YU ISSN 0015—3206
FZKAAA 15 (1) 83 (1983)

LONGITUDINAL OSCILLATIONS IN A RELATIVISTIC PLASMA

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Received 2 October 1981

Revised manuscript received 20 July 1982

UDC 533.95

Original scientific paper

Longitudinal oscillations in a relativistic plasma are studied. First a three-dimensional space + time approach (which is not manifestly covariant) is used. An approximate evaluation of the dispersion relation for the case of a weakly-relativistic plasma is then made. The problem of two-stream instability in a relativistic plasma is discussed, and the relativistic effects are shown to have a stabilising nature on the problem. Next, these treatments are repeated using a manifestly-covariant four dimensional space-time approach, and attention is drawn to discrepancies in the results deduced by these two approaches.

1. Introduction

Experiments with relativistic electron beams and observations of extremely energetic plasmas in space provide the raison d'etre for the theoretical study of relativistic plasmas. Clemmow and Willson[1], Silin[2], and Buti[3] considered this problem using a three-dimensional space + time approach which is not manifestly covariant (under a Lorentz transformation), and Kursunoglu[4] considered it using

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a manifestly covariant four-dimensional space-time approach. The present paper makes further considerations of linearised longitudinal oscillations in a relativistic plasma using both of the above approaches. Attention is drawn to disrepancies in the results deduced by these two approaches.

2. *Three-dimensional space + time approach*

a) *The dispersion relation*

If $\vec{V}$ and $\vec{F}$ denote the velocity of a particle and the force on it, introduce

$$\vec{u} = \frac{\vec{V}}{c\beta}, \quad \vec{\mathscr{F}} = \frac{\vec{F}}{\beta} \tag{1}$$

where,

$$\beta = \sqrt{1 - V^2/c^2}$$

c being the velocity of light.

The conservation of particles in phase space gives

$$f(t, \vec{x}, \vec{u})\, \mathrm{d}\vec{x}\, \mathrm{d}\vec{u} = f(t + \delta t, \vec{x} + \vec{\delta x}, \vec{u} + \vec{\delta u})\, \mathrm{d}(\vec{x} + \vec{\delta x})\, \mathrm{d}(\vec{u} + \vec{\delta u}) \tag{2}$$

from which

$$f(t, \vec{x}, \vec{u}) = \mathcal{J} f(t + \delta t, \vec{x} + \vec{\delta x}, \vec{u} + \vec{\delta u}) \tag{3}$$

where $\mathcal{J}$ is the Jacobian,

$$\mathcal{J} = \frac{\partial(\vec{x} + \vec{\delta x}, \vec{u} + \vec{\delta u})}{\partial(\vec{x}, \vec{u})}. \tag{4}$$

From (1) noting that

$$\vec{\delta x} = \beta c \vec{u}\, \delta t, \; \vec{\delta u} = \frac{1}{mc} \vec{F}\, \delta t \tag{5}$$

(4) becomes to $O(\delta t)$,

$$\mathcal{J} \approx 1 + \frac{1}{mc} \frac{\partial F_i}{\partial u_i} \delta t. \tag{6}$$

Using (5), one also has

$$f(t + \delta t, \vec{x} + \vec{\delta x}, \vec{u} + \vec{\delta u})$$

$$= f(t, \vec{x}, \vec{u}) + \frac{\partial f}{\partial t} \delta t + c\beta u_i \frac{\partial f}{\partial x_i} \delta t + \frac{F_i}{mc} \frac{\partial f}{\partial u_i} \delta t. \tag{7}$$

Using (3), (6), and (7), one obtains

$$\frac{1}{c\beta}\frac{\partial f}{\partial t} + u_l \frac{\partial f}{\partial x_l} + \frac{1}{mc^2\beta}\frac{\partial}{\partial u_l}(F_l f) = 0 \tag{8}$$

an equation first derived by Clemmow and Willson[5].

Consider an unperturbed state in which the plasma is electrically neutral. Assume that the positive ions are infinitely massive so that they remain motionless and merely make up a neutralising background. Look for one-dimensional longitudinal oscillations of the form

$$f = n_0 \left[f_0(\vec{u}) + f_1 \, e^{i(\omega t - kx)}\right] \tag{9}$$

where, $\vec{u} = (u_1, u_2, u_3)$.

