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DRIFT-WAVE INSTABILITIES IN AN ANISOTROPIC PLASMA

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Electrostatic drift-wave instabilities in a nonuniform low-plasma wherein the ion--temperature parallel to the magnetic field is much smaller that than of the ion--temperature in the perpendicular direction are studied. Specifically, three problems are treated:

(i) finite ion-Larmor-radius effects;

(ii) effects of curvature in the magnetic field;

(iii) drift-cyclotron instability.

The results show that the forementioned anisotropy makes available more free--energy sources to aggravate the various instabilities. Finally, a study is made of electromagnetic drift waves in a nonuniform, anisotropic, finite-β plasma. The results show that:

(1) the waves are unstable;

(2) the ions out of resonance with the waves produce some damping.

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A purely oscillatory transverse wave exists that derives it energy source from the kinetic energy of the electron-thermal-motion parallel to the unperturbed magnetic field.

1. Introduction

If a magnetised plasma is stratified, the particles have associated drifts. This drifting motion will not only modify the uniform-plasma waves like Alfven waves, ion-acoustic waves and ion-cyclotron waves, but will also excite a completely new type of wave — the drift wave, which moves across the magnetic field with a phase velocity of the order of the drift velocity of the electrons[1]. The electron drift waves have been observed in several Q-machine experiments[2-4].

Further, this drifting motion may also become a potential source of instability of these plasma waves.

A strongly magnetised plasma, such as the one in a mirror machine, ceases to have isotropic properties, and there must always exist an anisotropy in the velocity distribution perpendicular and parallel to the magnetic field. (The anisotropy may also be a consequence of the method of creating the plasma by injection perpendicular to the magnetic field.) Besides, it is such an anisotropy that makes a mirror-confinement of the plasma possible. This follows by adopting the Chew-Goldberger-Low[5] model, according to which the pressure tensor for an anisotropic plasma is given by

$$p_{ij} = p_{\perp}\,\delta_{ij} + (p_{\parallel} - p_{\perp})\, b_i\, b_j$$

where,

$$\vec{b} = \vec{B}/\,|\vec{B}\,|\,.$$

One then has for a static equilibrium

$$\nabla_{\parallel} p_{\parallel} + (p_{\perp} - p_{\parallel})\left(\frac{\nabla B}{B}\right)_{\parallel} = 0$$

from which, it is seen that the anisotropy in the plasma can maintain a pressure gradient along a field line in a direction of decreasing $|\vec{B}\,|$, and therefore, the field lines can leave the plasma-mirror confinement.

In an anisotropic plasma, free energy may always be gained by relaxing towards isotropy[6], and therefore, instabilities result. In a uniform plasma, one has the well-known loss-cone instabilities[7]. Furthermore, in a mirror machine the kinetic energy associated with the ion-thermal motion parallel to the magnetic field is much smaller than that in the perpendicular direction. In a uniform plasma, under these conditions the local constrictions or the »mirrors« produced by the change in the field strength of the perturbation will heavily concentrate the particles where the field is weakest so that the pressure there increases, pushing the field lines apart, and weakening the field further. Thus, »mirrors« keep growing, and one has the well-known mirror instability. It turns out, in the following, that in a nonuniform plasma under the forementioned conditions, further free-energy sources become available to aggravate the various drift-wave instabilities. Though this paper is primarily concerned with anisotropic-plasma effects on drift instabilities, the calculations given here also illustrate some features of drift instabilities in the case of an isotropic plasma that have been missed in the literature.

It may be mentioned that a combination of a loss cone and spatial inhomogeneties in the plasma drives what is called a »drift-cone« instability[8-10], which is known to constitute a serious threat to mirror confinement. However, this instability will not be considered in the following.

2. *Electrostatic drift-waves in an anisotropic plasma*

The analysis of drift-wave instabilities requires consideration of the kinetic model rather than the fluid model of a plasma. This is so because the resonant wave-particle interactions govern the instability mechanism and wavelengths smaller than the ion-Larmor radius are involved. The fluid model is known to follow from the kinetic model only in the limit of small Larmor radius and low frequency[12].

The dynamic effects due to the curvature of the magnetic field may be simulated by an effective gravitational force superimposed on a uniform magnetic field, the plasma now being appropriately stratified[13]. Since the gravitational field is to simulate the drift produced by a curvature in the magnetic field, it is appropriate to consider different gravitational fields acting on ions and electrons.

Consider a nonuniform, anisotropic, and low-β (β being the ratio of particle pressure to the magnetic pressure) plasma in a uniform magnetic field $\vec{B}_0 = B_0 \hat{i}_z$, and a gravitational field $\vec{g} = g \hat{i}_x$. The constants of motion in this case are

$$\left.\begin{aligned} E_s &= \frac{1}{2} m_s v^2 - m_s g_s x \\ P_{s\beta} &= m_s v_\beta \\ P_{sy} &= m_s v_y + \frac{e_s A_y}{c} \end{aligned}\right\} \qquad (1a)$$

where E denotes the total energy, $\vec{P}$ the generalised momentum, $\vec{v}$ the velocity, $\vec{A}$ the magnetic potential, m the mass, e the charge and c the velocity of light. The subscript s refers to the ions and electrons, and

$$A_y = \int B_0 (x) \, \mathrm{d} x.$$

For a weakly-varying static magnetic field, one obtains

$$P_{sy} = m_s v_y + \Omega_s m_s \varkappa \tag{1b}$$

where

$$\Omega_s = \frac{e_s B_0}{m_s c}.$$

Since any function of the constants of motion is a steady-state solution of the Vlasov equation, one may choose

