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LIE ALGEBRAS AND RELATIVISTIC WAVE-EQUATIONS

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Using the methods of Lie algebras the matrices L_k $(k = 0, 1, 2, 3)$, appearing in the Bhabha wave-equation based on the 16 and 20-dim representations of the group $SO\,(4,1)$ are given as linear combinations of the basis elements of the Lie algebra B_2. The hermitianizing matrix is also constructed and the masses of the particles determined. Finally the transformation properties of the hermitianizing matrix are studied.

1. Introduction

The field of relativistic wave equations is more than a half century old and aims at the description of particles in terms of wave-functions and equations of motion[1-23].

Bhabha in his effort to free the higher spin theories from the presence of the subsidiary conditions proposed an equation which is similar in appearance to the Dirac wave equation and which in the absence of interactions reads

$$L_0 \frac{\partial \Psi}{\partial x_0} + L_1 \frac{\partial \Psi}{\partial x_1} + L_2 \frac{\partial \Psi}{\partial x_2} + L_3 \frac{\partial \Psi}{\partial x_3} + \mathrm{i}\, \chi\, \Psi = 0 \qquad (1)$$

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where L_k, $k = 0, 1, 2, 3$ are four matrices of appropriate dimension depending on the representation according to which the wave function Ψ transforms and χ a constant related to the mass of the particle. The Bhabha field is a multimass and multispin field[11-15].

Making use of the methods of Lie algebras in this paper the matrices L_k $k = 0, 1, 2, 3$ appearing in the Bhabha wave equation based on the 16 and 20-dimensional representation of the group $SO(4, 1)$ are constructed[24-25].

2. *The matrices L_k as linear combinations of the basic elements of B_2*

The Lie algebra corresponding to $SO(4,1)$ is the complex Lie algebra B_2. Its generators are $\vec{h}_{\alpha_1}, \vec{h}_{\alpha_2}, \vec{e}_{\pm\alpha_1}, \vec{e}_{\pm\alpha_2}, \vec{e}_{\pm(\alpha_1+\alpha_2)}, \vec{e}_{\pm(\alpha_1+2\alpha_2)}$. We recall that for any Lie algebra $\mathscr{L}$:

i) any element $\vec{h} \in \mathscr{H}$ is given by

$$\vec{h} = \mu_1 \vec{h}_{\alpha_1} + \mu_2 \vec{h}_{\alpha_2} \tag{2}$$

where $\mathscr{H}$ the Cartan subalgebra of $\mathscr{L}$; $\vec{h}_{\alpha_1}, \vec{h}_{\alpha_2}$ the basis elements of the Cartan subalgebra and μ_1, μ_2 coefficients.

$$\text{ii)}\ [\vec{e}_\alpha, \vec{h}] = \alpha(\vec{h})\, \vec{e}_\alpha, \tag{3}$$

$$\text{iii)}\ [\vec{h}, \vec{h}'] = 0,\ \forall\, \vec{h}, \vec{h}' \in \mathscr{H}, \tag{4}$$

$$\text{iv)}\ [\vec{e}_\alpha, \vec{e}_{-\alpha}] = \vec{h}_\alpha, \tag{5}$$

$$\text{v)}\ [\vec{e}_\alpha, \vec{e}_\beta] = N_{\alpha,\beta}\, \vec{e}_{\alpha+\beta}, \tag{6}$$

where $N_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root of $\mathscr{L}$. $N_{\beta,\alpha} = -N_{\alpha,\beta}$ and by convention $N_{-\alpha,-\beta} = N_{\alpha,\beta}$. Moreover if the α-string of roots containing β is

$$\beta - r\alpha,\ \beta - (r - 1)\,\alpha, \ldots \beta \ldots \beta + q\alpha \tag{7}$$

then the magnitude of $N_{\alpha,\beta}$ is given by

$$(N_{\alpha,\beta})^2 = \frac{1}{2} g\,(r + 1)\,(\alpha, \alpha) \tag{8}$$

with the signs of $N_{\alpha,\beta}$ to some extent being arbitrary. ([,] = commutator, (,) = = inner product).

A five dimensional matrix realization of the complex Lie algebra of $SO(4,1)$ i. e. B_2 and hence of the canonical form of B_2 is given below[26].

For the basis elements of the Cartan subalgebra we have:

$$\vec{h}_{\alpha_1} = -\frac{1}{2}\{\vec{e}_{2,2} - \vec{e}_{4,4} - \vec{e}_{3,3} + \vec{e}_{5,5}\}, \tag{9}$$

$$\vec{h}_{\alpha_2} = -\frac{1}{6}\{\vec{e}_{3,3} - \vec{e}_{5,5}\}. \tag{10}$$

For the basis elements corresponding to the simple roots we have

$$\vec{e}_{\alpha_1} = \frac{1}{\sqrt{6}}\{\vec{e}_{2,3} - \vec{e}_{5,4}\}, \tag{11}$$

$$\vec{e}_{-\alpha_1} = -\frac{1}{\sqrt{6}}\{\vec{e}_{3,2} - \vec{e}_{4,5}\}, \tag{12}$$

$$\vec{e}_{\alpha_2} = \frac{1}{\sqrt{6}}\{\vec{e}_{1,5} - \vec{e}_{3,1}\}, \tag{13}$$

$$\vec{e}_{-\alpha_2} = -\frac{1}{\sqrt{6}}\{\vec{e}_{5,1} - \vec{e}_{1,3}\}. \tag{14}$$

For the other basis elements $e_{\pm(\alpha_1+\alpha_2)}$, $e_{\pm(\alpha_1+2\alpha_2)}$ of B_2 we find

$$\vec{e}_{(\alpha_1+\alpha_2)} = \frac{1}{\sqrt{6}}\{-\vec{e}_{2,1} + \vec{e}_{1,4}\}, \tag{15}$$

$$\vec{e}_{(-\alpha_1-\alpha_2)} = \frac{1}{\sqrt{6}}\{-\vec{e}_{4,1} + \vec{e}_{1,2}\}, \tag{16}$$

$$\vec{e}_{(\alpha_1+2\alpha_2)} = \frac{1}{\sqrt{6}}\{-\vec{e}_{3,4} + \vec{e}_{2,5}\}, \tag{17}$$

$$\vec{e}_{(\alpha_1+2\alpha_2)} = \frac{1}{\sqrt{6}}\{-\vec{e}_{5,2} + \vec{e}_{4,3}\}, \tag{18}$$

where in the above formulae $\vec{e}_{m,n}$ are 5-dim. square matrices in which the (m, n) element is unit and all the other elements are zero. The elements $\vec{e}_{\pm(\alpha_1+\alpha_2)}$ were determined using the elements $\vec{e}_{\pm\alpha_1}$, $e_{\pm c_2}$ and the commutation relations (6) taking for $N_{\alpha_1,\alpha_2} = N_{-\alpha_1,-\alpha_2} = \frac{1}{\sqrt{6}}$ and $(\alpha_1, \alpha_1) = \frac{1}{3}$. The elements $e_{\pm(\alpha_1+2\alpha_2)}$ were determined similarly using the elements $\vec{e}_{\pm\alpha_2}$, $\vec{e}_{\pm(\alpha_1'+\alpha_2)}$ and taking for $N_{\alpha_2,\alpha_1+\alpha_2} = N_{-\alpha_2,-\alpha_1-\alpha_2} = \frac{1}{\sqrt{6}}$ and $(a_2, a_2) = \frac{1}{6}$.

Bhabha in defining the five dimensional realizations of the matrices L_k, $k = 0, 1, 2, 3$ extended the group $SO(3, 1)$ to the group $SO(4, 1)$ by identifying

$$L_0 = I_{0,4}, \quad L_1 = I_{1,4}, \quad L_2 = I_{2,4}, \quad L_3 = I_{3,4} \tag{19}$$

where $I_{0,4}$, $I_{1,4}$, $I_{2,4}$, $I_{3,4}$ belong to the generators of the five dimensional Lorentz group.