Then, Eq. (8) gives, on linearisation

$$f_1 = \frac{i\, e\, E}{mc} \frac{\mathrm{d}f_o/\mathrm{d}u_1}{(\omega - k\,\beta\, c\, u_1)}. \tag{10}$$

Using (10) in the Maxwell equation

$$\frac{\partial E}{\partial t} + ec^2 \int \beta\, u_1 f\, \mathrm{d}\vec{u} = 0 \tag{11}$$

one obtains the dispersion relation,

$$\left(\frac{\omega}{\omega_p}\right)^2 = -\int \frac{c\,\beta\, u_1 \dfrac{\mathrm{d}E_o}{\mathrm{d}u}}{(1 - \mu\,\beta\, u_1)}\, \mathrm{d}\vec{u} \tag{12}$$

where,

$$\mu = \frac{ck}{\omega}, \quad \omega_p^2 = \frac{4\pi\, n_o\, e^2}{m}.$$

Note that (12) can be written as

$$1 - \frac{\omega_p^2}{k^2}\int \frac{(\mathrm{d}f_0/\mathrm{d}u_1)\,\mathrm{d}\vec{u}}{\left(\dfrac{c\,u_1}{\sqrt{1+u^2}} - w\right)} = 0 \tag{13}$$

where, we have used the fact that

$$\beta = \frac{1}{\sqrt{1+u^2}}$$

and have introduced $w = \omega/k$. (13) appears different from the one given by Clemmow and Dougherty[6], but gives the same result as the latter, at least for weak relativistic effects.

b) *Evaluation of the dispersion relation for weak relativistic effects*

If the relativistic effects are weak, one may expand the denominator in (13), and write

$$1 + \frac{\omega_p^2}{k^2}\frac{1}{w}\int \frac{\mathrm{d}f_0}{\mathrm{d}u_1}\left[1 + \frac{cu_1/w}{\sqrt{1+u^2}} + \frac{c^2\,u_1^2/w^2}{1+u^2} + \frac{c^3\,u_1^3/w^3}{(1+u^2)^{3/2}} + \dots\right]\mathrm{d}\vec{u} = 0 \tag{14}$$

On integrating by parts, (14) gives

$$1 = \frac{\omega_p^2}{k^2}\left[\frac{1}{w^2}\int \frac{(1+u_2^2+u_3^2)f_0}{(1+u^2)^{3/2}}\,c\,\mathrm{d}\vec{u} + \frac{1}{w^4}\int f_0 \frac{3u_1^2\,c^2}{(1+u^2)^{11/2}}\,c\,\mathrm{d}\vec{u} + \dots\right]. \tag{15}$$

Noting that

$$\int_{-\infty}^{\infty} f_0\,u_i^2\,c\,\mathrm{d}u_i = \frac{V_T^2}{c^2} \tag{16}$$

V_T being the thermal speed of the particles, (15) gives

$$\omega^2 \approx \omega_p^2\left[1 + \frac{3k^2\,V_T^2}{\omega_p^2} - \frac{5}{2}\frac{V_T^2}{c^2}\right]. \tag{17}$$

(17) agrees with the one given by Clemmow and Dougherty[6].

c) *Two-stream instability in a relativistic plasma*

Consider two unbounded homogeneous interpenetrating streams composed of identical charged particles with uniform velocities $+V_0$ and $-V_0$ along the x-direction. The charge and the current densities of the electrons in each stream in the unperturbed state are assumed to be neutralised by those of the ions, but the perturbational motion of the latter is ignored.

First note that, on integration by parts, (13) gives

$$1 = \frac{\omega_p^2}{k^2}\int \frac{f_0}{\left(\frac{cu}{\sqrt{1+u^2}} - w\right)^2}\,c\,\mathrm{d}\vec{u}. \tag{18}$$