$$f_{s0} = n_0 \left(\frac{m_s}{2\pi K T_{s\perp}}\right) \left(\frac{m_s}{2\pi K T_{s\parallel}}\right)^{1/2} \left[1 + \varepsilon_s \left(\varkappa + \frac{v_y}{\Omega_s}\right)\right] \times$$
$$\times\, e^{-\frac{m_s}{2}\left(\frac{v_\perp^2}{K T_{s\perp}} + \frac{v_\parallel^2}{K T_{s\parallel}}\right) + \frac{m_s g_s \varkappa}{K T_{s\perp}}} \tag{2}$$

so that at $x = 0$ (the »local approximation« which will be discussed shortly), one obtains

$$\varepsilon_s \approx \frac{1}{n_0}\frac{d n_0}{d \varkappa} - \frac{m_s g_s}{K T_{s\perp}} \ll \frac{1}{a_s} \tag{3}$$

where, T denotes the temperature, n the number density and

$$a_s = \frac{\sqrt{\dfrac{K T_{s\perp}}{m_s}}}{\Omega_s}.$$

If the equilibrium specified by (2) is perturbed, the perturbations evolve according to Vlasov's equation and Maxwell's equations,

$$\frac{\partial f_s}{\partial t} + \vec{v} \cdot \nabla f_s + \frac{e_s}{m_s}\left[\vec{E} + \frac{1}{c}\, \vec{v} \times \vec{B} + \frac{m_s}{e_s}\vec{g}_s\right] \cdot \nabla_v f_s = 0 \tag{4}$$

$$\nabla \cdot \vec{E} = 4\pi \sum_s e_s \int d\vec{v} f_s \tag{5}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \sum_s e_s \int d\vec{v}\, \vec{v} f_s + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \tag{6}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{7}$$

$$\nabla \cdot \vec{B} = 0. \tag{8}$$

Let

$$q = q_0 + q_1, \ | q_1 | \ll | q_0 | \tag{9}$$

where the subscripts 0 and 1 refer, respectively, to the state of equilibrium and perturbation. Upon linearising the perturbations with respect to the state of the equilibrium, Eq. (4) becomes

$$\frac{\partial f_{s1}}{\partial t} + \vec{v} \cdot \nabla f_{s1} + \frac{e_s}{m_s}\left[\frac{1}{c}\vec{v} \times \vec{B}_0 + \frac{m_s}{e_s}\vec{g}_s\right] \cdot \nabla_v f_{s1} =$$

$$= - \frac{e_s}{m_s}\left[\vec{E}_1 + \frac{1}{c}\vec{v} \times \vec{B}_1\right] \cdot \nabla_v f_s \tag{10}$$

One may formally solve Eq. (10) by introducing the parameter t, and defining the transformation

$$t, \vec{x}, \vec{v} \Rightarrow t', \vec{x}'(t), \vec{v}'(t)$$

by

$$\left.\begin{aligned} & t' = t \\ & \frac{d\vec{x}'}{dt} = \vec{v}'(t) \\ & \frac{d\vec{v}'}{dt} = \Omega_s(\vec{v}' \times \hat{i}_\beta) + g_s \hat{i}_x. \end{aligned}\right\} \tag{11}$$

Note that Eqs. (11) describe orbits of charged particles moving in the unperturbed force fields

$$\vec{F}_s = \frac{e_s}{c}\vec{v} \times \vec{B}_0 + m_s \vec{g}_s.$$

In terms of this transformation, Eq. (10) becomes

$$\frac{d}{dt'} f_{s1}(\vec{x}', \vec{v}', t') = - \frac{e_s}{m_s}\left[\vec{E}_1 + \frac{1}{c}\vec{v}' \times \vec{B}_1\right] \cdot \nabla_{v'} f_{s0} \tag{12}$$

and choosing the initial condition

$$t' = t, \ \vec{x}' = \vec{x}, \ \vec{v}' = \vec{v} \tag{13}$$

one obtains

$$f_{s1}(\vec{x}, \vec{v}, t) = - \frac{e_s}{m_s}\int_{-\infty}^{t} dt' \left[\vec{E}_1 + \frac{1}{c}\vec{v}' \times \vec{B}_1\right] \cdot \nabla_{v'} f_{s0}. \tag{14}$$

One now assumes that all perturbed quantities have space-time dependence given by $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$. (Now, because the equilibrium distribution function f_{s0} is a function of x, strictly speaking, it is not possible to express the fields and the perturbed distribution function exactly in terms of a single harmonic dependence, as in the foregoing. But, such complications can be avoided by adopting a »local approximation«[1]. This is a valid approximation if (3) is satisfied.) It will be assumed further that ω has a small positive imaginary part so that the perturbations vanish at $t = -\infty$. This would give one the prescription for handling some integrals which would otherwise be improper. It is equivalent to Landau's prescription[14]. The perturbed electro-magnetic field can be shown to be electrostatic (see section 7) for waves for which

$$\frac{\omega}{k_\beta} \ll v_A = \frac{B_0}{\sqrt{4\pi n_0 m_i}}$$

and this approximation is valid in the limit of small β[1]. Thus, putting

$$\vec{E}_1 = -\nabla \Phi_1 \tag{15}$$

the dispersion relation follows from Eq. (5)

$$\nabla^2 \Phi_1 = -4\pi \sum_s e_s \int d\vec{v} f_{s1}. \tag{16}$$

Noting

$$\left.\begin{aligned} \nabla \Phi_1 \cdot \nabla_v f_{s0} &= \Big[-\frac{m_s}{K T_{s\|}} \vec{v} \cdot \nabla \Phi_1 + \\ &+ i k y \left\{\frac{\varepsilon_s}{\Omega_s} - \frac{m_s}{K T_{s\perp}}\left(1 - \frac{T_{s\perp}}{T_{s\|}}\right) v_y\right\}\Phi_1\Big] f_{s0} \\ \frac{d\Phi_1}{dt} &= \frac{\partial \Phi_1}{\partial t} + \vec{v} \cdot \nabla \Phi_1 \end{aligned}\right\} \tag{17}$$

one obtains from (14)

$$\begin{aligned} f_{s1} = \frac{e_s}{m_s} f_{s0} \Big\{-\frac{m_s}{K T_{s\|}} + i \int_{-\infty}^{t} \Big[-\frac{\omega m_s}{K T_{s\|}} + k_y \left\{\frac{\varepsilon_s}{\Omega_s} - \frac{m_s}{K T_{s\perp}}\left(1 - \frac{T_{s\perp}}{T_{s\|}}\right) v_y'\right\}\Big] \times \\ \times e^{i[k_y(y'-y) + k_\beta(\beta'-\beta) - \omega(t'-t)]} dt'\Big\} \Phi_1. \end{aligned} \tag{18}$$

