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NATURAL COORDINATES FOR AN ACCELERATED OBSERVER

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The line element for an accelerated observer is derived by assuming linearity of transformation between his coordinates for any given time and a system of Galilean coordinates. It is noted that any 4 × 4 orthogonal matrix $[a]$ with a_{i4}, a_{4i} imaginary, a_{44} and other elements real represents a possible transformation from an inertial to an accelerated frame. Examples of such matrices are given and the associated angular velocities calculated. Finally, by a suitable interpretation of the Eulerian matrix giving the orientation of a rigid body in space the matrix representing the general motion of a frame on an inertial plane is obtained. The matrix contains a free parameter connected with the arbitrary rotational motion. This parameter becomes a definite function of time when the condition for irrotational motion is imposed and is represented by a time-integral directly related to the motion of the origin. The integral is evaluated in a closed form in special cases. The geodesic equations are then solved in a frame rotating with a constant angular velocity and moving with a constant acceleration along the axis of rotation and the motion of a free particle and a photon in such a frame is studied in arbitrary direction. Diagrams are drawn to illustrate the results.

1. Introduction.

The *General Relativity Theory* permits an accelerated observer to use any system of curvilinear coordinates in space-time even if he is moving in a region where no natural gravitational field exists. But, if the coordinate system used by him is too artifical his observations will give him no idea about the physical phenomena occurring around him. In Newtonian mechanics there is a perfectly definite way of determining the moving in an arbitrary manner in space. The primary requirement to be fulfilled by a coordinate system in *Relativity* is that it should reproduce the Newtonian results to a first approximation. We shall call the coordinate system of an accelerated observer *a natural coordinate system* when it fulfils this requirement. The object of the present paper is to derive the form of the line element[4)] corresponding to such a system and to study the geodesic motion as observed from a frame which has a uniform rotatory motion about an axis superposed on constant linear acceleration along the same axis.

Our derivation of the line element is based on the following assumptions:

(i) The accelerated observer is situated permanently at the origin of his space coordinates.

(ii) The world line of the accelerated observer is orthogonal to t = constant hypersurfaces.

(iii) For a fixed value of the time parameter t of the accelerated observer, transformation from the accelerated frame to an inertial frame is linear.

The common feature in the transformation equations for a linearly accelerated frame and a frame rotating about a fixed axis is that the equations are linear in the space coordinates for a fixed value of time. We have assumed this linearity to hold in the general case and, combining this with the other two assumptions, have derived a form of the line element in which the space part is Euclidean and the metric coefficients g_{i4}, g_{44} are the sums of the corresponding coefficients for a pure translatory and a pure rotatory motion[2)]. We have also probed the nature of the transformation and suggested a simple means of obtaining it. To illustrate the procedure some special cases of combined translatory and rotatory motion have been considered. In particular, the geodesic motion as observed from a frame which has a uniform rotatory motion superposed on a constant linear acceleration along the axis of rotation has been studied. In a permissible coordinate transformation certain restrictions are inherent which make only a part of the Minkowski space available to an accelerated observer[3)]. This gives rise to the so called *event horizon.* The equations of motion in terms of coordinates of the accelerated observer, obtained from the geodesic equations, show a typical relativistic damping. This prevents a particle from acquiring an infinite velocity and in fact the velocity of a particle and a photon vanishes as they approach event horizon. The behaviour of a particle and a photon projected in the upward direction is qualitatively different and is discussed in section 7.

In the following Greek indices will take the values 1, 2, 3, 4, Latin indices the values 1, 2, 3 and a repeated index will generally imply a summation. A subscript preceded by a comma will denote partial differentiation with respect to the corresponding coordinates. Prime and dot will denote differentiation with respect to $x_4 = ict$ and t, respectively. Further, g_{i4} and g_{44} will be abbreviated to h_i and h_4, respectively.