J. F. Cornwell[26] gives the following similarity transformation

$$S = \begin{bmatrix} 0 & i & 0 & -i & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & i & 0 & -i \\ 0 & 0 & 1 & 0 & 1 \\ i\sqrt{2} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

which maps the canonical form of B_2 to the Lie algebra $SO(4,1)$. Using this similarity transformation and constructing $S^{-1} L_k S$, $k = 0, 1, 2, 3$ we find

$$S^{-1}L_0S = S^{-1}I_{0,4}S = \begin{bmatrix} 0 & 0 & i/\sqrt{2} & 0 & i/\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 \\ -i/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i/\sqrt{2} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S^{-1}L_1S = S^{-1}I_{1,4}S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i/2 & 0 & i/2 \\ 0 & i/2 & 0 & -i/2 & 0 \\ 0 & 0 & -i/2 & 0 & -i/2 \\ 0 & i/2 & 0 & -i/2 & 0 \end{bmatrix},$$

$$S^{-1}L_2S = S^{-1}I_{2,4}S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & -1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & -1/2 & 0 & -1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \end{bmatrix}, \tag{21}$$

$$S^{-1}L_3S = S^{-1}I_{3,4}S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \end{bmatrix}.$$

The matrices $S^{-1}L_kS$ form a linear combination of the basis elements of the Lie algebra B_2. Namely

$$S^{-1}L_k S = a\,\Gamma_{5-dim}(\vec{h}_{\alpha_1}) + \beta\,\Gamma_{5-dim}(\vec{h}_{\alpha_2}) + \gamma\,\Gamma_{5-dim}(\vec{e}_{\alpha_1}) + \delta\,\Gamma_{5-dim}(\vec{e}_{-\alpha_1}) +$$

$$+ \varepsilon\,\Gamma_{5-dim}(\vec{e}_{\alpha_2}) + \zeta\,\Gamma_{5-dim}(\vec{e}_{-\alpha_2}) + \eta\,\Gamma_{5-dim}(\vec{e}_{\alpha_1+\alpha_2}) + \Theta\,\Gamma_{5-dim}(\vec{e}_{-\alpha_1-\alpha_2}) +$$

$$+ \varkappa\,\Gamma_{5-dim}(\vec{e}_{\alpha_1+2\alpha_2}) + \lambda\,\Gamma_{5-dim}(\vec{e}_{-\alpha_1-2\alpha_2}) \tag{22}$$

where Γ is used to denote the matrices associated with the elements $\vec{h}_{\alpha_1}, \vec{h}_{\alpha_2}, \vec{e}_{\pm\alpha_1}, \vec{e}_{\pm\alpha_2}, \vec{e}_{\pm(\alpha_1+\alpha_2)}, \vec{e}_{\pm(\alpha_1+2\alpha_2)}$ and $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \Theta, \varkappa, \lambda$ coefficients which for each $k = 0, 1, 2, 3$ can easily be determined by using (9) — (18) and (21). Performing these calculations we find:

$$S^{-1}L_0S = \mathrm{i}\sqrt{3}\,\Gamma_{5-dim}(\vec{e}_{\alpha_2}) + \mathrm{i}\sqrt{3}\,\Gamma_{5-dim}(\vec{e}_{-\alpha_2}) \tag{23}$$

$$S^{-1}L_1S = \frac{\mathrm{i}\sqrt{3}}{\sqrt{2}}\Gamma_{5-dim}(\vec{e}_{\alpha_1}) - \frac{\mathrm{i}\sqrt{3}}{\sqrt{2}}\Gamma_{5-dim}(\vec{e}_{-\alpha_1}) +$$

$$+ \frac{\mathrm{i}\sqrt{3}}{\sqrt{2}}\Gamma_{5-dim}(\vec{e}_{\alpha_1+2\alpha_2}) - \frac{\mathrm{i}\sqrt{3}}{\sqrt{2}}\Gamma_{5-dim}(\vec{e}_{-\alpha_1-2\alpha_2}) \tag{24}$$

$$S^{-1}L_2S = -\frac{\sqrt{3}}{\sqrt{2}}\Gamma_{5-dim}(\vec{e}_{\alpha_1}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma_{5-dim}(\vec{e}_{-\alpha_1}) -$$

$$- \frac{\sqrt{3}}{\sqrt{2}}\Gamma_{5-dim}(\vec{e}_{\alpha_1+2\alpha_2}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma_{5-dim}(\vec{e}_{-\alpha_1-2_2}) \tag{25}$$

$$S^{-1}L_3S = -6\mathrm{i}\,\Gamma_{5-dim}(\vec{h}_{\alpha_2}). \tag{26}$$

The 16 and 20-dimensional realizations of the matrices L_k are given by the same linear combinations of the basis elements of B_2 as for the 5-dimensional representation except that $\Gamma_{5-dim}(\vec{h}_{\alpha_1}) \dots \Gamma_{5-dim}(\vec{e}_{-\alpha_1-2\alpha_2})$ have to be replaced by the 16 or 20-dimensional matrices $\Gamma_{20-dim}(\vec{h}_{\alpha_1}) \dots \Gamma_{20-dim}(\vec{e}_{-\alpha_1-2\alpha_2})$. Thus for example for the 20-dimensional representation of the matrices L_k we have

$$L_0^{20-dim} = \mathrm{i}\sqrt{3}\,\Gamma_{20-dim}(\vec{e}_{\alpha_2}) + \mathrm{i}\sqrt{3}\,\Gamma_{20-dim}(\vec{e}_{-\alpha_2}) \tag{27}$$

$$L_1^{20-dim} = \frac{\mathrm{i}\sqrt{3}}{\sqrt{2}}\Gamma_{20-dim}(\vec{e}_{\alpha_1}) - \mathrm{i}\frac{\sqrt{3}}{\sqrt{2}}\Gamma_{20-dim}(\vec{e}_{-\alpha_1}) +$$

$$+ i\frac{\sqrt{3}}{\sqrt{2}} \Gamma_{20-dim}(\vec{e}_{\alpha_1+2\alpha_2}) - i\frac{\sqrt{3}}{\sqrt{2}} \Gamma_{20-dim}(\vec{e}_{-\alpha_1-2\alpha_2}) \tag{28}$$

$$L_2^{20-dim} = -\frac{\sqrt{3}}{\sqrt{2}} \Gamma_{20-dim}(\vec{e}_{\alpha_1}) - \frac{\sqrt{3}}{\sqrt{2}} \Gamma_{20-dim}(\vec{e}_{-\alpha_1}) -$$

$$-\frac{\sqrt{3}}{\sqrt{3}} \Gamma_{20-dim}(\vec{e}_{\alpha_1+2\alpha_2}) - \frac{\sqrt{3}}{\sqrt{2}} \Gamma_{20-dim}(\vec{e}_{-\alpha_1-2\alpha_2}) \tag{29}$$

$$L_3^{20-dim} = -6i\, \Gamma_{20-dim}(\vec{h}_{\alpha_2}). \tag{30}$$

3. 20-dimensional basis of B_2

We give now in this paragraph the 20-dimensional basis elements of B_2. The fundamental weights of B_2 are

$$\lambda_1 = \alpha_1 + \alpha_2, \quad \lambda_2 = \frac{1}{2}\alpha_1 + \alpha_2. \tag{31}$$