The lowest-order solutions to (11) are

$$\left.\begin{aligned} v_y' &= v_\perp \cos(\Omega_s \tau + \varphi) - \frac{g_s}{\Omega_s} \\ v_\beta' &= v_\beta \end{aligned}\right\} \tag{19}$$

and

$$\left.\begin{aligned} y' - y &= \frac{v_\perp}{\Omega_s}\left[\sin(\Omega_s \tau + \varphi) - \sin\varphi\right] - \frac{g_s}{\Omega_s}\tau \\ \beta' - \beta &= v_\beta \tau \end{aligned}\right\} \tag{20}$$

where

$$\tau = t' - t.$$

Noting

$$e^{i\Lambda \sin\Theta} = \sum_{n=-\infty}^{\infty} J_n(\Lambda)\, e^{in\Theta} \tag{21}$$

J_n being the Bessel function of order n, and using (19) and (20), (18) becomes

$$f_{s1} = \frac{n_0 e_s}{m_s}\Phi_1 \left(\frac{m_s}{2\pi K T_{s\perp}}\right)\left(\frac{m_s}{2\pi K T_{s\parallel}}\right)^{1/2} \times$$

$$\times \left\{-\frac{m_s}{K T_{s\parallel}} - \left[\frac{\omega + \left(n\Omega_s - \frac{k_\perp g_s}{\Omega_s}\right)\left(\frac{T_{s\parallel}}{T_{s\perp}} - 1\right)}{K T_{s\parallel}/m_s} - \frac{\varepsilon_s k_\perp}{\Omega_s}\right] \times\right.$$

$$\times \sum_m \sum_n \frac{J_m\left(\frac{k_\perp v_\perp}{\Omega_s}\right) J_n\left(\frac{k_\perp v_\perp}{\Omega_s}\right)}{-\omega + k_\parallel v_\parallel + n\Omega_s - \frac{k_\perp g_s}{\Omega_s}}\, e^{i(n-m)\varphi} \times$$

$$\left.\times \left(1 + \frac{\varepsilon_s n}{k_\perp}\right) e^{-\frac{m_s}{2}\left(\frac{v_\perp^2}{K T_{s\perp}} + \frac{v_\parallel^2}{K T_{s\parallel}}\right)}\right\} \tag{22}$$

where the subscripts $\|$ and $\perp$ denote, respectively, subscripts z and y.

The perturbed number density is then given by

$$n_{s1} = \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} d v_\parallel \int_0^{\infty} d v_\perp\, v_\perp f_{s1}. \tag{23}$$

Using

$$\int_0^\infty 2\pi v_\perp\, d v_\perp \left(\frac{m_s}{2\pi K T_{s\perp}}\right) e^{-\frac{m_s v_\perp^2}{2 K T_{s\perp}}} J_n^2\left(\frac{k_\perp v_\perp}{\Omega_s}\right) =$$

$$= I_n\left(\frac{k_\perp^2 K T_{s\perp}}{m_s \Omega_s^2}\right) e^{-\frac{k_\perp^2 K T_{s\perp}}{m_s \Omega_s^2}} \tag{24}$$

I_n being the modified Bessel function of order n, one obtains

$$n_{s1} = -\frac{e_s n_0}{K T_{s\|}} \Phi_1 \left\{ 1 + \sum_n I_n \left(\frac{k_\perp^2 K T_{s\perp}}{m_s \Omega_s^2} \right) e^{-\frac{k_\perp^2 K T_{s\perp}}{m_s \Omega_s^2}} \times \right.$$

$$\left. \times \left(1 + \frac{\varepsilon_s n}{k_\perp}\right) \frac{\omega + \left(n \Omega_s - \frac{k_\perp g_s}{\Omega_s}\right)\left(\frac{T_{s\|}}{T_{s\perp}} - 1\right) - \frac{\varepsilon_s k_\perp K T_{s\|}}{m_s \Omega_s}}{k_\| \sqrt{\frac{2 K T_{s\|}}{m_s}}} \times Z(\zeta_s) \right\} \quad (25)$$

where $Z(\zeta_s)$ is the Fried-Conte dispersion function

$$Z(\zeta_s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\varkappa \frac{e^{-\varkappa^2}}{\varkappa - \zeta_s}$$

and

$$\zeta_s = \frac{\omega - n \Omega_s + \frac{k_\perp g_s}{\Omega_s}}{k_\| \sqrt{\frac{2 K T_{s\|}}{m_s}}}. \quad (26)$$

Note[15]

$$Im\,[Z(\zeta)] = \sqrt{\pi}\, e^{-\zeta^2}$$

$$Re\,[Z(\zeta)] = \begin{cases} -2\zeta \left[1 - \frac{2\zeta^2}{3} + \dots\right], & |\zeta| \ll 1 \\ -\frac{1}{\zeta}\left[1 + \frac{1}{2\zeta^2} + \dots\right], & |\zeta| \gg 1. \end{cases} \quad (27)$$