2. *Sufficient conditions for the linearity of the transformation*

It will be seen from the the following example that the Euclidean character of the space part of the line element alone does not ensure te linearity of the transformation for a fixed value of t. To illustrate this point we consider the following somewhat artificial example of coordinate transformation given by

$$-(X^1)^2 + (T - t)^2 a^2 c^2 = 1, \; X^1 = \xi, X^2 = x^2, X^3 = x^3. \tag{2.1}$$

From (2.1)

$$X^4 = x_4 + \frac{i}{a} \sqrt{1 + \xi^2}.$$

Therefore,

$$\mathrm{d}X^a \, \mathrm{d}X^a = \left[1 + \left(\frac{i}{a} \frac{\xi}{\sqrt{1 + \xi^2}} \right)^2 \right] \mathrm{d}\xi^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 +$$

$$+ 2 \frac{i}{a} \frac{\xi}{\sqrt{1 + \xi^2}} \mathrm{d}\xi \, \mathrm{d}x^4 + (\mathrm{d}x^4)^2. \tag{2.2}$$

It suffices to take $a = 1$. If the following change of variable is made

$$x^1 = \int \left[1 - \frac{\xi^2}{1 + \xi^2} \right]^{1/2} \mathrm{d}\xi,$$

the line element (2.2) becomes

$$dX^a \, dX^a = dx^a \, dx^a + \mathrm{i} \sinh x^1 \, dx^1 + (\mathrm{d}x^4)^2. \tag{2.3}$$

We see that although the space part of the line element is Euclidean the transformation is not linear.

The other condition needed for the linearity of transformation is the vanishing of $h_{i,i}$ for each i. To show this we assume the condition to hold and write the line element in the form

$$ds^2 = dx^a \, dx^a + 2 \, h_i \, dx^i \, dx^4 + h_4 \, (dx^4)^2. \tag{2.4}$$

The Christoffel symbols and other relevant quantities associated with this line element are

$$\Gamma^4_{ab} = g^{\mu 4} \frac{1}{2} \, [h_{a,b} + h_{b,a}] \tag{2.5}$$

$$g^{pq} = (g \delta_{pq} + h_p h_q)/g, \; g^{i4} = - \frac{h_i}{g}, \; g^{44} = \frac{1}{g} \tag{2.6}$$

$$g \equiv |g_{\mu\nu}| = (h_4 - h_i \, h_i). \tag{2.7}$$

Using the standard expression for Riemann tensor R and (2.5) we obtain

$$R_{ijij} = \frac{1}{3} g^{44} [(h_{j,i} + h_{i,j})^2 - 4 h_{j,j} h_{i,i}]. \tag{2.8}$$

Since we are dealing with flat space-time, $R_{ijij} = 0$, and $h_{i,i} = 0$ for each i by hypothesis. Equation (2.8) therefore reduces to

$$h_{j,i} + h_{i,j} = 0. \tag{2.9}$$

Substituting (2.9) in (2.5) we get

$$\Gamma^{\mu}_{ab} = 0. \tag{2.10}$$

The equations of the four-dimensional geodesics in the coordinates of the accelerated observer are

$$\frac{d^2 x_i}{ds^2} + \Gamma^{i}_{ab} \frac{dx^a}{ds} \frac{dx^b}{ds} + \Gamma^{i}_{4\mu} \frac{dx^4}{ds} \frac{dx^{\mu}}{ds} = 0. \tag{2.11}$$

On the hypersurfaces, t = constant, the equations of geodesics are

$$\frac{d^2 x_i}{d\sigma^2} = 0 \tag{2.12}$$

where $d\sigma$ is the spatial distance. For t = constant $ds = d\sigma$ and also the third term of (2.11) vanishes. Therefore, by (2.10), the geodesics described by (2.12) become geodesic in the four-dimensional space. Thus the hypersurfaces, t = constant, are geodesic hypersurfaces in the Minkowski space. The inertial observer describes the same geodesics by the equations

$$\frac{d^2 X^{\mu}}{ds^2} = 0 \tag{2.13}$$

where X^{μ} are the coordinates used by the inertial observer. As we shall show below, this implies that the transformation $x^i \to X^{\mu}$ for fixed value of t is linear. For a constant value of t

$$\frac{d^2 X^{\mu}}{ds^2} = \frac{\partial^2 X^{\mu}}{\partial x^a \partial x^b} \frac{dx^a}{d\sigma} \frac{dx^b}{d\alpha} = 0.$$