Using

$$f_0 = [\delta(V+V_0) + \delta(V-V_0)]$$

$$= \frac{1}{c\beta}[\delta(u+\mathscr{V}_0) + \delta(u-\mathscr{V}_0)] \tag{19}$$

where from (1),

$$\mathscr{V}_0 = \frac{V_0/c}{\sqrt{1 - V_0^2/c^2}} \tag{20}$$

(18) gives

$$1 = \omega_p^2 \sqrt{1 + \mathscr{V}_0^2} \left[\frac{1}{\left(\omega - \frac{c\, k \mathscr{V}_0}{\sqrt{1 + \mathscr{V}_0^2}}\right)^2} + \frac{1}{\left(\omega + \frac{c\, k \mathscr{V}_0}{\sqrt{1 + \mathscr{V}_0^2}}\right)} \right],$$

$$1 = \frac{\omega_p^2}{\sqrt{1 - V_0^2/c^2}} \left[\frac{1}{(\omega - k\, V_0)^2} + \frac{1}{(\omega + k\, V_0)^2} \right] \tag{21}$$

from which,

$$\omega^2 = k^2\, V_0 + \frac{\omega_p^2}{\sqrt{1 - V_0^2/c^2}} \pm \left[\left(k^2\, V_0^2 + \frac{\omega_p^2}{\sqrt{1 - V_0^2/c^2}} \right)^2 + \right.$$

$$\left. + \frac{2\, \omega_p^2\, k^2\, V_0^2}{\sqrt{1 - V_0^2\, c^2}} - k^4\, V_0^4 \right]^{1/2} \tag{22}$$

which demonstrates the stabilising nature of the relativistic effects.

3. *Manifestly covariant four-dimensional space-time approach*

a) *The dispersion relation*

Introduce a distribution function

$$f = f\,(x^\mu, u^\mu) \tag{23}$$

where,

$$x^\mu = (ct, \vec{x}), \quad u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

$$u^\mu\, u_\mu = c^2, \; u_\mu = g_{\mu\nu}\, u^\nu$$

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{24}$$

and τ is the proper time (measured in the rest frame of the particle).

The 4-flux of particles is given by

$$\mathscr{J}^\mu = \int f\, u^\mu\, \mathrm{d}\vec{u} = (nc, n\vec{V}). \tag{25}$$

Consider a distribution function of the form

$$f = A\, \mathrm{e}^{-S_\mu u^\mu}, \quad S_\mu = S_\mu(u_\mu). \tag{26}$$

In order to determine the constant A, note that in the rest frame,

$$J^\mu = (n_0, c, \vec{0}) \tag{27}$$

so that from (25) and (27), one obtains

$$n_0 c = A \int \mathrm{e}^{-S_0 u_0}\, u_0\, \delta(u_0^2 - V^2 - c^2)\, \mathrm{d}u_0\, \mathrm{d}\vec{V}$$

$$= \frac{A}{2} \int \mathrm{e}^{-S_0 \sqrt{V^2 + c^2}}\, \mathrm{d}\vec{V}$$

$$= 2\pi A \int V^2\, \mathrm{d}V\, \mathrm{e}^{-S_0 \sqrt{V^2 + c^2}}$$

$$= 2\pi c^2 \cdot A \cdot \frac{K_2(S_0 c)}{S_0 c}$$

where $K_2(\varkappa)$ is modified Bessel's function of the second kind.

Thus,

$$A = \frac{n_0}{2\pi c^2} \frac{S_0 c^2}{K_2(S_0 c)} \tag{28}$$

so one has the relativistic Maxwellian distribution function,

$$f_M = \frac{n_0 S_0}{2\pi K_2(S_0 c)} \mathrm{e}^{-S_\mu u^\mu} \delta(u^\mu u_\mu - c^2) \tag{29}$$

if,

$$S_0 c = \frac{m c^2}{k_B T}$$

k_B being the Boltzmann constant.

The relativistic form of Vlasov's equation is

$$u^\mu \frac{\partial f}{\partial x^\mu} + \frac{e}{m c} u^\nu F^\sigma_\nu \frac{\partial f}{\partial u^\sigma} = 0 \tag{30}$$

where F^σ_ν is Maxwell's field tensor,

$$F^{\sigma}_{\nu} = g^{\sigma\mu} (A_{\mu\nu} - A_{\nu\mu}) \tag{31}$$

A^{μ} being the 4-vector potential of the electromagnetic field,

$$A^{\mu} = (\Phi, c\vec{A}).$$

Assume that the electromagnetic fields are small and can be treated as a perturbation, and that the plasma is uniform and electrically neutral in the absence of these fields, as before. Writing.

