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# ON A PROPERTY OF CANONICAL TRANSFORMATIONS FOR DEGENERATE SYSTEMS

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Canonical transformations are considered for degenerate systems in Dirac's sense, and a detailed analysis of the method of evaluation of the corresponding constraint multipliers is given. It is shown that, under very general conditions, these multipliers are obtained as linear combinations of the constraint functions. The results are subsequently generalised to classical field theory. It is concluded that canonical transformations for degenerate systems are, after the terminology suggested by Dirac, weakly equal to those for the corresponding nondegenerate system or field, obtained from the given one by omitting the constraint equations.

## *Introduction*

The beginning of the study of degenerate systems in analytical mechanics and classical field theory dates back to the fifties of this century. This study was primarily related to the needs of the theory of relativity and quantum mechanics, and the fundamental works in this domain are due to *P. Dirac*[1-5]. Dirac defined degenerate systems as those for which the Hessian matrix of their Lagrangian is singular, so that the equations defining the generalized momenta do not yield single-valued solutions for the generalized velocities, which makes impossible the usual transition from Lagrangian to Hamiltonian formalism. For such systems Dirac developed the appropriate generalized Hamiltonian formalism and showed that the generalized coordinates and momenta are connected by a number of constraints, appearing also in the general equation of motion expressed in terms of Poisson brackets, but with a number of constraint multipliers remaining entirely arbitrary. Moreover, Dirac defined generalized Poisson brackets in such a way as to reduce effectively the number of degrees of freedom, and formulated the rules for transition from classical to quantum mechanics.

Further contributions to this problem are due to *P. Bergmann* et al.[6-10]. These authors have studied systems of this kind in covariant field theory, where invariant relations among the canonical variables appear as a consequence of the invariance of the theory with respect to groups of transformations. Attention has also been given to the above mentioned Dirac brackets, and to the quantization structure of such systems. *S. Shanmugadhasan*[11-12] considered the systems in question from the standpoint of the Lagrangian formalism and of the invariance of the system. The influence of degeneracy on the Lagrange equations has been given special attention, and it has been shown that certain linear combinations of these equations degenerate to first or zeroth order differential equations. The corresponding Hamiltonian formalism has also been given from this standpoint, it differs from the Dirac approach, above all with respect to the origin of constraints and their influence on the motion of the system. The problem of quantization of these systems had also been considered.

Besides these authors, *N. Mukunda* and *E. Sudarshan*[13-14] also dealt with the analysis of these systems with constraints and the properties of Dirac brackets on the basis of group theory. *H. Kunzle*[15] has also studied these degenerate systems in modern mathematical language and showed that the space of these variables has symplectic structure. Moreover, several authors have recently applied the Dirac theory of degenerate systems to a series of problems in classical and contemporary theoretical physics, as can be seen from the monography by *A. Hanson*, *T. Regge* and *C. Teitelboim*[16].

Finally, this problem was lately studied also by *D. Djokić-Ristanović* and the author of this paper[17-19]. The generalization of the Dirac method to the case where time explicitly appears in the Lagrangian was developed, starting from Shanmugadhasan's approach. The thus formulated Dirac theory of degenerate systems was systematically generalized to classical field theory by D. Djokić-Ristanović, who used *Volterra's* calculus of functionals[20] and took electrodynamics as an example. As far as canonical transformations for degenerate systems are concerned, they have been, to our knowledge, studied only in the second and the third of the above mentioned papers, both in mechanics and in field theory. The corresponding relations with generating functions were thus obtained, and a comparison with the case of S. Lie was carried out. Generalized Lagrange brackets were also formulated, as well as the corresponding differential and integral invariants.

## *1. Canonical transformations in mechanics*

The degenerate systems are defined, after Dirac, as those for which the Hessian determinant of the Lagrangian is identically equal to zero

$$\Delta \equiv \left| \frac{\partial^2 L}{\partial \dot{q}_i \, \partial \dot{q}_k} \right| = 0. \tag{1.1}$$