The direction cosines $\frac{dx^i}{d\alpha}$ are arbitrary, subject to the restriction $\frac{dx^a}{d\sigma} \frac{dx^a}{d\sigma} = 1$. From this it can be easily shown that

$$\frac{\partial^2 X^{\mu}}{\partial x^a \partial x^b} = 0, \text{ for all } a,b. \tag{2.14}$$

Thus X^μ are linear functions of x^1, x^2, x^3, with the coefficients depending on t, and we can write

$$X^\mu = a_k^\mu (t)\, x^k + f^\mu (t). \tag{2.15}$$

3. Derivation of the line element for the accelerated observer

Substitution of (2.15) in the expression for the interval in the inertial frame gives

$$ds^2 = a_i^\mu a_k^\mu \, dx^i \, dx^k + 2\, [(a_k^\mu)' x^k + (f^\mu)'] \, a_i^\mu \, dx^i \, dx^4 + [(a_k^\mu)' x^k + (f^\mu)']^2 \, (dx^4)^2. \tag{3.1}$$

By a time dependent linear transformation of space coordinates the first term of (3.1) can be brought into the form $dx^a \, dx^a$. After this has been done the new $a_k^\mu (t)$ will satisfy the conditions

$$a_i^\mu a_k^\mu = \delta_{ik} \tag{3.2}$$

and $h_{i,k}$ will take the form

$$h_{i,k} = (a_k^\mu)' \, a_i^\mu = \frac{\omega_{kl}}{ic} \tag{3.3}$$

where ω_{kl} is the component of angular velocity of the frame along the x^j-axis. The line element (without the bars) can now be written as

$$ds^2 = dx^a \, dx^a + dt \sum_{i,j=12,23,31} \omega_{ij} \, (x^i \, dx^j - x^j \, dx^i) -$$

$$- c^2 \, dt^2 \left[(1 - x^k \, a_k^a \, f^{a''})^2 - \frac{\omega^2 \varrho^2}{c^2} \right] \tag{3.4}$$

where ϱ is the perpendicular distance of the point (x^1, x^2, x^3) from the axis of rotation.

It is obvious from the mode of its derivation that this line element is invariant under time-dependent orthogonal transformations of the x's. If

$$x^i = R_a^i \, \overline{x^a} \tag{3.5}$$

be such a transformation, then

$$a_i^\mu = R_i^a \, a_a^\mu$$

and

$$\dot{\overline{a_i^\mu}} \, \overline{a_j^\mu} = R_i^a \, R_j^b \, \omega_{ab} + \dot{R}_i^a \, R_j^a. \tag{3.6}$$

Now, $\dot{R}^a_i$ can always be written as

$$\dot{R}^a_i = \hat{\omega}_{ib} R^b_a \text{ with some } \hat{\omega}_{ib}.$$

Substituting this in (3.3) we have

$$\overline{\omega_{ij}} = R^a_i R^b_j \omega_{ab} + \hat{\omega}_{ij}.$$

Thus, ω_{ij} does not transform as an antisymmetrical tensor of the second rank if R is time-dependent. Since the ω_{ij} are completely at our disposal we can chose them so as to make h_i vanish.

5. *The transformation matrix in special cases*

Since the space part of the line element has been brought into the form $dx^a \, dx^a$, the transformation coefficients $a^\mu_\varkappa$ of equation (2.15) satisfy

$$\delta_{ab} = \frac{\partial X}{\partial x^a} \frac{\partial X}{\partial x^b} \bar{g}_{\alpha\beta} = a^a_a \, a^a_b \tag{4.1}$$

where $\overline{g}_{\alpha\beta}$ is the metric tensor in Galilean coordinates. Equation (4.1) and assumption (iii) show that the coefficients $a^\mu_\varkappa$ and $(f^\mu)'$ form a 4×4 orthogonal matrix. Conversely, any such matrix with a^i_4, a^4_i, pure imaginary and other elements real will correspond to a possible translatory and rotatory motion of a frame.