$$f = f_0 (u^{\mu}) + \delta f (x^{\mu}, u^{\mu}) \tag{32}$$

δf being the perturbation, and Fourier analysing the perturbations as

$$\delta f, A^{\nu} \sim e^{ik_{\mu}x^{\mu}} \tag{33}$$

one obtains upon linearising Eq. (30),

$$ik_{\mu} u^{\mu} \delta f = \frac{e}{mc} i [k_{\mu} A_{\nu} - k_{\nu} A_{\mu}] g^{\mu\sigma} u^{\nu} \frac{\partial f_0}{\partial u^{\sigma}}. \tag{34}$$

Assume, as before, that the ions are so massive that they remain motionless and merely make up a neutralising background. Using (32) and (34), (25) gives

$$j^{\lambda} = \frac{e^2}{mc^2} \int d\vec{u} \frac{(k_{\mu} A_{\nu} - k_{\nu} A_{\mu}) u^{\lambda} u^{\nu}}{k_{\alpha} u^{\alpha}} g^{\sigma\mu} \frac{\partial f_0}{\partial u^{\sigma}} =$$

$$= \frac{e^2}{mc^2} \int d\vec{u} \left[\frac{u^{\lambda} u^{\nu}}{k_{\alpha} u^{\alpha}} k^{\sigma} \frac{\partial f_0}{\partial u^{\sigma}} A_{\nu} - u^{\lambda} A^{\sigma} \frac{\partial f_0}{\partial u^{\sigma}}\right] =$$

$$= \frac{e^2}{mc^2} \int d\vec{u} f_0 A^{\lambda} + \frac{e^2}{mc^2} \int d\vec{u} \frac{u^{\lambda} u^{\nu}}{k_{\alpha} u^{\alpha}} k^{\sigma} \frac{\partial f_0}{\partial u^{\sigma}} A_{\nu}. \tag{35}$$

Now, using the Lorentz condition

$$k_{\lambda} A^{\lambda} = 0 \tag{36}$$

Maxwell's equations give

$$k^{\mu} k_{\mu} A^{\lambda} = 4\pi j^{\lambda}. \tag{37}$$

Using (35), Eq. (37) gives

$$\left(k^{\mu} k_{\mu} - \frac{\omega_p^2}{c^2}\right) A^{\lambda} = \left(\frac{\omega_p^2}{c^2} \int d\vec{u} \frac{u^{\lambda} u_{\tau}}{k_{\alpha} u^{\alpha}} k^{\sigma} \frac{\partial f_0}{\partial u^{\sigma}}\right) A^{\tau}$$

$$= \left(\frac{\omega_p^2}{c^2} \int \mathrm{d}\vec{u} f_0 \frac{u^\lambda u_\tau}{(k_\alpha u^\alpha)^2} k^\sigma k_\sigma\right) A^\tau$$

$$- \left(\frac{\omega_p^2}{c^2} \int \mathrm{d}\vec{u} f_0 \frac{k^\lambda u_\tau}{k_\alpha u^\alpha}\right) A^\tau$$

or

$$\left(\frac{\omega_p^2}{c^2 k^\mu k_\mu} - 1\right) A^\lambda = \frac{\omega_p^2}{c^2} \left(I_\tau^\lambda + \frac{k^\lambda I_\tau}{k^\mu k_\mu}\right) A^\tau \tag{38}$$

where,

$$I_\tau^\lambda = - \int \frac{u^\lambda u_\tau}{(k_\alpha u^\alpha)^2} f_0 \, \mathrm{d}u$$

$$I_\tau = \int \frac{u_\tau}{k_\alpha u^\alpha} f_0 \, \mathrm{d}u.$$

Corresponding to longitudinal oscillations $A^\lambda = (A^0, \vec{0})$, one obtains from Eq. (38), the dispersion relation

$$\frac{\omega_p^2 - \omega^2 + k^2 c^2}{\omega^2 - k^2 c^2} = \omega_p^2 \frac{\partial I_0}{\partial \omega} + \frac{\omega \, \omega_p^2}{\omega^2 - k^2 c^2} I_0 \tag{39}$$

where, for a'Maxwellian distribution (29),

$$I_0 = \frac{n_0 S_0 c}{2\pi K_2 (S_0 c)} \int \frac{\sqrt{V^2/c^2 + 1}\, \mathrm{e}^{-S_0 c \sqrt{1 + V^2/c^2}}}{\omega \sqrt{1 + V^2/c^2} - \vec{k} \cdot \vec{V}} \, \mathrm{d}\vec{V}. \tag{40}$$