The highest weight is given by

$$\Lambda = q_1\lambda_1 + q_2\lambda_2 \tag{32}$$

where q_1, q_2 are integers satisfying the relation

$$N = \frac{1}{6}(q_1 + 1)(q_2 + 1)(q_1 + q_2 + 2)(2q_1 + q_2 + 3). \tag{33}$$

N gives the degree of the representation with highest weight Λ. For $N = 20$ the formula is satisfied with $q_1 = 0$, $q_2 = 3$. The highest weight then is

$$\Lambda = \frac{3}{2}\alpha_1 + 3\alpha_2. \tag{34}$$

The other weights are

$$\Lambda = \Lambda_1 = \frac{3}{2}\alpha_1 + 3\alpha_2, \; \Lambda_2 = \frac{3}{2}\alpha_1 + 2\alpha_2, \; \Lambda_3 = \frac{3}{2}\alpha_1 + \alpha_2, \; \Lambda_4 = \frac{3}{2}\alpha_1,$$

$$\Lambda_5 = \frac{1}{2}\alpha_1 + 2\alpha_2, \; \Lambda_6 = \Lambda_7 = \frac{1}{2}\alpha_1 + \alpha_2, \; \Lambda_8 = \Lambda_9 = \frac{1}{2}\alpha_1, \; \Lambda_{10} = \frac{1}{2}\alpha_1 - \alpha_2,$$

$$\Lambda_{11} = -\frac{1}{2}\alpha_1 + \alpha_2,\ \Lambda_{12} = \Lambda_{13} = -\frac{1}{2}\alpha_1,\ \Lambda_{14} = \Lambda_{15} = -\frac{1}{2}\alpha_1 - \alpha_2,$$

$$\Lambda_{16} = -\frac{3}{2}\alpha_1,\ \Lambda_{17} = -\frac{1}{2}\alpha_1 - 2\alpha_2,\ \Lambda_{18} = -\frac{3}{2}\alpha_1 - \alpha_2,$$

$$\Lambda_{19} = -\frac{3}{2}\alpha_1 - 2\alpha_2,\ \Lambda_{20} = -\frac{3}{2}\alpha_1 - 3\alpha_2. \tag{35}$$

In the determination of these weights we have made use of the Cartan matrix of B_2 defined by

$$(A_{jk}) = \frac{2(\alpha_j, \alpha_k)}{(\alpha_j, \alpha_j)} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}. \tag{36}$$

The multiplicities of the weighst were determined using Freudentals recursion formula. Using the test of reflections one can check that the above weights are all the weights of the representation.

Using the weights calculated above and the elements of the Cartan matrix (36) we find for the matrices $\Gamma(\vec{h}_{\alpha_1})$, $\Gamma(\vec{h}_{\alpha_2})$ forming the basis of the Cartan subalgebra

$$\Gamma(\vec{h}_{\alpha_1}) = \frac{1}{6}\, diag\,(0, 1, 2, 3, -1, 0, 0, 1, 1, 2, -2, -1, -1, 0, 0, -3, 1, -2, -1, 0), \tag{37}$$

$$\Gamma(\vec{h}_{\alpha_2}) = \frac{1}{12}\, diag\,(3, 1, -1, -3, 3, 1, 1, -1, -1, -3, 3, 1, 1, -1, -1, 3, -3, 1, -1, -3). \tag{38}$$

The other basis elements of B_2.

Constructing the differences $\Lambda_p - \Lambda_q$ between the weights and finding those for which $\Lambda_p - \Lambda_q = -\alpha_1$ we find that the matrix $\Gamma(\vec{e}_{\alpha_2})$ has elements e_{pq} different from zero at the positions

$$(p, q) = (5,2), (6,3), (7,3), (8,4), (9,4), (11,6), (11,7), (12,8), (12,9), (13,8), (13,9), (14,10), (15,10), (16,12), (16,13), (18,14), (18,15), (19,17). \tag{39}$$

Similarly from the differences $\Lambda_p - \Lambda_q = -\alpha_2$ we find that the matrix $\Gamma(\vec{e}_{\alpha_2})$ has elements ε_{pq} different from zero at the positions

$$(p, q) = (2,1), (3,2), (4,3), (6,5), (7,5), (8,6), (8,7), (9,6), (9,7), (10,8), (10,9), (12,11), (13,11), (14,12), (14,13), (15,12), (15,13), (17, 14), (17,15), (18,16), (19,18), (20,19). \quad (40)$$

The matrix $\Gamma(\vec{e}_{-\alpha_1})$ is taken equal to $-\Gamma^{tr}(\vec{e}_{\alpha_1})$ and the matrix $\Gamma(\vec{e}_{-\alpha_2})$ equal to $-\Gamma^{tr}(\vec{e}_{\alpha_2})$ where tr = transpose.

The matrix $\Gamma(\vec{e}_{\alpha_1+\alpha_2})$ is given by the formula

$$\Gamma(\vec{e}_{\alpha_1+\alpha_2}) = \frac{1}{N_{\alpha_1,\alpha_2}} [\Gamma(\vec{e}_{\alpha_1}), \Gamma(\vec{e}_{\alpha_2})] \quad (41)$$

where $N_{\alpha_1,\alpha_2} = \frac{1}{\sqrt{6}}$ and the matrix $\Gamma(\vec{e}_{\alpha_1+2\alpha_1})$ by the formula

$$\Gamma(\vec{e}_{\alpha_1+2\alpha_2}) = \frac{1}{N_{\alpha_2,\alpha_1+\alpha_2}} [\Gamma(\vec{e}_{\alpha_2}), \Gamma(\vec{e}_{\alpha_1+\alpha_2})] \quad (42)$$

where $N_{\alpha_2,\alpha_1+\alpha_2} = \frac{1}{\sqrt{6}}$. The matrix $\Gamma(\vec{e}_{-\alpha_1-\alpha_1})$ is taken equal to $-\Gamma^{tr}(\vec{e}_{\alpha_1+\alpha_2})$ and the matrix $\Gamma(\vec{e}_{-\alpha_1-2\alpha_2})$ is taken equal to $-\Gamma^{tr}(\vec{e}_{\alpha_1+2\alpha_2})$. Thus all the basis elements $\Gamma(\vec{e}_{\pm\alpha_1})$, $\Gamma(\vec{e}_{\alpha_2\pm})$, and $\Gamma(\vec{e}_{\pm(\alpha_1+2\alpha_2)})$ are function of the quantities e_{ij}, ε_{ij}. To determine these quantities we make use of the commutation relations of the Lie algebra, namely:

$$[\Gamma(\vec{e}_{\alpha_1}), \Gamma(\vec{e}_{-\alpha_1})] = \Gamma(\vec{h}_{\alpha_1}), [\Gamma(\vec{e}_{\alpha_2}), \Gamma(\vec{e}_{-\alpha_2})] = \Gamma(\vec{h}_{\alpha_2}), \quad (43)$$

$$[\Gamma(\vec{e}_{\alpha_1}), \Gamma(\vec{e}_{-\alpha_2})] = 0, [\Gamma(\vec{e}_{\alpha_1+\alpha_2}), \Gamma(\vec{e}_{-\alpha_1-\alpha_2})] = \Gamma(\vec{h}_{\alpha_1}) + \Gamma(\vec{h}_{\alpha_2}), \quad (44)$$

$$[\Gamma(\vec{e}_{\alpha_1}), \Gamma(\vec{e}_{\alpha_1+\alpha_2})] = 0, [\Gamma(\vec{e}_{-\alpha_1}), \Gamma(\vec{e}_{\alpha_1+2\alpha_2})] = 0, \quad (45)$$

$$[\Gamma(\vec{e}_{\alpha_2}), \Gamma(\vec{e}_{\alpha_1+2\alpha_2})] = 0, [\Gamma(\vec{e}_{\alpha_1}), \Gamma(\vec{e}_{\alpha_1+2\alpha_2})] = 0, \quad (46)$$

$$[\Gamma(\vec{e}_{\alpha_1+\alpha_2}), \Gamma(\vec{e}_{\alpha_1+2\alpha_2})] = 0, [\Gamma(\vec{e}_{\alpha_1+2\alpha_2}), \Gamma(\vec{e}_{-\alpha_1-2\alpha_2})] = \Gamma(\vec{h}_{\alpha_1}) + 2\Gamma(\vec{h}_{\alpha_2}), \quad (47)$$

from each one of these relations we derive a set of equations satisfied by the quantities e_{ij}, ε_{ij}. Solving these equations simultaneously we find

$$\varepsilon_{2,1} = \pm\frac{1}{2},\ \varepsilon_{4,3} = \pm\frac{1}{2},\ \varepsilon_{18,16} = \pm\frac{1}{2},\ \varepsilon_{20,19} = \pm\frac{1}{2},\ \varepsilon_{3,2} = \pm\frac{1}{\sqrt{3}},$$