In this case the generalized coordinates and momenta are related by certain constraints of the form

$$\Theta_a (q_i, p_i, t) = 0 \qquad (a = 1, 2, ..., B), \tag{1.2}$$

the number of which is equal to the degree of the degeneracy of the systems. Let the canonical transformations of the degenerate system be determined by the relations

$$Q_i = Q_i (q_k, p_k, t), \qquad P_i = P_i (q_k, p_k, t) \qquad (i = 1, 2, ..., N) \tag{1.3}$$

such that their inversion is possible and that the canonical form of the Hamilton equations is conserved. The necessary and sufficient condition for the transformation to be canonical, in this case, is

$$p_i \,\mathrm{d}q_i - H_T \,\mathrm{d}t = c\,(P_i \,\mathrm{d}Q_i - K_T \,\mathrm{d}t) + \mathrm{d}F, \tag{1.4}$$

where $H_T$ and $K_T$ are the total Hamiltonians in old and new variables, respectively, namely $H_T = H_0 + \lambda_\beta \Theta_\beta$ and likely for $K_T$, and $F$ is the generating function of this transformation. Solving the first group of equations (1.3) with respect to

$$p_i = p_i (q_k, Q_k, t) \quad (i = 1, 2, ..., N), \tag{1.5}$$

the generating function, as well as the constraint functions, can be expressed in terms of old and new generalized coordinates

$$\begin{aligned} F (q_i, p_i, t) &= F\, [q_i\, p_i (q_k, Q_k, t), t] \equiv W (q_i, Q_i, t), \\ \Theta_a (q_i, p_i, t) &= \Theta_a\, [q_i, p_i (q_k, Q_k, t), t] \equiv \Omega_a (q_i, Q_i, t). \end{aligned} \tag{1.6}$$

Using the method of the constraint multipliers, these transformations are shown to be determined, for $c = 1$, by the following system of equations

$$\begin{aligned} p_i &= \frac{\partial W}{\partial q_i} + \lambda_\beta \frac{\partial \Omega_\beta}{\partial q_i}, \qquad P_i = -\frac{\partial W}{\partial Q_i} - \lambda_\beta \frac{\partial \Omega_\beta}{\partial Q_i}, \\ K_T &= H_T + \frac{\partial W}{\partial t} + \lambda_\beta \frac{\partial \Omega_\beta}{\partial t}, \qquad \Omega_\beta (q_i, Q_i, t) = 0. \end{aligned} \tag{1.7}$$

This system consists of $2N + B + 1$ equations, and can be used to determine the equal number of unknown quantities

$$Q_1, ..., Q_N; \quad P_1, ..., P_N; \quad \lambda_1, ..., \lambda_B; \quad K_T.$$

If we represent $p_i$ and $P_i$ in the form of sums of two terms

$$p_i = p_{i0} + p_i', \qquad P_i = P_{i0} + P_i', \tag{1.8}$$

where $p_{i0}$ and $P_{i0}$ stand for that part of the generalized momenta which would have existed if the constraints were absent, the above system of equations is reduced to

$$p_i' = \lambda_\beta \frac{\partial \Omega_\beta}{\partial q_i}, \qquad P_i' = -\lambda_\beta \frac{\partial \Omega_\beta}{\partial Q_i}, \qquad \Theta_a\left(q_i, \frac{\partial W}{\partial q_i} + p_i', t\right) = 0. \qquad (1.9)$$

Here the unknown quantities are $p_i'$, $P_i'$ and $\lambda_\beta$, and they are obtained in the form

$$p_i' = p_i'(q_k, Q_k, t), \quad P_i' = P_i'(q_k, Q_k, t), \quad \lambda_\beta = \lambda_\beta(q_k, Q_k, t). \qquad (1.10)$$

The basic problem here can be formulated in the following manner. Given the functions $W(q_i, Q_i, t)$ and $\Omega_a(q_i, Q_i\ t)$, for which it is known to be the consequence of the presence of the constraints, determine the canonical transformations themselves, viz. the above listed quantities. To solve it, we assume that the system (1.2) may be solved with respect to $B$ generalized momenta. Since $B \leqslant N$, we have

$$p_a = \varphi_a(q_1, \ldots, q_N, p_{B+1}, \ldots, p_N, t) \quad (a = 1, 2, \ldots, B). \qquad (1.11)$$