3. *Finite-ion-Larmor-radius effects*

Consider a plasma in the absence of any effective gravitational field, with hot electrons ($\zeta_e \ll 1$), and cold ions ($\zeta_i \gg 1$). Since the collisional relaxation times for the electrons are shorter than those for the ions, the electrons in mirror machine are very likely to have an isotropic velocity distribution[11], i. e., $T_{e\perp} = T_{e\|} = T_e$. One then obtains from (25)—(27)

$$n_{e1} \approx \frac{n_0 e}{K T_e} \Phi_1 \left[1 + i \sqrt{\frac{\pi}{2}} \frac{(\omega - \hat{\omega})}{k_\| \sqrt{\frac{K T_e}{m_e}}} \right] \quad (28)$$

$$n_{i1} \approx \frac{n_0 e}{K T_e} \Phi_1 \left\{ \frac{\hat{\omega}}{\omega} - \frac{k_\perp^2 a_i^2}{1 - \omega^2/\Omega_i^2} + \frac{k_\|^2 c_s^2}{\omega^2} + \right.$$

$$+\frac{k_\perp^2 a_i^2}{2}\frac{i\sqrt{\frac{\pi}{2}}\left[\omega+\Omega_i\left(\frac{T_{i\parallel}}{T_{i\perp}}-1\right)+\hat{\omega}\frac{T_{i\parallel}}{T_e}\right]}{k_\parallel\sqrt{\frac{KT_{i\parallel}}{m_i}}}\times e^{-\frac{m_i(\omega-\Omega_i)^2}{2k_\parallel^2 KT_{i\parallel}}}\Bigg\} \tag{29}$$

where

$$c_s^2=\frac{KT_e}{m_i},\ a_i^2=\frac{c_s^2}{\Omega_i^2},\ \hat{\omega}=\frac{\frac{1}{n_0}\frac{\mathrm{d}n_0}{\mathrm{d}x}k_\perp KT_e}{m_e\Omega_e}$$

and we have assumed

$$n_{e0}=n_{i0}=n_0.$$

Using the quasineutrality condition, one then obtains from (28) and (29)

$$\omega_r\approx-\hat{\omega} \tag{30}$$

$$\omega_i\approx\frac{\sqrt{\frac{\pi}{2}}}{k_\parallel\sqrt{\frac{KT_e}{m_e}}}\frac{2\hat{\omega}^2}{1+\frac{k_\perp a_i^2}{1-\hat{\omega}^2/\Omega_i^2}}+$$

$$-k_\perp a_i^2\frac{(T_{i\perp}/T_{i\parallel})\hat{\omega}}{1+\frac{k_\perp^2 a_i^2}{1-\hat{\omega}^2/\Omega_i^2}}\frac{\sqrt{\frac{\pi}{2}}\left[\hat{\omega}\left(\frac{T_{i\parallel}}{T_e}-1\right)+\Omega_i\left(\frac{T_{i\parallel}}{T_{i\perp}}-1\right)\right]}{2k_\parallel\sqrt{\frac{KT_{i\parallel}}{m_i}}}\times$$

$$\times e^{-\frac{m_i(\hat{\omega}+\Omega_i)^2}{2k_\parallel^2 KT_{i\parallel}}}. \tag{31}$$

It is seen that:

(1) finite-ion-Larmor-radius effects, which become operative for waves with wavelengths less than the ion-Larmor-radius ($k_\perp a_i \lesssim 1$), have a stabilising nature on the drift waves, a result which is well known[16]; the physical origin of this effect being the $\frac{\vec{E}\times\vec{B}}{B^2}$ drift characterising the hydrodynamic motion which must be averaged over the orbit of a particle; because of their finite Larmor radius, the ions see a slightly different perturbed electric field than the electrons, causing a slight-out-of-phase motion which may lead to stabilisation;

(2) the finite ion-Larmor-radius effects become stronger as $|\hat{\omega}|$ gets closer to Ω_i — advent of a drift-cyclotron resonance; though, according to (31), the convective instability then disappears, as we shall see later, a flute instability with $k_\parallel = 0$ then arises;

(3) the ion response (represented by the second term on the right in (31)) further destabilises the drift waves if $T_{i\perp} > T_{i\parallel}$;

(4) the term $\dfrac{k_\perp a_i^2}{\left(1 - \dfrac{\hat{\omega}^2}{\Omega_i^2}\right)}$ in the denominators in (31) was missed previously in the literature for the case of an isotropic plasma, this is essential in treating high-frequency drift waves (see Shivamoggi[17]).

4. *Effect of curvature in the magnetic field*

Consider a plasma confined by an outwardly curved magnetic field. One then simulates this by introducing an effective gravitational force in a direction opposite to that of the plasma stratification, $\hat{g} = -g\,\hat{i}_x$. Consider a flute with $k_\parallel = 0$. If we restrict ourselves to $\omega \ll \Omega_s$ only the $n = 0$ term need be retained in (25). Using (25) in (15), one obtains the dispersion relation

$$k_\perp^2 = \sum_s \frac{1}{\lambda_{Ds\parallel}^2}\left[-1 + \left\{1 - \frac{k_\perp K T_{s\parallel}}{m_s \Omega_s}\left(\frac{1}{n_0}\frac{\mathrm{d}n_0}{\mathrm{d}x}\right)\frac{1}{\omega - \dfrac{k_\perp g_s}{\Omega_s}}\right\} \times \right.$$

$$\left. \times\, I_0\left(\frac{k_\perp^2 K T_{s\perp}}{m_s \Omega_s^2}\right) \mathrm{e}^{-\frac{k_\perp^2 K T_{s\perp}}{m_s \Omega_s^2}}\right] \tag{32}$$

where

$$\lambda_{Ds\parallel}^2 = \frac{K T_{s\parallel}}{m_s \omega_{ps}^2}.$$

Let

$$\frac{k_\perp^2 K T_{s\perp}}{m_s \Omega_s^2} \ll 1. \tag{33}$$

Further, since the actual forces on the particles due to a curvature in the magnetic field are proportional to particle energy, let

$$m_e g_e = m_i g_i \frac{T_e}{T_{i\parallel}}. \tag{34}$$

Putting

$$\omega = -\frac{k_\perp g_i}{\Omega_i}\left(\sigma + \frac{T_e}{T_{i\parallel}}\right) \tag{35}$$