$$\varepsilon_{19,18} = \pm\frac{1}{\sqrt{3}},\ \varepsilon_{17,15} = 0,\ \varepsilon_{17,14} = \pm\frac{1}{2},\ \varepsilon_{12,11} = 0,\ \varepsilon_{13,11} = \pm\frac{1}{2},$$

$$\varepsilon_{14,12} = 0,\ \varepsilon_{15,12} = \pm\frac{1}{2\sqrt{3}},\ \varepsilon_{15,13} = 0,\ \varepsilon_{14,11} = \pm\frac{1}{\sqrt{3}},\ \varepsilon_{6,5} = 0,\ \varepsilon_{7,5} = \pm\frac{1}{2},$$

$$\varepsilon_{8,6} = 0,\ \varepsilon_{9,6} = \pm\frac{1}{2\sqrt{3}},\ \varepsilon_{9,7} = 0,\ \varepsilon_{8,7} = \pm\frac{1}{\sqrt{3}},\ \varepsilon_{10,8} = \pm\frac{1}{2},\ \varepsilon_{10,9} = 0, \tag{48}$$

$$e_{5,2}\pm = \frac{1}{\sqrt{6}},\ e_{19,17} = \pm\frac{1}{\sqrt{6}},\ e_{18,14} = \pm\frac{\sqrt{2}}{3},\ e_{18,15} = \pm\frac{1}{3},\ e_{15,10} = \pm\frac{1}{3},$$

$$e_{11,17} = \pm\frac{\sqrt{2}}{3},\ e_{11,6} = \pm\frac{1}{3},\ e_{7,3} = \pm\frac{\sqrt{2}}{3},\ e_{6,3} = \pm\frac{1}{3},\ e_{16,13} = \pm\frac{1}{\sqrt{6}},$$

$$e_{16,12} = \pm\frac{1}{\sqrt{3}},\ e_{8,4} = \pm\frac{1}{\sqrt{6}},\ e_{9,4} = \pm\frac{1}{\sqrt{3}},\ e_{14,10} = \pm\frac{\sqrt{2}}{3},\ e_{13,9} = \pm\frac{1}{3\sqrt{3}},$$

$$e_{12,9} = \pm\frac{5}{3\sqrt{6}},\ e_{12,8} = \pm\frac{1}{3\sqrt{3}},\ e_{13,8} = \pm\frac{2\sqrt{2}}{3\sqrt{2}}. \tag{49}$$

Choosing for the ε_{ij} the positive sign this fixes sign of the e_{ij} to positive values. Hence we have for the basis matrices of B_2 the following matrices which we give in terms of their non zero elements.

$\Gamma(\vec{e}_{\alpha_1})$:

$$\Gamma_{5,2} = \frac{1}{\sqrt{6}},\ \Gamma_{6,3} = -\frac{1}{3},\ \Gamma_{7,3} = \frac{\sqrt{2}}{3},\ \Gamma_{8,4} = \frac{1}{\sqrt{6}},\ \Gamma_{9,4} = \frac{1}{\sqrt{3}},\ \Gamma_{11,6} = \frac{1}{3},$$

$$\Gamma_{11,7} = \frac{\sqrt{2}}{3},\ \Gamma_{12,8} = \frac{1}{3\sqrt{3}},\ \Gamma_{12,9} = \frac{5}{3\sqrt{6}},\ \Gamma_{13,8} = \frac{2\sqrt{2}}{3\sqrt{3}},\ \Gamma_{13,9} = \frac{1}{3\sqrt{3}},$$

$$\Gamma_{14,10} = \frac{\sqrt{2}}{3},\ \Gamma_{15,10} = \frac{1}{3},\ \Gamma_{16,12} = \frac{1}{\sqrt{3}},\ \Gamma_{16,13} = \frac{1}{\sqrt{6}},\ \Gamma_{18,14} = \frac{\sqrt{2}}{3},$$

$$\Gamma_{18,15} = \frac{1}{3},\ \Gamma_{19,17} = \frac{1}{\sqrt{6}}. \tag{50}$$

$\Gamma(\vec{e}_{-\alpha_1})$:

$$\Gamma_{2,5} = -\frac{1}{\sqrt{6}},\ \Gamma_{3,6} = -\frac{1}{3},\ \Gamma_{3,7} = -\frac{\sqrt{2}}{3},\ \Gamma_{4,8} = -\frac{1}{\sqrt{6}},\ \Gamma_{4,9} = -\frac{1}{\sqrt{3}},$$

$$\Gamma_{6,11} = -\frac{1}{3},\ \Gamma_{7,11} = -\frac{\sqrt{2}}{3},\ \Gamma_{8,12} = -\frac{1}{3\sqrt{3}},\ \Gamma_{8,13} = -\frac{2\sqrt{2}}{3\sqrt{3}},$$

$$\Gamma_{9,12} = -\frac{5}{3\sqrt{6}},\ \Gamma_{9,13} = -\frac{1}{3\sqrt{3}},\ \Gamma_{10,14} = -\frac{\sqrt{2}}{3},\ \Gamma_{10,15} = -\frac{1}{3},$$

$$\Gamma_{12,16} = -\frac{1}{\sqrt{3}},\ \Gamma_{13,16} = -\frac{1}{\sqrt{6}},\ \Gamma_{14,18} = -\frac{\sqrt{2}}{3},\ \Gamma_{15,18} = -\frac{1}{3},$$

$$\Gamma_{17,19} = -\frac{1}{\sqrt{6}}. \tag{51}$$

$\Gamma(\vec{e}_{\alpha_2})$:

$$\Gamma_{2,1} = \frac{1}{2},\ \Gamma_{3,2} = \frac{1}{\sqrt{3}},\ \Gamma_{4,3} = \frac{1}{2},\ \Gamma_{7,5} = \frac{1}{2},\ \Gamma_{9,6} = \frac{1}{2\sqrt{3}},\ \Gamma_{8,7} = \frac{1}{\sqrt{3}},$$

$$\Gamma_{10,8} = \frac{1}{2},\ \Gamma_{13,11} = \frac{1}{2},\ \Gamma_{15,12} = \frac{1}{2\sqrt{3}},\ \Gamma_{14,13} = \frac{1}{\sqrt{3}},\ \Gamma_{17,14} - \frac{1}{2},$$

$$\Gamma_{18,16} = \frac{1}{2},\ \Gamma_{19,18} = \frac{1}{\sqrt{3}},\ \Gamma_{20,19} = \frac{1}{2}. \tag{52}$$

$\Gamma(\vec{e}_{-\alpha_2})$:

$$\Gamma_{1,2} = -\frac{1}{2},\ \Gamma_{2,3} = -\frac{1}{\sqrt{3}},\ \Gamma_{3,4} = -\frac{1}{2},\ \Gamma_{5,7} = -\frac{1}{2},\ \Gamma_{7,8} = -\frac{1}{\sqrt{3}},$$

$$\Gamma_{6,9} = -\frac{1}{2\sqrt{3}},\ \Gamma_{8,.0} = -\frac{1}{2},\ \Gamma_{11,13} = -\frac{1}{2},\ \Gamma_{13,14} = -\frac{1}{\sqrt{3}},$$

$$\Gamma_{12,15} = -\frac{1}{2\sqrt{3}},\ \Gamma_{14,17} = -\frac{1}{2},\ \Gamma_{16,18} = -\frac{1}{2},\ \Gamma_{18,19} = -\frac{1}{\sqrt{3}},$$

$$\Gamma_{19,20} = -\frac{1}{2}. \tag{53}$$

$\Gamma(\vec{e}_{\alpha_1+\alpha_2})$: Its non-zero elements apart from the mutiplying factor $\sqrt{6}$ are:

$$\Gamma_{5,1} = \frac{1}{2\sqrt{6}},\ \Gamma_{6,2} = \frac{1}{3\sqrt{3}},\ \Gamma_{7,2} = \frac{1}{6\sqrt{6}},\ \Gamma_{8,3} = -\frac{1}{6\sqrt{6}},\ \Gamma_{9,3} = \frac{1}{3\sqrt{3}},$$