The initial system of constraint equations may then be substituted by

$$\Theta_a^*(q_i, p_i, t) \equiv p_a - \varphi_a(q_1, \ldots, q_N, p_{B+1}, \ldots, p_N, t) = 0$$

and inserting the functions (1.5) for $p_i$, these may be rewritten as

$$\Omega_a^*(q_i, Q_i, t) \equiv p_a(q_i, Q_i, t) - \varphi_a^*(q_i, Q_i, t) = 0, \qquad (1.12)$$

where $\varphi_a^*(q_i, Q_i, t)$ stand for the result of substitution of the above functions in the second term of the previous equation. On the other hand, if in the previous equation $p_a$ is decomposed according to (1.8), use made of the relations $p_{a0} = \partial W/\partial q_a$, and $p_a'$ substituted by the corresponding expressions (1.9), we obtain a system of equations for the determination of the multipliers

$$\Theta_a^*\left(q_i, \frac{\partial W}{\partial q_i} + p_i', t\right) \equiv \frac{\partial W(q_i, Q_i, t)}{\partial q_a} + \lambda_\beta \frac{\partial \Omega_\beta^*}{\partial q_i} - \varphi_a^*(q_i, Q_i, t) = 0. \qquad (1.13)$$

The generating function $W(q_i, Q_i, t)$ depends on the canonical transformations only, and not on the equations of the constraints, since its evaluation from the canonical transformation is based on the condition (1.4), i. é. on the requirement that

$$p_i\,\mathrm{d}q_i - H_T\,\mathrm{d}t - c\,(P_i\,\mathrm{d}Q_i - K_T\,\mathrm{d}t)$$

be a total differential, irrespective of independence or interdependence of $q_i$'s and $Q_i$'s. The first term in the right-hand side of (1.13) may, therefore, be transformed as if the constraints were not present and denoting the corresponding generating function by $W_0(q_i, Q_i, t)$, we have

$$\frac{\partial W(q_i, Q_i, t)}{\partial q_a} = \frac{\partial W_0(q_i, Q_i, t)}{\partial q_a} = p_a(q_i, Q_i, t). \tag{1.14}$$

The independent term of these equations then becomes, on the ground of (1.12)

$$\frac{\partial W(q_i, Q_i, t)}{\partial q_a} - \varphi_a^*(q_i, Q_i, t) = p_a(q_i, Q_i, t) - \varphi_a^*(q_i, Q_i, t) = \Omega_a^*(q_i, Q_i, t),$$

so that the system of equations (1.13) for the constraint multipliers $\lambda_\beta$ may be expressed in terms of the constraint functions only

$$\frac{\partial \Omega_\beta^*}{\partial q_a} \lambda_\beta = -\Omega_a^*(q_i, Q_i, t) \quad (a = 1, 2, \ldots, B). \tag{1.15}$$

If we assume, furthermore, that the determinant of this system of linear equations is different from zero

$$\Delta \equiv \left| \frac{\partial \Omega_\beta^*}{\partial q_a} \right| \neq 0, \tag{1.16}$$

the above system will have unique solution, and this can be expressed, in accordance with the Kramer's rule, as

$$\lambda_\beta = \frac{\Delta_\beta}{\Delta} = \frac{\begin{vmatrix} \frac{\partial \Omega_1^*}{\partial q_1} \cdots -\Omega_1^* \cdots \frac{\partial \Omega_B^*}{\partial q_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial \Omega_1^*}{\partial q_B} \cdots -\Omega_B^* \cdots \frac{\partial \Omega_B^*}{\partial q_B} \end{vmatrix}}{\begin{vmatrix} \frac{\partial \Omega_1^*}{\partial q_1} \cdots \frac{\partial \Omega_\beta^*}{\partial q_1} \cdots \frac{\partial \Omega_B^*}{\partial q_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial \Omega_1^*}{\partial q_B} \cdots \frac{\partial \Omega_\beta^*}{\partial q_B} \cdots \frac{\partial \Omega_B^*}{\partial q_B} \end{vmatrix}}. \tag{1.17}$$

Developing the determinant in the numerator with respect to the elements of the $\beta$-th column, we obtain

$$\Delta_\beta = -\Omega_1^* A_{1\beta} - \Omega_2^* A_{2\beta} - \ldots - \Omega_B^* A_{B\beta} = -\Omega_a^* A_{a\beta},$$

where $A_{\alpha\beta}$ are the corresponding cofactors of the determinant $\Delta$ of the system. Hence, the constraint multipliers $\lambda_\beta$ are obtained in the form

$$\lambda_\beta = -\frac{A_{\alpha\beta}}{\Delta}\Omega_\alpha^*(q_i, Q_i, t), \tag{1.18}$$

i. e. as linear combinations of the constraint functions.