Eq. (32) becomes

$$\sigma^2 + \sigma(\tau + \alpha k_\perp^2 a_i^2) + \alpha\tau = 0 \tag{36}$$

where, we have assumed

$$I_0(\delta_i)\,e^{-\delta_i} \approx 1 - \delta_i, \text{ if } \delta_i \ll 1$$

and

$$\tau = 1 + \frac{T_e}{T_{i\parallel}}$$

$$a = \frac{K T_{i\parallel}}{m_i g_i}\left(\frac{1}{n_0}\frac{\mathrm{d} n_0}{\mathrm{d} x}\right)\frac{1}{k_\perp \lambda^2_{D i\parallel}(1 + a_i^2/\lambda^2_{D i\parallel})}$$

$$a_i^2 = \frac{K T_{i\perp}}{m_i \Omega_i^2}.$$

The growth rate is then given by

$$\omega_i \approx \frac{k_\perp g_i}{\Omega_i}\left[a\tau - \frac{1}{4}(\tau + a k_\perp^2 a_i^2)^2\right]^{1/2}. \tag{37}$$

Note the stabilisation caused again by the finite ion-Larmor-radius effects. This stabilising effect, however, diminishes, if $T_{i\perp}$ becomes much larger than $T_{i\parallel}$ a trend we also saw in the previous section.

One obtains the hydrodynamic result by ignoring the stabilising terms, and letting $a_i^2 \gg \lambda^2_{D i\parallel}$

$$\omega_i \approx \sqrt{\tau g_i \frac{1}{n_0}\frac{\mathrm{d} n_0}{\mathrm{d} x}} \tag{38}$$

which is the well-known Rayleigh-Taylor instability induced by a density gradient in the presence of an unfavourable gravitational force[18].

5. *Drift-cyclotron instability*

It turns out an otherwise stable drift wave becomes unstable when it becomes degenerate with the stable cyclotron oscillation, $\hat{\omega} \approx \Omega_s$[19]. Consider a flute mode with

$$k_\parallel = 0,\ \omega \ll \Omega_e,\ \omega \approx \hat{\omega} \approx \Omega_i$$

$$k_\perp = k,\ T_e = T_{i\parallel},\ n_{e0} = n_{i0} = n_0.$$

If

$$\frac{k^2 K T_e}{m_e \Omega_e^2} \ll 1,\ \frac{k_\perp^2 K T_{i\perp}}{m_i \Omega_i^2} \gg 1$$

using (25) in (16), one obtains

$$1 + \frac{1}{k^2 \lambda_D^2}\left[1 - \frac{\omega + \hat{\omega}}{\omega}\right] +$$

$$+ \frac{1}{k^2 \lambda_D^2}\left[1 - \frac{\omega + \Omega_i \left(\frac{T_{i\parallel}}{T_{i\perp}} - 1\right) - \hat{\omega}}{\omega - \Omega_i} I_1 (k^2 a_i^2)\, e^{-k^2 a_i^2}\right] = 0 \qquad (39)$$

from which one obtains the condition for stability

$$\left[\Omega_i (1 + k^2 \lambda_D^2) + \hat{\omega} (1 - \Lambda_1) + \Omega_i \left(\frac{T_{i\parallel}}{T_{i\perp}} - 1\right) \Lambda_1\right]^2 >$$

$$> 4 \hat{\omega}\, \Omega_i (1 + k^2 \lambda_D^2 - \Lambda_1) \qquad (40)$$

where,

$$\Lambda_1 = I_1 (k^2 a_i^2) e^{-k^2 a_i^2} \ll 1.$$

Only for $\hat{\omega} \approx \Omega_i$ can (40) be violated, and drift-cyclotron instability would then arise. It is noticed again that $T_{i\perp} > T_{i\parallel}$ favours this instability.

In the presence of an inwardly curved magnetic field simulated by the introduction of an effective gravitational force in the direction of plasma stratification, one obtains in place of (39)

$$1 + \frac{1}{k^2 \lambda_D^2}\left[1 - \frac{\omega + \hat{\omega}}{\omega - \frac{k g_i}{\Omega_i}}\right] + \frac{1}{k^2 \lambda_D^2}\left[1 +\right.$$

$$\left. + \frac{\omega + \left(\Omega_i - \frac{k g_i}{\Omega_i}\right)\left(\frac{T_{i\parallel}}{T_{i\perp}} - 1\right) - \hat{\omega}}{\omega + \frac{k g_i}{\Omega_i} - \Omega_i} I_1 (k^2 a_i^2)\, e^{-k^2 a_i^2}\right] = 0 \qquad (41)$$

and the condition for stability becomes

$$\left[\Omega_i (1 + k^2 \lambda_D^2) + \left(\hat{\omega} + \frac{k g_i}{\Omega_i}\right)(1 - \Lambda_1) + \left(\Omega_i - \frac{k g_i}{\Omega_i}\right)\left(\frac{T_{i\parallel}}{T_{i\perp}} - 1\right) \Lambda_1\right]^2 >$$

$$> 4 \left\{\left(\hat{\omega} + \frac{k g_i}{\Omega_i}\right)\left(\Omega_i - \frac{k g_i}{\Omega_i}\right) - \frac{k^2 g_i^2}{\Omega_i^2} (1 + k^2 \lambda_D^2) -\right.$$

$$\left. - \frac{k g_i}{\Omega_i} \Lambda_1 \left[\hat{\omega} - \left(\Omega_i - \frac{k g_i}{\Omega_i}\right)\left(\frac{T_{i\parallel}}{T_{i\perp}} - 1\right)\right]\right\} \times (1 + k^2 \lambda_D^2 - \Lambda_1). \qquad (42)$$

Note the stabilising effect of g on the flute mode. This is in accord with the result of Krall and Fowler[20]. The drift-cyclotron instabilities have been observed in experiments[21,22].