$$\Gamma_{10,4} = -\frac{1}{2\sqrt{6}}, \ \Gamma_{11,5} = \frac{\sqrt{2}}{6}, \ \Gamma_{12,6} = \frac{5}{6\sqrt{18}}, \ \Gamma_{12,7} = \frac{1}{9}, \ \Gamma_{13,6} = -\frac{1}{9},$$

$$\Gamma_{13,7} = \frac{\sqrt{2}}{18}, \ \Gamma_{14,8} = -\frac{\sqrt{2}}{18}, \ \Gamma_{14,9} = -\frac{1}{9}, \ \Gamma_{15,8} = \frac{1}{9}, \ \Gamma_{15,9} = -\frac{5}{6\sqrt{18}},$$

$$\Gamma_{17,10} = -\frac{\sqrt{2}}{6}, \ \Gamma_{16,11} = \frac{1}{2\sqrt{6}}, \ \Gamma_{18,12} = -\frac{1}{3\sqrt{3}}, \ \Gamma_{18,13} = \frac{1}{6\sqrt{6}},$$

$$\Gamma_{19,14} = -\frac{1}{6\sqrt{6}}, \ \Gamma_{19,15} = -\frac{1}{3\sqrt{3}}, \ \Gamma_{20,17} = -\frac{1}{2\sqrt{6}}. \tag{54}$$

$\Gamma(\vec{e}_{-\alpha_1-\alpha_2})$: Its non-zero elements apart from the multiplying factor $\sqrt{6}$ are:

$$\Gamma_{1,5} = -\frac{1}{2\sqrt{6}}, \ \Gamma_{2,6} = -\frac{1}{3\sqrt{3}}, \ \Gamma_{2,7} = -\frac{1}{6\sqrt{6}}, \ \Gamma_{3,8} = \frac{1}{6\sqrt{6}}, \ \Gamma_{3,9} = -\frac{1}{3\sqrt{3}},$$

$$\Gamma_{4,10} = \frac{1}{2\sqrt{6}}, \ \Gamma_{5,11} = -\frac{\sqrt{2}}{6}, \ \Gamma_{6,12} = -\frac{5}{6\sqrt{18}}, \ \Gamma_{6,13} = \frac{1}{9}, \ \Gamma_{7,12} = -\frac{1}{9},$$

$$\Gamma_{7,13} = -\frac{\sqrt{2}}{18}, \ \Gamma_{8,14} = \frac{\sqrt{2}}{18}, \ \Gamma_{8,15} = -\frac{1}{9}, \ \Gamma_{9,14} = \frac{1}{9}, \ \Gamma_{9,15} = \frac{5}{6\sqrt{18}},$$

$$\Gamma_{11,16} = -\frac{1}{2\sqrt{6}}, \ \Gamma_{10,17} = \frac{\sqrt{2}}{6}, \ \Gamma_{12,18} = \frac{1}{3\sqrt{3}}, \ \Gamma_{13,18} = -\frac{1}{6\sqrt{6}},$$

$$\Gamma_{14,19} = \frac{1}{6\sqrt{6}}, \ \Gamma_{15,19} = \frac{1}{3\sqrt{3}}, \ \Gamma_{17,20} = \frac{1}{2\sqrt{6}}. \tag{55}$$

$\Gamma(\vec{e}_{\alpha_1+2\alpha_2})$: Its non-zero elements apart from the multiplying factor 6 are:

$$\Gamma_{6,1} = -\frac{1}{6\sqrt{3}}, \ \Gamma_{7,1} = \frac{1}{6\sqrt{6}}, \ \Gamma_{8,2} = \frac{1}{3\sqrt{18}}, \ \Gamma_{9,2} - \frac{1}{18}, \ \Gamma_{10,3} = \frac{1}{6\sqrt{6}},$$

$$\Gamma_{12,5} = -\frac{1}{18}, \ \Gamma_{13,5} = \frac{1}{3\sqrt{18}}, \ \Gamma_{14,6} = -\frac{1}{18\sqrt{3}}, \ \Gamma_{14,7} = \frac{2}{9\sqrt{6}}, \ \Gamma_{15,6} = \frac{5}{18\sqrt{6}},$$

$$\Gamma_{15,7} = -\frac{1}{18\sqrt{3}}, \ \Gamma_{17,8} = \frac{1}{9\sqrt{2}}, \ \Gamma_{17,9} = -\frac{1}{18}, \ \Gamma_{18,11} = \frac{1}{6\sqrt{6}}, \ \Gamma_{19,12} = -\frac{1}{18},$$

$$\Gamma_{19,13} = \frac{1}{3\sqrt{18}}, \ \Gamma_{20,14} = \frac{1}{6\sqrt{6}}, \ \Gamma_{20,15} = -\frac{1}{6\sqrt{3}}. \tag{56}$$

$\Gamma(\vec{e}_{-\alpha_1-2\alpha_2})$: Its non-zero elements apart from the multiplying factor 6 are:

$$\Gamma_{1,6}=\frac{1}{6\sqrt{3}},\ \Gamma_{1,7}=-\frac{1}{6\sqrt{6}},\ \Gamma_{2,8}=-\frac{1}{3\sqrt{18}},\ \Gamma_{2,9}=\frac{1}{18},\ \Gamma_{3,10}=-\frac{1}{6\sqrt{6}},$$

$$\Gamma_{5,12}=\frac{1}{18},\ \Gamma_{5,13}=-\frac{1}{3\sqrt{18}},\ \Gamma_{6,14}=\frac{1}{18\sqrt{3}},\ \Gamma_{6,15}=-\frac{5}{18\sqrt{6}},\ \Gamma_{7,14}=-\frac{2}{9\sqrt{6}},$$

$$\Gamma_{7,15}=\frac{1}{18\sqrt{3}},\ \Gamma_{8,17}=-\frac{1}{9\sqrt{2}},\ \Gamma_{9,17}=\frac{1}{18},\ \Gamma_{11,18}=-\frac{1}{6\sqrt{6}},\ \Gamma_{12,19}=\frac{1}{18},$$

$$\Gamma_{13,19}=-\frac{1}{3\sqrt{18}},\ \Gamma_{14,20}=-\frac{1}{6\sqrt{6}},\ \Gamma_{15,20}=\frac{1}{6\sqrt{3}}. \tag{57}$$

4. *The matrices L_k^{20-dim} $(k=0,1,2,3)$*

We are now in the position to construct the matrices L_k^{20-dim} for which we have:

$$L_0^{20-dim}=\mathrm{i}\sqrt{3}\,\Gamma(\vec{e}_{\alpha_2})+\mathrm{i}\sqrt{3}\,\Gamma(\vec{e}_{-\alpha_2})=$$

$$-\mathrm{i}\sqrt{3}\left[\begin{array}{cccccccccccccccccccc}
\cdot & -1/2 & \cdot & \cdot & \cdot & & & & & & & & & & & & & & & \\
1/2 & \cdot & -1/\sqrt{3} & \cdot & \cdot & & & & & & & & & & & & & & & \\
\cdot & 1/\sqrt{3} & \cdot & -1/2 & \cdot & & & & & & & & & & & & & & & \\
\cdot & \cdot & 1/2 & \cdot & \cdot & & & & & & & & & & & & & & & \\
 & & & & \cdot & \cdot & -1/2 & \cdot & \cdot & \cdot & & & & & & & & & & \\
 & & & & \cdot & \cdot & \cdot & \cdot & -1/2\sqrt{3} & \cdot & & & & & & & & & & \\
 & & & & 1/2 & \cdot & \cdot & -1/\sqrt{3} & \cdot & \cdot & & & & & & & & & & \\
 & & & & \cdot & \cdot & 1/\sqrt{3} & \cdot & \cdot & -1/2 & & & & & & & & & & \\
 & & & & \cdot & 1/2\sqrt{3} & \cdot & \cdot & \cdot & \cdot & & & & & & & & & & \\
 & & & & \cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & & & & & & & & & \\
 & & & & & & & & & & \cdot & \cdot & -1/2 & \cdot & \cdot & \cdot & & & & \\
 & & & & & & & & & & \cdot & \cdot & \cdot & \cdot & -1/2\sqrt{3} & \cdot & & & & \\
 & & & & & & & & & & 1/2 & \cdot & \cdot & -1/\sqrt{3} & \cdot & \cdot & & & & \\
 & & & & & & & & & & \cdot & \cdot & 1/\sqrt{3} & \cdot & \cdot & \cdot & -1/2 & \cdot & \cdot & \cdot \\
 & & & & & & & & & & \cdot & 1/2\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1/2 & \cdot & \cdot \\
 & & & & & & & & & & \cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & & & & & & & & & 1/2 & \cdot & \cdot & -1/\sqrt{3} & \cdot \\
 & & & & & & & & & & & & & & & \cdot & \cdot & 1/\sqrt{3} & \cdot & -1/2 \\
 & & & & & & & & & & & & & & & \cdot & \cdot & \cdot & 1/2 & \cdot
\end{array}\right]. \tag{58}$$