Thus, upon solving the system (1.9) which determines the canonical transformations of the degenerate system, both the multipliers $\lambda_\beta$ and the additional terms $p_i'$ and $P_i'$ are obtained in the form of linear combinations of the constraint functions $\Omega_\alpha^*(q_i, Q_i, t)$. With constraints taken into account, all these quantities become weakly equal to zero, according to the terminology of Dirac, so that it can be deduced that the canonical transformations of the degenerate system are weakly equal to the canonical transformations of the corresponding nondegenerate system, obtained by omitting the constraints.

## 2. *Canonical transformations in field theory*

Let us generalize the above results to the case of the classical field theory. We shall assume that the field considered is determined by a given set of field functions $\psi_i(x, t)$ $(i = 1, 2, ..., N)$ in a given domain $V$, where $x = (x_1, x_2, x_3)$, and that it can be described by a given Lagrangian $L[\dot{\psi}_i(x, t), \psi_i(x, t), t]$. On the basis of the calculus of functionals developed by V. Volterra, it is well known that a close correspondence may be established between analytical mechanics and classical field theory, with functionals corresponding to functions, functional derivatives corresponding to partial derivatives, and the summation over discrete indices being replaced by the summation over continuous indices $x$, i. e. by the integration over the domain $V$. Then, the considered physical field will be degenerate if

$$\Delta \equiv \left| \frac{\delta^2 L}{\delta\dot{\psi}_i(x)\,\delta\dot{\psi}_k(x)} \right| = 0 \tag{2.1}$$

holds in every point, and the constraints between field functions and the corresponding generalized momenta $\pi_i(x) = \delta L/\delta\dot{\psi}_i(x)$ will, in general, be of the form of functionals

$$\Theta_\alpha[\psi_i(x, t), \pi_i(x, t), t] = 0 \quad (\alpha = 1, 2, ..., B), \tag{2.2}$$

which depend also on the position $x$, for instance

$$\pi(x) - \int K(x, x')\,\psi(x')\,\mathrm{d}^3x' = 0.$$

In the case of the local field theory, where the Lagrangian is of the form $L = \int \mathscr{L} \, \mathrm{d}^3x$, these constraints reduce to functions relating the values of $\psi_i(x, t)$ and $\pi_i(x, t)$ in the same point $x$

$$\Theta_a(\psi_i, \pi_i, t) = 0 \qquad \forall x \in V \ (a = 1, 2, \ldots, B), \tag{2.3}$$

so that now there is an infinity of these relations, $B$ for every point $x$.

If we define the canonical transformations in the field theory by functional relations

$$\begin{aligned} \Psi_i(x, t) &= \Psi_i[\psi_k(x', t), \pi_k(x', t); t], \\ \Pi_i(x, t) &= \Pi_i[\psi_k(x', t), \pi_k(x', t); t], \end{aligned} \tag{2.4}$$

which allow the inversion and maintain the form of the corresponding Hamilton equations invariant, the necessary and sufficient condition for the transformation to be canonical will be

$$\pi_i \, \mathrm{d}\psi_i - \mathscr{H}_T \, \mathrm{d}t = c(\Pi_i \, \mathrm{d}\Psi_i - k_T \, \mathrm{d}t) + \mathrm{d}F. \tag{2.5}$$

Here $\mathscr{H}_T$ and $k_T$ are the respective total Hamiltonian densities in old and new variables. If we assume that the first system of the functional equations (2.4) can be solved with respect to the generalized momenta

$$\pi_i(x, t) = \pi_i[\psi_k(x', t), \Psi_k(x', t); t],$$

and insert these solutions into the functionals $F$ and $\Theta_a$, we shall then have these quantities expressed in terms of old and new field variables

$$\begin{aligned} F[\psi_i(x, t), \pi_i(x, t); t] &= F[\psi_i(x, t), \pi_i[\psi_k(x', t), \Psi_k(x', t); t]; t] \equiv W, \\ \Theta_a[\psi_i(x, t), \pi_i(x, t); t] &= \Theta_a[\psi_i(x, t), \pi_i[\psi_k(x', t), \Psi_k(x', t); t]; t] \equiv \Omega_a. \end{aligned} \tag{2.6}$$

In view of the above mentioned correspondence between the mechanics and the field theory, we substitute equations (1.7) by the following system of equations for the constraint multipliers

$$\begin{aligned} \pi_i(x) &= \frac{\delta W}{\delta \psi_i(x)} + \int \lambda_\beta(x') \frac{\delta \Omega_\beta(x')}{\delta \psi_i(x)} \, \mathrm{d}^3x', \\ \Pi_i(x) &= -\frac{\delta W}{\delta \Psi_i(x)} - \int \lambda_\beta(x') \frac{\delta \Omega_\beta(x')}{\delta \Psi_i(x)} \, \mathrm{d}^3x'. \end{aligned} \tag{2.7}$$

This is a system of integral equations with respect to unknown functions $\lambda_\beta(x')$, pertaining to the type of Fredholm equations of the first kind.