First, let us consider a matrix in the block form

$$\begin{pmatrix} \cos\Omega & \sin\Omega & & \\ -\sin\Omega & \cos\Omega & & \\ & & (f^4)' & -(f^3)' \\ & & (f^3)' & (f^4)' \end{pmatrix}. \tag{4.2}$$

With $(f^3)'$ imaginary and Ω, $(f^4)'$ real it represents an arbitrary translatory motion of a frame along the z-axis superposed on an arbitrary rotatory motion about the same axis. The angular velocity, $\omega_{12} = \dot{a}^1_1 \, a^1_2 + \dot{a}^2_1 \, a^2_2$, is seen to be equal to $\dot{\Omega}$. In particular, the origin may be executing a hyperbolic motion and ω_{12} may be constant. Relativistic effects in the motion of a particle and a photon in such a frame have been discussed in sections 6, 7 and 8.

Next, we take the Eulerian matrix given by Synge and Griffith[4]. If we take Θ to be pure imaginary and set $\Phi = \frac{\pi}{2}$, $(f^3)' = \sin\Theta$, $(f^4)' = \cos\Theta$, the matrix takes the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\sin\Psi & (f^4)'\cos\Psi & -(f^3)'\cos\Psi \\ 0 & -\cos\Psi & -(f^4)'\sin\Psi & (f^3)'\sin\Psi \\ 0 & 0 & (f^3)' & (f^4)' \end{pmatrix}. \tag{4.3}$$

This corresponds to an arbitrary translatory motion of a frame along the z-axis superposed on an arbitrary rotatory motion about the x-axis. The angular velocity is given by

$$\omega_{23} = \dot{a}_2^a \, \dot{a}_3^a = \dot{\psi}. \tag{4.4}$$

Finally, we substitute

$$\sin \Theta \cos \Phi = (f^1)', \; \sin \Theta \sin \Phi = (f^2)', \; \cos \Theta = (f^4)', \quad \beta = \frac{\pi}{2} - \Psi \tag{4.5}$$

in the Eulerian matrix and obtain a matrix representing a general two dimentional motion in the XY-plane. The motion of the origin in the inertial frame is given by

$$X = f^1(\tau), \; Y = f^2(\tau), \; X^4 = f^4(\tau)$$

and the velocity by

$$u^2 = \left(\frac{dX}{dT}\right)^2 + \left(\frac{dY}{dT}\right)^2 = [(f^1)'^2 + (f^2)'^2] \left(\frac{d\tau}{dT}\right)^2 = c^2 \left[1 - \frac{1}{(f^4)'^2}\right]$$

or,

$$(f^4)'^2 = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \equiv \gamma. \tag{4.6}$$

Further,

$$(f^1)'^2 + (f^2)'^2 = 1 - (f^4)'^2 = - \frac{u^2}{c^2} \gamma^2 \tag{4.7}$$

or,

$$(\dot{f}^1)^2 + (\dot{f}^2)^2 = u^2 \gamma^2. \tag{4.8}$$

To get the most general irrotational motion of the frame in two dimension we set $\omega_{12} = 0$, obtaining

$$\frac{1}{u^2 \gamma} (\dot{f}^1 \ddot{f}^2 - \ddot{f}^1 \dot{f}^2) - \frac{d\beta}{dt} = 0 \tag{4.9}$$

where

$$\beta = \frac{1}{u^2 \gamma} \int (\dot{f}^1 \ddot{f}^2 - \ddot{f}^1 \dot{f}^2) \, dt + \text{a constant}. \tag{4.10}$$

This fixes the value of β for rotation free motion. For constant u, f^2 and $u^{(2)}$ the integral is seen to have the following values, respectively

$$\beta = - \gamma \sin^{-1} \frac{f^1}{u} + A_1 \tag{4.11}$$

$$\beta = \tan^{-1} \frac{\gamma \dot{f}^2}{\dot{f}^1} + \frac{1}{2} \frac{\dot{f}^2}{c} \ln \frac{c\gamma - \dot{f}^1}{c\gamma + \dot{f}^2} + A_2 \tag{4.12}$$

and

$$\beta = - \tan^{-1} \frac{\gamma \dot{f}^1}{\dot{f}^2} + A_3 \tag{4.13}$$

where *A's* are constants of integration.