The matrix L_0^{20-dim} is hermitian.

$$L_1^{20-dim}=\mathrm{i}\frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{\alpha_1})-\mathrm{i}\frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{-\alpha_1})+\mathrm{i}\frac{\sqrt{3}}{\sqrt{2}}(\vec{e}_{\alpha_1+2\alpha_2})-\mathrm{i}\frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{-\alpha_1-2\alpha_2})=$$

$$
-i\frac{\sqrt{3}}{\sqrt{2}}\left[\begin{array}{cccccccccccccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & -1/\sqrt{3} & 1/\sqrt{6} & & & & & & & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & 1/\sqrt{6} & \cdot & \cdot & 2/\sqrt{18} & -1/3 & & & & & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1/3 & \sqrt{2}/3 & \cdot & \cdot & 1/\sqrt{6} & & & & & & & & & & \\
\cdot & \cdot & \cdot & & & & & 1/\sqrt{6} & 1/\sqrt{3} & \cdot & \cdot & \cdot & & & & & & & & \\
\cdot & 1/\sqrt{6} & \cdot & & & & & \cdot & \cdot & \cdot & \cdot & -1/3 & 2/\sqrt{18} & & & & & & & \\
-1/\sqrt{3} & \cdot & 1/3 & & & & & & & & 1/3 & \cdot & \cdot & -1/3\sqrt{3} & 5/3\sqrt{6} & & & & & \\
1/\sqrt{6} & \cdot & \sqrt{2}/3 & & & & & & & & \sqrt{2}/3 & \cdot & \cdot & 4/3\sqrt{6} & -1/3\sqrt{3} & & & & & \\
\cdot & 2/\sqrt{18} & \cdot & 1/\sqrt{6} & & & & & & & \cdot & 1/3\sqrt{3} & \frac{2\sqrt{2}}{3\sqrt{3}} & \cdot & \cdot & \cdot & 2/3\sqrt{2} & & & \\
\cdot & -1/3 & \cdot & 1/\sqrt{3} & \cdot & & & & & & & 5/3\sqrt{6} & 1/3\sqrt{3} & \cdot & \cdot & \cdot & -1/3 & & & \\
 & & 1/\sqrt{6} & \cdot & \cdot & & & & & & & \cdot & \cdot & \sqrt{2}/3 & 1/3 & \cdot & \cdot & \cdot & & \\
 & & & & \cdot & 1/3 & \sqrt{2}/3 & \cdot & & & & & & & & \cdot & \cdot & 1/\sqrt{6} & & \\
 & & & & -1/3 & \cdot & \cdot & 1/3\sqrt{3} & 5/3\sqrt{6} & & & & & & & 1/\sqrt{3} & \cdot & \cdot & -1/3 & \\
 & & & & 2/\sqrt{18} & \cdot & \cdot & \frac{2\sqrt{2}}{3\sqrt{3}} & 1/3\sqrt{3} & & & & & & & 1/\sqrt{6} & \cdot & \cdot & 2/\sqrt{18} & \\
 & & & & & -1/3\sqrt{3} & 4/3\sqrt{6} & \cdot & \cdot & \sqrt{2}/3 & & & & & & \cdot & \cdot & \sqrt{2}/3 & \cdot & 1/\sqrt{6} \\
 & & & & & 5/3\sqrt{6} & -1/3\sqrt{3} & \cdot & \cdot & 1/3 & \cdot & & & & & & & 1/3 & \cdot & -1/\sqrt{3} \\
 & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/\sqrt{3} & 1/\sqrt{6} & & & & & \cdot & \cdot & \cdot \\
 & & & & & & & 2/3\sqrt{2} & -1/3 & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \cdot & 1/\sqrt{6} & \cdot \\
 & & & & & & & & & & 1/\sqrt{6} & \cdot & \cdot & \sqrt{2}/3 & 1/3 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & & & & & -1/3 & 2/\sqrt{18} & \cdot & \cdot & \cdot & 1/\sqrt{6} & \cdot & \cdot & \cdot \\
 & & & & & & & & & & & & & 1/\sqrt{6} & -1/\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right] \tag{59}
$$

$$
L_2^{20-dim} = -\frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{\alpha_1}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{-\alpha_1}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{\alpha_1+2\alpha_2}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma(\vec{e}_{-\alpha_1-2\alpha_2}) =
$$

$$
= -i\frac{\sqrt{3}}{\sqrt{3}}\left[\begin{array}{cccccccccccccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & 1/\sqrt{3} & -1/\sqrt{6} & \cdot & \cdot & \cdot & \cdot & & & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & -1/\sqrt{6} & \cdot & \cdot & -2/\sqrt{18} & 1/3 & \cdot & \cdot & & & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & -1/3 & -\sqrt{2}/3 & \cdot & \cdot & -1/\sqrt{6} & \cdot & & & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1/\sqrt{6} & -1/\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & \\
\cdot & 1/\sqrt{6} & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/3 & -2/\sqrt{18} & \cdot & \cdot & \cdot & & & & \\
-1/\sqrt{3} & \cdot & -1/3 & \cdot & & & & & & \cdot & -1/3 & \cdot & \cdot & 1/3\sqrt{3} & -5/3\sqrt{6} & \cdot & & & & \\
1/\sqrt{6} & \cdot & \sqrt{2}/3 & \cdot & \cdot & & & & & \cdot & -\sqrt{2}/3 & \cdot & \cdot & -4/3\sqrt{6} & 1/3\sqrt{3} & \cdot & \cdot & \cdot & & \\
\cdot & 2/\sqrt{18} & \cdot & 1/\sqrt{6} & \cdot & & & & & \cdot & \cdot & -1/3\sqrt{3} & \frac{-2\sqrt{2}}{3\sqrt{3}} & \cdot & \cdot & \cdot & -2/3\sqrt{2} & \cdot & & \\
\cdot & -1/3 & \cdot & 1/\sqrt{3} & \cdot & \cdot & \cdot & & & & \cdot & -5/3\sqrt{6} & -1/3\sqrt{3} & \cdot & \cdot & \cdot & 1/3 & \cdot & & \\
\cdot & \cdot & 1/\sqrt{6} & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & -\sqrt{2}/3 & -1/3 & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1/3 & \sqrt{2}/3 & \cdot & \cdot & \cdot & & & & \cdot & \cdot & \cdot & \cdot & -1/\sqrt{6} & \cdot & \cdot \\
 & & & \cdot & -1/3 & \cdot & \cdot & 1/3\sqrt{3} & 5/3\sqrt{6} & \cdot & & & & \cdot & \cdot & -1/\sqrt{3} & \cdot & \cdot & 1/3 & \cdot \\
 & & & \cdot & 2/\sqrt{18} & \cdot & \cdot & \frac{2\sqrt{2}}{3\sqrt{3}} & 1/3\sqrt{3} & \cdot & \cdot & & & & \cdot & -1/\sqrt{6} & \cdot & \cdot & -2/\sqrt{18} & \cdot \\
 & & & \cdot & \cdot & -1/3\sqrt{3} & 4/3\sqrt{6} & \cdot & \cdot & \sqrt{2}/3 & \cdot & \cdot & & & \cdot & \cdot & \cdot & -\sqrt{2}/3 & \cdot & -1/\sqrt{6} \\
 & & & & \cdot & 5/3\sqrt{6} & -1/3\sqrt{3} & \cdot & \cdot & 1/3 & \cdot & \cdot & \cdot & & & & \cdot & -1/3 & \cdot & 1/\sqrt{3} \\
 & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/\sqrt{3} & 1/\sqrt{6} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & \cdot & 2/3\sqrt{2} & -1/3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1/\sqrt{6} & \cdot \\
 & & & & & & & \cdot & \cdot & \cdot & 1/\sqrt{6} & \cdot & \cdot & \sqrt{2}/\sqrt{3} & 1/3 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & & & & \cdot & -1/3 & 2/\sqrt{18} & \cdot & \cdot & \cdot & 1/\sqrt{6} & \cdot & \cdot & \cdot \\
 & & & & & & & & & & & \cdot & \cdot & 1/\sqrt{6} & -1/\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right] \tag{60}
$$