6. *Electromagnetic drift-waves in an anisotropic plasma*

The assumption of electrostatic nature for the drift-waves made in the previous sections is valid for low-β plasma or for phase velocities much less than the Alfven velocity

$$\frac{\omega}{k_{||}} \ll v_A \equiv \frac{B_0}{\sqrt{4\pi n_0 m_i}}.$$

We will now relax this restriction, and consider electromagnetic waves in a non-uniform anisotropic plasma.

Consider a plasma in a uniform magnetic field $\vec{B}_0 = B_0 \hat{i}_\beta$. Proceeding as in section 2, one obtains

$$E_{s1} = -\frac{e_s}{m_s}\Bigg\{-\frac{m_s}{K T_{s||}}\frac{E_{1y}}{\mathrm{i}\, k_\perp} + \Bigg[\frac{m_s}{K T_{s||}}\frac{(-\omega + k_{||} v_\beta)}{k_\perp} E_{1y} + $$

$$-\frac{m_s}{K T_{s||}} v_\beta E_{1\beta} + \left(\frac{\varepsilon}{\Omega_s} - \frac{m_s}{K T_{s\perp}}\left\{1 - \frac{T_{s\perp}}{T_{s||}}\right\}\frac{n\,\Omega_s}{k_\perp}\right) \times$$

$$\times \left(E_{1y} - \frac{v_\beta}{\omega}\{k_{||} E_{1y} - k_\perp E_{1\beta}\}\right)\Bigg] \times$$

$$\times \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{J_m\left(\frac{k_\perp v_\perp}{\Omega_s}\right) J_n\left(\frac{k_\perp v_\perp}{\Omega_s}\right)}{\mathrm{i}\,(-\omega + k_{||} v_\beta + n\,\Omega_s)}\, \mathrm{e}^{i(n-m)\varphi}\Bigg\} E_{s0}. \qquad (43)$$

Using (43), Eq. (5) and the z-component of Eq. (6) give

$$\mathrm{i}\, k_\perp E_{1y} + \mathrm{i}\, k_{||} E_{1\beta} = \sum_s \frac{m_s}{K T_{s||}}\frac{\omega_{ps}^2}{i\, k_\perp} E_{1y} -$$

$$-\sum_s \omega_{ps}^2 \left[\int_{-\infty}^{\infty} \mathrm{d}v_\beta \int_0^{\infty} 2\pi v_\perp \,\mathrm{d}v_\perp\, J_n^2\left(\frac{k_\perp v_\perp}{\Omega_s}\right) \times\right.$$

$$\times \left\{\left[\frac{m_s}{K T_{s||}}\frac{(-\omega + k_{||} v_\beta)}{k_\perp} E_{1y} - \frac{m_s}{K T_{s||}} v_\beta E_{1\beta}\right] \times\right.$$

$$\times \left[1 + \frac{n\Omega_s}{\omega}\left(\frac{T_{s\parallel}}{T_{s\perp}} - 1\right)\right] +$$

$$+ \frac{\varepsilon}{\Omega_s}\left[E_{1y} - \frac{v_\beta}{\omega}(k_\parallel E_{1y} - k_\perp E_{1\beta})\right]\Bigg\} \times$$

$$\times \frac{\left(\frac{m_s}{2\pi K T_{s\perp}}\right)\left(\frac{m_s}{2\pi K T_{s\parallel}}\right)^{1/2} e^{-\frac{m_s}{2}\left(\frac{v_\perp^2}{K T_{s\perp}} + \frac{v_\parallel^2}{K T_{s\parallel}}\right)}}{i(-\omega + k_\parallel v_\beta + n\Omega_s)}\Bigg] \tag{44}$$

$$-\frac{i k_\perp c}{\omega}(k_\perp E_{1\beta} - k_\parallel E_{1y}) = -\frac{i\omega}{c}E_{1\beta} -$$

$$-\sum_s \omega_{ps}^2 \Bigg[\int_{-\infty}^{\infty} dv_\beta \int_0^\infty 2\pi v_\perp \, dv_\perp \, J_n^2\left(\frac{k_\perp v_\perp}{\Omega_s}\right) \times$$

$$\times \Bigg\{\left[\frac{m_s}{K T_{s\parallel}} \frac{(-\omega + k_\parallel v_\beta)}{k_\perp} E_{1y} - \frac{m_s}{K T_{s\parallel}} v_\beta E_{1\beta}\right] \times$$

$$\times \left[1 + \frac{n\Omega_s}{\omega}\left(\frac{T_{s\parallel}}{T_{s\perp}} - 1\right)\right] +$$

$$+ \frac{\varepsilon}{\Omega_s}\left[E_{1y} - \frac{v_\beta}{\omega}(k_\parallel E_{1y} - k_\perp E_{1\beta})\right]\Bigg\} \times$$

$$\times \frac{\left(\frac{m_s}{2\pi K T_{s\perp}}\right)\left(\frac{m_s}{2\pi K T_{s\parallel}}\right)^{1/2} e^{-\frac{m_s}{2}\left(\frac{v_\perp^2}{K T_{s\perp}} + \frac{v_\parallel^2}{K T_{s\parallel}}\right)}}{i(-\omega + k_\parallel V_\beta + n\Omega_s)}\Bigg]. \tag{45}$$