(39) reduces in the nonrelativistic limit to the well-known Vlasov formula.

b) *Evaluation of the dispersion relation for weak relativistic effects*

Consider a weakly-relativistic plasma so that

$$\omega^2, \omega_p^2 \ll k^2 c^2. \tag{41}$$

Then, as a first approximation in (39a), one may use from (40),

$$I_0 \approx \frac{1 + \dfrac{3k_B T/m}{\omega_p^2} k^2}{\omega}. \tag{42}$$

Such an approximativn was also made by Clemmow and Willson[5]. (39) then gives

$$-\left(1+\frac{\omega_p^2}{k^2 c^2}\right)=-\omega_p^2 \frac{\left(1+\frac{3k_B T/m}{\omega_p^2} k^2\right)}{\omega^2}+$$

$$-\frac{\omega\,\omega_p^2}{k^2 c^2} \frac{\left(1+\frac{3k_B T/m}{\omega_p^2} k^2\right)}{\omega} \tag{43}$$

from which, one obtains

$$\omega^2 \approx \omega_p^2+\frac{3k_B T}{m} k^2+\frac{\omega_p^2}{c^2}\frac{3k_B T}{m}. \tag{44}$$

Note that the dispersion relation (44) differs from (17) deduced previously using a non-manifestly covariant approach.

c) *Two-stream instability in a relativistic plasma*

Consider two unbounded homogeneous interpenetrating streams composed of identical charged particles with uniform velocities V_0 and $-V_0$ along the x-direction. The charge and current densities of the electrons in each stream in the unperturbed state are assumed to be neutralised by those of the ions, but the perturbational motion of the latter is ignored, as before.

Using now a properly drifted Maxwellian distribution for the electrons, one writes as a first approximation

$$I_0 \approx \frac{1}{\omega \pm k V_0}. \tag{45}$$

(39) then gives

$$-1-\frac{\omega_p^2}{k^2 c^2}=-\frac{\omega_p^2}{(\omega-k_0 V)^2}-\frac{\omega_p^2}{(\omega+k V_0)^2}-\frac{2\,\omega_p^2}{k^2 c^2} \tag{46}$$

which differs from (21) deduced previously using a non-manifestly covariant approach. (46) gives

$$\omega^2=k^2 V_0^2+\frac{\omega_p^2}{1-\omega_p^2/k^2 c^2} \pm \left[\left(k^2 V_0^2+\frac{\omega_p^2}{1-\omega_p^2/k^2 c^2}\right)^2\right.$$

$$\left.+\frac{2\omega_p^2 k^2 V_0^2}{1-\omega_p^2/k^2 c^2}-k^4 V_0^4\right], \tag{47}$$

which nontheless demonstrates the stabilising nature of the relativistic effects, as before.

4. *Discussion*

In order to understand the discrepancies between the results deduced respectively from the non-manifestly convariant and manifestly-covariant approaches, let us examine the approximations introduced in the two approaches. The approximation involved in (14) and that in (42) lead to compatible results in the non-relativistic limit so that these approximations are not likely to be the source of the discreapancy. The source of the discrepancy then seems to be the approximation involved in (6) in the non-manifestly covariant approach; this approximation has no counterpart in the manifestly-covariant approach. However, it must be noted that the approximation involved in (6) leads to some differences in the details but not in the trend of the relativistic effects, as demonstrated by consideration of the problem of the two-stream instability.

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Originalni znanstveni rad

Proučavano je longitudinalno titranje u klasičnoj plazmi. Formulirajući problem nekovarijantno, izveden je aproksimativan oblik disperzione relacije za slučaj slabe relativističke plazme. Diskutiran je problem dvostrujne nestabilnosti u relativističkoj plazmi i ustanovljeno je da relativistički efekti imaju stabilizirajući karakter. Ponavljajući račun u kovarijantnoj četvero-dimenzionalnoj formulaciji, ukazano je na neslaganje rezultata izvedenih pomoću ta dva pristupa.