5. *Irrotational motion*

In the case of irrotational motion the transformation coefficients are found to satisfy the coupled differential equations

$$(a^{\mu}_{R})' = -a^{\alpha}_{R} (f^{\alpha})' (f^{\mu})'. \tag{5.1}$$

These equation were derived by Möller[2] by applying successive infinitesimal Lorentz transformations without rotation to the rest frame of the accelerated observer. It is easy to verify that, if these equations are satisfied by the elements a^{μ}_{i} of A, then they are also satisfied by the corresponding elements of BA and AC, where B is a time independent rotation matrix and C is a Lorentz matrix. These are two invariance properties of the equations for irrotational motion.

Equation (5.1) can be given a three-dimentional form by writting A as a product of a time-dependent matrix L having the form of Lorentz transformation without rotation and a matrix R of spatial rotation with

$$R_{ij} = b_{ij}, R_{i4} = R_{4i} = 0, R_{44} = 1.$$

Since

$$u_i = ic (f^1)'/(f^4)', \text{and } \gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} = (f^4)'$$

by equation (4.6), the elements of L are

$$L'_{ij} = \delta_{ij} - \frac{(f^i)' (f^j)'}{1 + \gamma}, \; L_{4i} = - L_{i4} = (f^i)', \; L_{44} = \gamma. \tag{5.2}$$

Forming the general 4×4 orthogonal matrix A by multiplying L on the left by R, and setting $\omega_{ij} = ic \, (a^{\mu}_{i})' \, (a^{\mu}_{j}) = 0$, we obtain the equations of irrotational motion in the form

$$b'_{ij} = (\gamma + 1)^{-1} \, [b_{ia} (f_a)' (f^j)'' - b_{ia} (f_a)'' (f^j)'] \tag{5.3}$$

With $b_{ij} = v_j$ these can be written in the vectorial form

$$\vec{v}' = \vec{h} \times \vec{v} \tag{5.4}$$

where

$$h_i = (\gamma + 1)^{-1} \, [(f^j)' (f^k)'' - (f^k)' (f^j)''].$$

These are the familiar equations expressing the time variation of a vector fixed in a rotating body and are easier to handle than Möller's original equations. For example, $(f^3)' = 0$ gives

$$v_1 = \cos(\Lambda + k) \tag{5.5}$$

and

$$v_2 = \sin(\Lambda + k)$$

where

$$\Lambda = \int h_3 \, d\tau .$$

The constant of integration $k = 0, \frac{\pi}{2}$ gives, respectively, the first and second row of the matrix b_{ij}. The parameter β in (4.8) is related to Λ by

$$\Lambda = -(\beta + \Phi) \tag{5.6}$$

where

$$\tan \Phi = \frac{(f^1)'}{(f^2)'} .$$

6. *Equation of motion*

Motion of a free particle as observed from the accelerated frame can be determined either by using the transformation connecting the inertial with the accelerated frame or by solving the geodesic equations. We have adopted the second method in which certain features of the problem become more apparent.

In the motion described by the matrix (4.2) with constant ω and with the origin executing a hyperbolic motion the line element in terms of coordinates of the accelerated observer becomes.

$$ds^2 = dx^2 + dy^2 + dz^2 + 2\omega(-y dx + x dy)\, dt - \left[\left(1 + \frac{gz}{c^2}\right)^2 - \frac{\omega^2 \varrho^2}{c^2}\right] c^2 \, dt^2 \tag{6.1}$$

where

$$\varrho^2 = x^2 + y^2 . \tag{6.2}$$

Suitable combinations of standard equations for geodesics, with respect to ds^2 specified in (6.1), gives

$$x \frac{d^2 y}{ds^2} - y \frac{d^2 x}{ds^2} = -\omega \frac{d}{ds}\left(\varrho^2 \frac{dt}{ds}\right) \tag{6.3}$$