$$
L_3^{20-dim} = -6i\,\Gamma(h_{\alpha_2}) = -\frac{i}{2}\,diag\,\{3, 1, -1, -3, 3, 1, 1, -1, -1, -3, 3, 1, 1,
$$

$$
-1, -1, 3, -3, 1, -1, -3\}. \tag{61}
$$

The matrices L_1^{20-dim}, L_2^{20-dim}, L_3^{20-dim} are antihermitian.

5. *Hermitianizing matrix A*

Using the hermiticity of the matrix L_0 and the antihermiticity of the matrices L_1, L_2, L_3 a matrix A called the hermitianizing matrix can be found satisfying the following properties

$$A^2 = 1,\ [L_o, A] = 0,\ \{L_m, A\}_+ = 0, (m = 1, 2, 3) \tag{62}$$

where $\{,\}_+$ indicates the anticommutator.

The matrix A can be expressed in terms of the matrix L_0. A general formula for the matrix A and for any value of the spin S is given by Madavarao, Thiruvenkatachar and Venkatachalienger[27–28]. This is

$$A(s) = f(L_0, s) \tag{63}$$

with

$$f(x, s) = \frac{(x-s)(x-s+1)\dots(x+s-1)(x+s)}{2s!} \sum_{n=0}^{2s} \binom{2s}{n} \frac{1}{(x-s+\eta)}. \tag{64}$$

If s is half integer this formula reduces to

$$f(x, s = \textit{half integer}) =$$

$$= \frac{2x\left(x^2 - \frac{1}{4}\right)\left(x^2 - \frac{9}{4}\right)\dots(x^2 - s^2)}{2s!} \sum_{n=1}^{s+1/2} \binom{2s}{s+1/2-n} \frac{1}{[x^2 - (n-1/2)^2]} \tag{65}$$

while if s is integer the formula becomes

$$f(x, s = \textit{integer}) = \frac{(x^2-1)(x^2-2^2)\dots(x^2-s^2)}{(s!)^2} +$$

$$+ \frac{2x^2(x^2-1)\dots(x^2-s^2)}{(2s)!} \sum_{n=1}^{s} \binom{2s}{s-n} \frac{1}{[x^2-n^2]}. \tag{66}$$

Specific examples are:

for $s = \frac{1}{2}$, $A = L_o$ (case of the Dirac equation) (67)

for $s = 1$, $A = 2(L_0)^2 - 1$ (case of the Kemmer equation) (68)

$$\text{for } s = \frac{3}{2},\quad A = \frac{1}{3} L_0 \{4(L_0)^2 - 7\}, \tag{69}$$

$$\text{for } s = 2,\ A = \frac{2}{3}(L_0)^4 - \frac{8}{3}(L_0)^2 + 1. \tag{70}$$

Thus in the case $s = \frac{3}{2}$ using the formula (69) and L_0^{20-dim} we find for the hermitianizing matrix:

$$A_{20-dim} = \frac{1}{3} L_0^{20-dim} \{4 (L_0^{20-dim})^2 - 7\} =$$

$$- \begin{bmatrix} \begin{matrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{matrix} & & \\ & \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} & \\ & & \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{matrix} \end{bmatrix}. \tag{71}$$

A_{20-dim} satisfies the relations

$$(A_{20-dim})^2 = 1, \; [L_0^{20-dim}, A] = 0, \; \{L_0^{20-dim}, A\}_+ = 0, \; m \; . \, 1, 2, 3. \tag{72}$$

6. Eigenvalues of L_0^{20-dim}

The eigenvalues of L_0^{20-dim} give the masses with which the particles of spin 3/2 may appear. These are the roots of the polynomial

$$\det (L_0^{20-dim} - \lambda) = 0. \tag{73}$$

Evaluating this determinant by standard methods we find for the eigenvalues:

$$\begin{aligned} \lambda_j &= \frac{3}{2}, \quad j = 1, 2, 3, 4, \\ \lambda_l &= -\frac{3}{2}, \quad l = 5, 6, 7, 8, \\ \lambda_m &= \frac{1}{2}, \quad m = 9, 10, 11, 12, 13, 14, \\ \lambda_n &= -\frac{1}{2}, \quad n = 15, 16, 17, 18, 19, 20. \end{aligned} \tag{74}$$

The masses are given by the formula $m_i = \frac{\chi}{\lambda_i}$ and thus for a spin 3/2 field which we are considering here the possible different masses are

$$m_{1,2} = \pm \frac{2\chi}{3}, m_{3,4} = \pm 2\chi. \tag{75}$$

7. *16-dimensional representation of B_2*

Using the dimensionality formula (33) with $N = 16$ we find $q_1 = 1$ and $q_2 = 1$ and hence the highest weight of the representation is

$$\Lambda_1 = \frac{3}{2} \alpha_1 + 2 \alpha_2. \tag{76}$$

The other weights are:

$$\Lambda_2 = \frac{3}{2} \alpha_1 + \alpha_2, \; \Lambda_3 = \frac{1}{2} \alpha_1 + 2 \alpha_2, \; \Lambda_4 = \Lambda_5 = \frac{1}{2} \alpha_1 + \alpha_2,$$

$$\Lambda_6 = \Lambda_7 = \frac{1}{2} \alpha_1, \; \Lambda_8 = \frac{1}{2} \alpha_1 - \alpha_2, \; \Lambda_9 = -\frac{1}{2} \alpha_1 + \alpha_2,$$

$$\Lambda_{10} = \Lambda_{11} = -\frac{1}{2} \alpha_1, \quad \Lambda_{12} = \Lambda_{13} = -\frac{1}{2} \alpha_1 - \alpha_2, \quad \Lambda_{14} - \frac{1}{2} \alpha_1 - 2\alpha_2.$$

$$\Lambda_{15} = -\frac{3}{2} \alpha_1 - \alpha_2, \; \Lambda_{16} = -\frac{3}{2} \alpha_1 - 2\alpha_2. \tag{77}$$

The basis elements of the Cartan subalgebra are

$$\Gamma(\vec{h}_{\alpha_1}) = \frac{1}{6} \, diag \, \{1, 2, -1, 0, 0, 1, 1, 2, -2, -1, -1, 0, 0, 1, -2, -1\}, \tag{78}$$

$$\Gamma(\vec{h}_{\alpha_2}) = \frac{1}{12} \, diag \, \{1, -1, 3, 1, 1, -1, -1, -3, 3, 1, 1, -1, -1, -3, 1, -1\}. \tag{79}$$