Consider the low-frequency behaviour

$$\frac{\omega}{\Omega_s} \ll 1, \quad \frac{k_\perp^2 K T_{s\perp}}{m_s \Omega_s^2} \ll 1$$

so that only $n = 0$ terms need be retained in Eqs. (44) and (45). Using (24), (44) and (45) become

$$\Bigg[k_\perp^2 + \sum_s \omega_{ps}^2 \frac{\varepsilon k_\perp}{\Omega_s \omega} I_0(\sigma_s) e^{-\sigma_s} +$$

$$+ \sum_s \omega_{ps}^2 \frac{m_s}{K T_{s\parallel}} \{1 - I_0(\sigma_s) e^{-\sigma_s}\}\Bigg] \frac{E_{1y}}{k_\perp} +$$

$$+\left[k_{\|} - \sum_s \omega_{ps}^2 \left(\frac{k_\perp \varepsilon}{\Omega_s \omega} - \frac{m_s}{K T_{s\|}}\right) I_0(\sigma_s)\, e^{-\sigma_s} W_{1s}\right] E_{1\beta} = 0 \tag{46}$$

$$\frac{k_{\|}^2 k_\perp^2 c^2}{\omega^2} \frac{E_{1y}}{k_\perp} + \left\{k_{\|} - \frac{k_{\|} k_\perp^2 c^2}{\omega^2} - \right.$$

$$\left. - \sum_s \omega_{ps}^2 \left(\frac{\varepsilon k_\perp}{\Omega_s \omega} - \frac{m_s}{K T_{s\|}}\right) I_0(\sigma_s)\, e^{-\sigma_s} W_{2s}\right\} E_{1\beta} = 0 \tag{47}$$

where

$$\sigma = \frac{k_\perp^2 K T_{s\perp}}{m_s \Omega_s^2} = k_\perp^2 a_s^2,$$

$$W_{1s} = \sqrt{\frac{m_s}{2\pi K T_{s\|}}} \int_{-\infty}^{\infty} \mathrm{d}v_\beta \frac{v_\beta\, e^{-\frac{m_s v_\beta^2}{2 K T_{s\|}}}}{-\omega + k_{\|} v_\beta},$$

$$W_{2s} = \sqrt{\frac{m_s}{2\pi K T_{s\|}}} \int_{-\infty}^{\infty} \mathrm{d}v_\beta \frac{v_\beta\, e^{-\frac{m_s v_\beta^2}{2 K T_{s\|}}}}{-\omega + k_{\|} v_\beta} \left(\frac{k_{\|} v_\beta}{\omega}\right).$$

From (47)

$$\frac{E_{1\beta}}{E_{1y}} = \frac{k_{\|}}{k_\perp} \left[1 + \sum_s \frac{\omega^2 \omega_{ps}^2}{k_\perp^2 k_{\|}^2 c^2} \left(\frac{\varepsilon k_\perp}{\Omega_s \omega} - \frac{m_s}{K T_{s\|}}\right) \times I_0(\sigma_s)\, e^{-\sigma_s} W_{2s}\right. \tag{48}$$

so that if $\frac{\omega}{k_{\|} v_A} \ll 1$,

$$\frac{E_{1\beta}}{E_{1y}} \approx \frac{k_{\|}}{k_\perp} \tag{49}$$

and one has then electrostatic waves.

From Eqs. (46) and (47), the dispersion relation follows:

$$\left[k_\perp^2 + \sum_s \omega_{ps}^2 \frac{\varepsilon k_\perp}{\Omega_s \omega} I_0(\sigma_s)\, e^{-\sigma_s} + \right.$$

$$\left. + \sum_s \omega_{ps}^2 \frac{m_s}{K T_{s\|}} \{1 - I_0(\sigma_s)\, e^{-\sigma_s}\}\right] +$$

$$+ \left[k_{\|} - \sum_s \omega_{ps}^2 \left(\frac{k_\perp \varepsilon}{\Omega_s \omega} - \frac{m_s}{K T_{s\|}}\right) I_0(\sigma_s) \, e^{-\sigma_s} W_{1s}\right] \times$$

$$\times \left[-\frac{\omega^2}{k_{\|} k_\perp^2 c^2}\right] \left[-\frac{k_{\|}^2 k_\perp^2 c^2}{\omega^2} + k_\perp^2 + \sum \omega_{ps}^2 \frac{\varepsilon k_\perp}{\Omega_s \omega} I_0(\sigma_s) \, e^{-\sigma_s} + \right.$$

$$\left. + \sum_s \omega_{ps}^2 \frac{m_s}{K T_{s\|}} \{1 - I_0(\sigma_s) \, e^{-\sigma_s}\}\right] = 0. \tag{50}$$

Consider the limit

$$\sqrt{\frac{K T_{i\|}}{m_i}} \ll \frac{\omega}{k_{\|}} \ll \sqrt{\frac{K T_e}{m_e}} \tag{51}$$

and let, as before

$$T_{e\perp} = T_{e\|} = T_e$$

so that one obtains from (50)

$$\left\{\omega \left[1 + \frac{k_{\|}^2 v_{Te}^2}{\omega_{pe}^2} - \frac{k_{\|}^2 c_s^2}{\hat{\omega}_e^2}\left(1 + \frac{T_e}{T_{i\|}}\right)\right] - \hat{\omega}_e + \right.$$

$$\left. + \frac{i \sqrt{\pi} \, \hat{\omega}_e / k_{\|}}{\sqrt{\frac{K T_e}{m_e}}} (\omega - \hat{\omega}_e)\right\} \left\{\omega^2 - \omega \hat{\omega}_i - \frac{T_{i\|}}{T_{i\perp}} k_{\|}^2 v_A^2\right\} =$$

$$= k_\perp^2 a_i^2 k_{\|}^2 v_A^2 \frac{T_e}{T_{i\|}} (\omega + \hat{\omega}_i) \tag{52}$$

where

$$\hat{\omega}_s = \frac{\varepsilon k_\perp K T_{s\|}}{m_s \Omega_s}, \quad c_s^2 = \frac{K T_e}{m_i}.$$