If the generalized momenta $\pi_i$ and $\Pi_i$ are again decomposed in two terms, as above

$$\pi_i(x) = \pi_{i0}(x) + \pi_i'(x), \qquad \Pi_i(x) = \Pi_{i0}(x) + \Pi_i'(x), \tag{2.8}$$

a system of functional equations for the determination of the unknown quantities $\pi_i'$, $\Pi_i'$ and $\lambda_\beta$ will be obtained

$$\pi_i'(x) = \int \lambda_\beta(x') \frac{\delta \Omega_\beta(x')}{\delta \psi_i(x)} d^3x',$$

$$\Pi_i'(x) = - \int \lambda_\beta(x') \frac{\delta \Omega_\beta(x')}{\delta \Psi_i(x)} d^3x'. \tag{2.9}$$

Assuming, again as above, that the system of functional constraints (2.2) may be solved with respect to $B$ generalized momenta

$$\pi_a(x, t) = \varphi_a[\psi_1(x', t), \dots, \psi_N(x', t), \pi_1(x', t), \dots, \pi_N(x', t); t],$$

the equations for the constraints will be found to be of the form

$$\Omega_a^*[\psi_i(x, t), \Psi_i(x, t); t] \equiv \pi_a[\psi_i(x, t), \Psi_i(x, t); t] - \\ - \varphi_a^*[\psi_i(x, t), \Psi_i(x, t); t] = 0, \tag{2.10}$$

where $\varphi_a^*[\ ]$ denotes the result obtained by inserting the functionals $\pi_i = \pi_i[\psi_k, \Psi_k; t]$ into $\varphi_a[\ ]$. Using an analogous procedure as above, the following system of integral equations for constraint multipliers will be obtained instead of (1.15)

$$\int \frac{\delta \Omega_\beta^*(x')}{\delta \psi_a(x)} \lambda_\beta(x') d^3x' = - \Omega_a^*[\psi_i, \Psi_i; t]. \quad (a = 1, 2, \dots, B). \tag{2.11}$$

If the conditions of existence of the solution of these Fredholm integral equations of the first kind are met, the solution will be of the form

$$\lambda_\beta(x') = \int K_{\beta a}^{-1}(x', x) \Omega_a^*[\psi_i(x, t), \Psi_i(x, t); t] d^3x, \tag{2.12}$$

where $K_{\beta a}^{-1}(x', x)$ denotes the kernel of the integral transformation inverse to the integral transformation (2.11) with the kernel $K_{\beta a}(x', x) = \delta \Omega_\beta^*(x')/\delta \psi_a(x)$.

For the case of the local field theory, where the constraint equations are of the form (2.3), the system (2.9) reduce to a system of algebraic equations

$$\pi_i'(x) = \lambda_\beta \frac{\partial \Omega_\beta}{\partial \psi_i(x)}, \qquad \Pi_i'(x) = - \lambda_\beta \frac{\partial \Omega_\beta}{\partial \Psi_i(x)}, \tag{2.13}$$

and equations (2.11) become

$$\frac{\partial \Omega_\beta^*}{\partial \psi_a(x)} \lambda_\beta = - \Omega_a^*(\psi_i, \Psi_i, t) \quad (a = 1, 2, \dots, B). \tag{2.14}$$

The corresponding solutions (2.12) of these linear algebraic equations then assume the following form

$$\lambda_\beta (x, t) = - a_{\beta a} \Omega_a^* (\psi_t, \Psi_t, t), \qquad (2.15)$$

where $a_{\beta a}$ denotes the ratio of the corresponding cofactor and the determinant of the system $\Delta$, all these quantities being taken in the same field point $x$.

Hence, both in the general case and in the local field theory, the constraint multipliers are obtained as linear combinations of the constraint functionals or functions, respectively, and are weakly equal to zero in view of the constraint equations. Analogously to the situation in analytical mechanics, the canonical transformations for the degenerate physical field are, therefore, also weakly equal to the canonical transformations for the corresponding nondegenerate physical field.

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