Working as for the 20-dimensional representation we find by solving the set of simultaneous equations in the quantities e_{ij}, ε_{ij} that a possible solution for the ε_{ij} is:

$$\varepsilon_{2,1} = \pm \frac{1}{2\sqrt{3}}, \; \varepsilon_{5,3} = 0, \quad \varepsilon_{4,3} = \pm \frac{1}{2}, \; \varepsilon_{6,4} = 0, \quad \varepsilon_{7,4} = \pm \frac{1}{\sqrt{3}},$$

$$\varepsilon_{7,5} = 0, \; \varepsilon_{6,5} = \pm \frac{1}{2\sqrt{3}}, \; \varepsilon_{8,6} = 0, \; \varepsilon_{8,7} = \pm \frac{1}{2}, \; \varepsilon_{11,9} = 0, \; \varepsilon_{10,9} = \pm \frac{1}{2},$$

$$\varepsilon_{12,10} = 0, \; \varepsilon_{14,13} = \pm \frac{1}{2}, \; \varepsilon_{18,10} = \pm \frac{1}{\sqrt{3}}, \; \varepsilon_{13,11} = 0, \; \varepsilon_{12,11} = \pm \frac{1}{2\sqrt{3}}, \tag{80}$$

$$\varepsilon_{14,12} = 0.$$

Choosing the positive sign for the ε_{ij} we find for

$$L_0^{16-dim} = \mathrm{i}\sqrt{3}\,(\vec{e}_{\alpha_2}) + \mathrm{i}\sqrt{3}\,\Gamma(\vec{e}_{-\alpha_2}) =$$

$$-1\sqrt{3}\left[\begin{array}{cccccccccccccccc}
\cdot & -1/2\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1/2\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & -1/\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -1/2\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1/2\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1/\sqrt{3} & \cdot & \cdot & \cdot & -1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & -1/\sqrt{3} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1/2\sqrt{3} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/2\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/\sqrt{3} & \cdot & \cdot & \cdot & -1/2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1/2\sqrt{3} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/2\sqrt{3} & \cdot
\end{array}\right]. \tag{81}$$

Its eigenvalues are

$$\lambda_j = \frac{3}{2}, \; j = 1, 2, \; \lambda_l = -\frac{3}{2}, \; l = 3, 4$$

$$\lambda_m = \frac{1}{2}, \; m = 5, 6, 7, 8, 9, 10 \tag{82}$$

$$\lambda_n = -\frac{1}{2}, \; n = 11, 12, 13, 14, 15, 16$$

and the different masses associated with the field are

$$m_{1,2} = \pm \frac{2\chi}{3}, \; m_{3,} = \pm 2\chi. \tag{83}$$

8. *Transformation properties of the hermitianizing matrix*

For every transformation T of $SO\,(4,1)$ which belongs also to the group $SO\,(3,1)$ the hermitianizing matrix A satisfies the following relation

$$T^{+} A\, T = A \tag{84}$$

where T^+ is the hermitian conjugate of T. We demonstrate the validity of this relation in the case of the spin 3/2 Bhabha field based on the 20-dimensional representation. It is sufficient to prove (84) for the infinitesimal transformations $T = 1 + \varepsilon I_{ij}$ where ε the infinitesimal parameter and I_{ij} the infinitesimal generators. The generators I_{ij} can be expressed by their 20-dimensional representation matrices in terms of the basis elements of the Lie algebra as follows:

$$\Gamma_{20-dim}(I_{1,2}) = \mathrm{i}\,\Gamma_{20-dim}(\vec{h}_{\alpha_1}) + \frac{\mathrm{i}}{2}\Gamma_{20-dim}(\vec{h}_{\alpha_2}), \tag{85}$$

$$\Gamma_{20-dim}(I_{1,3}) = -\frac{\sqrt{3}}{\sqrt{2}}\Gamma_{20-dim}(\vec{e}_{\alpha_1}) - \frac{\sqrt{3}}{\sqrt{2}}\Gamma_{20-dim}(\vec{e}_{-\alpha_1}) + $$
$$+\frac{\sqrt{3}}{\sqrt{2}}\Gamma_{20-dim}(\vec{e}_{\alpha_1+2\alpha_2}) + \frac{\sqrt{3}}{\sqrt{2}} + \Gamma_{20-dim}(\vec{e}_{-\alpha_1-2\alpha_2}), \tag{86}$$

$$\Gamma_{20-dim}(I_{1,0}) = -\sqrt{3}\,\Gamma_{20-dim}(\vec{e}_{\alpha_1+\alpha_2}) + \sqrt{3}\,\Gamma_{20-dim}(\vec{e}_{-\alpha_1-\alpha_2}), \tag{87}$$

$$\Gamma_{20-dim}(I_{2,3}) = -\mathrm{i}\frac{\sqrt{3}}{\sqrt{2}}\,\Gamma_{20-dim}(\vec{e}_{\alpha_1}) + \mathrm{i}\frac{\sqrt{3}}{\sqrt{2}}\Gamma_{20-dim}(\vec{e}_{-\alpha_1}) + $$
$$+\mathrm{i}\frac{\sqrt{3}}{\sqrt{2}}\Gamma_{20-dim}(\vec{e}_{\alpha_1+2\alpha_2}) - \mathrm{i}\frac{\sqrt{3}}{\sqrt{2}}\Gamma_{20-dim}(\vec{e}_{-\alpha_1-2\alpha_2}), \tag{88}$$

$$\Gamma_{20-dim}(I_{2,0}) = -\mathrm{i}\sqrt{3}\,\Gamma_{20-dim}(\vec{e}_{\alpha_1+\alpha_2}) + \mathrm{i}\sqrt{3}\,\Gamma_{20-dim}(\vec{e}_{-\alpha_1-\alpha_2}), \tag{89}$$

$$\Gamma_{20-dim}(I_{3,0}) = -\sqrt{3}\,\Gamma_{20+dim}(\vec{e}_{\alpha_2}) + \sqrt{3}\,\Gamma_{20-dim}(\vec{e}_{-\alpha_2}). \tag{90}$$

In order that (84) holds it is necessary that

$$\{\Gamma^{+}_{20-dim}(I_{i,j})\,A + A\,\Gamma_{20-dim}(I_{i,j})\} = 0, \tag{91}$$

be satisfied. Thus using (71) and (85) to (90) we verify that (91) is satisfied and hence (84) holds true for the particular case of the 20-dimensional representation. Similarly it is verified that (84) holds for any other representation of the group $SO(4, 1)$ which belongs also to the group $SO(3, 1)$.

9. *Summary and discussion*

In the present paper we have concentrated on the Bhabha equation based on the 16 and 20-dimensional representations of the group $SO(4, 1)$ describing particles of maximum spin 3/2. By using the methods of Lie algebras we gave matrix representations of the basis elements $\Gamma(\vec{h}_{\alpha_1})$, $\Gamma(\vec{h}_{\alpha_2}) \ldots \Gamma(\vec{e}_{\pm(\alpha_1+2\alpha_2)})$ of the complex Lie algebra B_2 associated with the 16 and 20-dimensional representations of $SO(4, 1)$. It was shown also that the matrices L_k, $k = 0, 1, 2, 3$ appearing in the Bhabha equation can be expressed in terms of the basis elements of B_2 by the linear combinations (23) to (26) (where in the general case the subscript 5-dim must be removed). These matrices were given explicitly. Then by means of the matrix L_0 the hermitianizing matrix A was constructed employing (69). Finding the eigenvalues of L_0 the masses of the particles were determined. Finally it was shown that the hermitianizing matrix A satisfies (84).

Using the results of this paper it can be shown that the charge associated with the spin 3/2 Bhabha field is indefinite. Also by generalizing the ideas involved here to higher representations we can show that the charge of the half integer spin Bhabha fields is indefinite; (exception to this forms the Dirac equation).

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LIEVE ALGEBRE I RELATIVISTIČKE VALNE JEDNADŽBE

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U radu su metodama Lievih algebri konstruirane matrice L_k ($k = 0, 1, 2, 3$) koje se pojavljuju u Bhabhainoj valnoj jednadžbi. Metoda se zasniva na korištenju reprezentacija dimenzije 16 i 20 grupe $SO(4, 1)$, a same matrice L_k su linearna kombinacija elemenata baze u algebri B_2 u danoj reprezentaciji. U radu je konstruirana hermitizirajuća matrica i određene mase čestica. Studirana su transformaciona svojstva hermitizirajuće matrice.