In the limit $k_\perp^2 a_i^2 \ll 1$, one may drop the right hand side of (52). Also, assuming

$$\frac{k_{\|} \sqrt{\frac{K T_e}{m_e}}}{\omega_{pe}} \ll 1, \quad \frac{k_{\|} c_s}{\omega_e} \ll 1 \tag{53}$$

one obtains from (52)

$$\omega = \hat{\omega}_e + i \sqrt{\pi} \frac{\hat{\omega}_e^2 / k_{\|}}{\sqrt{\frac{K T_e}{m_e}}} \left[\frac{k_{\|}^2 \frac{K T_e}{m_e}}{\omega_{pe}^2} - \left(1 + \frac{T_e}{T_{i\|}}\right) \frac{k_{\|}^2 c_s^2}{\hat{\omega}_e^2}\right]. \tag{54}$$

Note that:

(1) if the electron response (represented by the first term in the bracket on the right) dominates the ion response (represented by the second term on the right), waves are unstable;

(2) ion response contributes damping;

(3) interestingly, the ions out of phase with the waves (because of (51)) produce this damping;

(4) ion-thermal-motion parallel to the static magnetic field primarily sustains this ion response;

(5) for the case of an isotropic plasma, (54) indicates the presence of a wave growth, whereas Krall's[1] calculation for the same case does not show any.

7. *A purely oscillatory transverse wave*

Consider a flute mode $k_{\parallel} = 0$. Using Eqs. (6), (7) and (15), one obtains

$$(k_{\perp}^2 c^2 - \omega^2) = \sum_s \omega_{ps}^2 \frac{T_{s\parallel}}{T_{s\perp}} \times \left[\frac{\omega + n\,\Omega_s \left(\frac{T_{s\parallel}}{T_{s\perp}} - 1 \right)}{2\,K\,T_{s\parallel}/m_s} - \frac{\varepsilon\, k_{\perp}}{\Omega_s} \right] \times$$

$$\times \int \frac{\sum_n J_n^2 \left(\frac{k_{\perp} v_{\perp}}{\Omega_s} \right) e^{-\frac{m_s v_{\perp}^2}{2 K T_{s\perp}}}}{- \omega + n\,\Omega_s} v_{\perp}\, \mathrm{d}v_{\perp}. \tag{55}$$

This represents a purely transverse mode, propagating perpendicular to the static magnetic field with its associated electric field vector along the static magnetic field.

Consider the limit

$$k_{\perp}^2 a_s^2 \ll 1$$

so that neglecting the $n = 0$ terms in (55), one obtains

$$k_{\perp}^2 c^2 = - \sum_s \frac{\omega_{ps}^2}{\omega} \left(\omega - \frac{\varepsilon\, k_{\perp}}{\Omega_s} \frac{K\,T_{s\parallel}}{m_s} \right). \tag{56}$$

The electron term will dominate, being larger by $\frac{m_i}{m_e}$ than the ion term. Thus

$$\omega \approx \frac{2\,\varepsilon\, k_{\perp}\, K\, T_{e\parallel}/m_e\,\Omega_e}{1 + k_{\perp}^2 c^2/\omega_{pe}^2}. \tag{57}$$

It is seen that:

(1) these waves are purely oscillatory;

(2) the energy source of these transverse waves is, interestingly enough, primarily associated with the kinetic energy of the electron-thermal-motion parallel to the static magnetic field, (unlike the Bernstein modes[23], in a uniform plasma);

(3) these waves can be driven unstable only in the presence of an inhomogeneity in the static magnetic field[24].

8. *Conclusions*

In some plasmas of practical interest (ex. mirror machine), the kinetic energy associated with the ion-thermal-motion parallel to the magnetic field is much smaller than that of the ion-cyclotron-motion in the perpendicular direction. First, electrostatic drift-wave instabilities in a nonuniform low-plasma under these conditions are studied. Specifically, three problems are treated:

(1) finite ion-Larmor-radius effects;

(2) effects of curvature in the magnetic field;

(3) drift-cyclotron instability.

The results show that the forementioned anisotropy makes available more free-energy sources to agrravate the various instabilities. Further, this paper also illustrates some features of drift instabilities in the case of an isotropic plasma that have been previously missed in the literature — some of these additional features are essential in treating high-frequency drift waves.

Finally, a study is made of electromagnetic drift waves in a nonuniform, anisotropic, finite-β plasma. The results show that:

(i) the wave are unstable;

(ii) the ions out of resonance with the waves produce some damping.

A purely oscillatory transverse wave exists that derives its energy source from the kinetic energy of the electron-thermal-motion parallel to the unperturbed magnetic field. Further, for the case of an isotropic plasma, the calculations given in this paper indicate instabilities that were missed previously in the literature.

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NESTABILNOST DRIFTNIH VALOVA U ANIZOTROPNOJ PLAZMI

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Proučavana je elektrostatska nestabilnost u neuniformnoj plazmi u kojoj je ionska temperatura u smjeru magnetskog polja mnogo niža od ionske temperature u smjeru okomitom na magnetsko polje. Posebno su proučavani efekt konačnog ionskog Larmorovog radijusa, efekt zakrivljenosti u magnetskom polju i driftna ciklotronska nestabilnost.