta^2(y)} \tag{14}$$

where:

$$a(x, g) = \int_0^\infty \frac{y}{x} A\left(\frac{x}{y}, g\right) \frac{\alpha(y)\,\mathrm{d}y}{\alpha^2(y) + y\,\beta^2(y)},$$
$$b(x, g) = 1 + \int_0^\infty \left(\frac{y}{x}\right)^2 B\left(\frac{x}{y}, g\right) \frac{\beta(y)\,\mathrm{d}y}{\alpha^2(y) + y\,\beta^2(y)}. \tag{15}$$

The importance of Eqs. (14) is, in fact, that they can be considered not only in the spacelike ($x > 0$) region but also in the timelike ($x < 0$) one. This is particularly important from the point of view of an investigation of dynamical mass generation.

At the end, let us also write Eqs. (15) as one equation, which is in the more compact form

$$U(x) = U_0(x) + \int_0^x V\left(\frac{y}{x}, g\right) U^{-1}(y)\,\mathrm{d}y \tag{16}$$

where:

$$U(x) = \begin{pmatrix} u_+(x) \\ u_-(x) \end{pmatrix}, \qquad U_0(x) = \begin{pmatrix} u_+^0(x) \\ u_-^0(x) \end{pmatrix}, \qquad U^{-1}(x) = \begin{pmatrix} u_+^{-1}(x) \\ u_-^{-1}(x) \end{pmatrix},$$

$$V\left(\frac{y}{x}, g\right) = \begin{pmatrix} v_-\left(\frac{y}{x}, g\right) & v_+\left(\frac{y}{x}, g\right) \\ v_+\left(\frac{y}{x}, g\right) & v_-\left(\frac{y}{x}, g\right) \end{pmatrix}, \tag{17}$$

$$u_\pm(x) = \alpha(x) \pm i\sqrt{x}\,\beta(x), \qquad \bar{A}\left(\frac{y}{x}, g\right) = A\left(\frac{y}{x}, g\right) - \frac{y}{x} A\left(\frac{x}{y}, g\right),$$

$$u_\pm^0(x) = a(x, g) \pm i\sqrt{x}\,b(x, g), \qquad \bar{B}\left(\frac{y}{x}, g\right) = B\left(\frac{y}{x}, g\right) - \left(\frac{y}{x}\right)^2 B\left(\frac{x}{y}, g\right),$$

$$u_\pm^{-1}(x) = \frac{\alpha(x) \mp i\sqrt{x}\,\beta(x)}{\alpha^2(x) + x\,\beta^2(x)}, \qquad v_\pm\left(\frac{y}{x}, g\right) = \frac{1}{2}\bar{A}\left(\frac{y}{x}, g\right) \pm \frac{1}{2}\sqrt{\frac{x}{y}}\,\bar{B}\left(\frac{y}{x}, g\right).$$

Eq. (16) can be useful in further mathematical investigations like existence and uniqueness of its solutions.

References

1) K. Johnson, M. Baker and R. Willey, Phys. Rev. **136** (1964) B 1111; K. Johnson and M. Baker, Phys. Rev. D**8** (1973) 1110, (and references quoted therein);

2) B. G. Dragović, D. P. Mavlo and A. T. Filippov, FIZIKA **10** (1978) 51, (and references quoted therein);

3) D. Atkinson and D. W. E. Blatt, Nucl. Phys. B **151** (1979) 342; A. I. Alekseev, B. A. Arbuzov and A. Ya. Rodionov, Theor. Math. Phys. **42** (1980) 291, (Sov. J.); B. Sazdović and B. Dragović, FIZIKA **13** (1981) 241;

4) R. Haag and Th. A. J. Maris, Phys. Rev. **132** (1963) 2325;

5) N. N. Bogolubov and D. V. Shirkov, *Vvedenie v teoriyu kvantovannih polej*, Moskva 1976;

6) B. Sazdović and B. Dragović, to be published in J. Phys. G.

NEKA OPŠTA SVOJSTVA SCHWINGER-DYSONOVE JEDNAČINE ZA KONAČAN ELEKTRONSKI PROPAGATOR

B. DRAGOVIĆ i B. SAZDOVIĆ

Institut za fiziku, P. P. 57, 11001 *Beograd*

UDK 539.12

Originalni naučni rad

Na osnovu invarijantnosti kvantne elektrodinamike u odnosu na konjugaciju naboja izvedena su neka opšta svojstva Schwinger-Dysonove jednačine za konačan elektronski propagator.