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2} + 2i\omega\frac{\mathrm{d}\xi}{\mathrm{d}s}\frac{\mathrm{d}t}{\mathrm{d}s} + i\omega\xi\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right) - \omega^2\xi\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = 0 \tag{6.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right) = -g\left(1 + \frac{gz}{c^2}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 \tag{6.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\left(1 + \frac{gz}{c^2}\right)^2 c\frac{\mathrm{d}t}{\mathrm{d}s}\right] = 0 \tag{6.6}$$

where

$$\xi = x + iy. \tag{6.7}$$

From (6.6) we get

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{k}{\left(1 + \frac{gz}{c^2}\right)^2}, \tag{6.8}$$

where k is a constant of integration. We shall see that the expression (6.8) gives important information regarding the velocity of particle. To get equations of motion with t as independent variable we substitute the expression for dt from (6.8) in (6.4), (6.5) and obtain

$$\ddot{\xi} = \omega^2\xi - 2i\omega\dot{\xi} - f(t)\,[\dot{\xi} + i\omega\xi] \tag{6.9}$$

$$\ddot{z} = \frac{2g/c^2}{1 + \frac{gz}{c^2}}\dot{z}^2 - g\left(1 + \frac{gz}{c^2}\right) \tag{6.10}$$

where

$$f(t) = -\frac{2g}{c^2}\frac{\dot{z}}{\left(1 + \frac{gz}{c^2}\right)}. \tag{6.11}$$

The solution of (6.10) is

$$1 + \frac{gz}{c^2} = \frac{(1 + gb_3/c^2)}{\cosh\frac{gt}{c} - \sigma\sinh\frac{gt}{c}} \tag{6.12}$$

where

$$b_3 = z(0)$$

and

$$\sigma = \frac{a_3}{c} = \frac{\dot{z}(0)/c}{\left(1 + \frac{gz(0)}{c^2}\right)} \tag{6.13}$$

are two constants of integration. From (6.12) we obtain

$$2\frac{g}{c^2}\frac{\dot{z}}{(1+gz/c^2)} = -2\frac{g}{c}\frac{\tanh(gt/c)-\sigma}{1-\sigma\tanh(gt/c)} \equiv -f(t). \tag{6.14}$$

Eliminating t from (6.12) and (6.14) we get

$$\frac{\dot{z}^2}{(1+gz/c^2)^2} = c^2\left[1-\frac{(1+gz/c^2)^2}{(1+gb_3/c^2)^2}(1-\sigma^2)\right] \tag{6.15}$$

The term proportional to ξ in equation (6.9) with real time dependent coefficient $f(t)$ indicates the presence of damping in the motion. This is a purely relativistic effect. The damping is initially negative, passes through zero at time given by

$$\tanh\frac{gt}{c} = \sigma \tag{6.16}$$

and then becomes positive. Finally, it tends to a constant value $+2g/c$ as t tends to infinity. This prevents a particle from acquiring infinite velocity under constant acceleration and indeed causes the coordinates to attain a limiting value. For motion along z direction a similar role is played by the first term of right hand side of (6.10). Damping arises on account of linear acceleration of the frame and effects equally all the three components of motion; the first term on the right hand side of (6.10) is obviously $-f(t)\,\dot{z}$. The term with imaginary coefficient does no work like magnetic field in electrodynamics. Equation (6.9) factorizes into the form

$$[D+i\omega+f(t)]\,[D+i\omega]\,\xi = 0, \tag{6.17}$$

where $D = \frac{\mathrm{d}}{\mathrm{d}t}$. The solution of (6.17) is given by

$$\xi = \frac{K_1\tanh(gt/c)}{1-\sigma\tanh(gt/c)}\,\mathrm{e}^{-i\omega t} + K_2\mathrm{e}^{-i\omega t}. \tag{6.18}$$

Separating real and imaginary parts of (6.18) we get

$$x(t) = a(t)\,(a_1\cos\omega t + a_2\sin\omega t) + (b_1\cos\omega t + b_2\sin\omega t) \tag{6.19}$$

$$y(t) = a(t)\,(-a_1\sin\omega t + a_2\cos\omega t) + (-b_1\sin\omega t + b_2\cos\omega t) \tag{6.20}$$

where

$$a(t) = \frac{c}{g}\left(1+\frac{gb_3}{c^2}\right)\frac{\tanh gt/c}{(1-\sigma\tanh(gt/c))} \tag{6.21}$$

and

$$K_1 = \left(1+\frac{gb_3}{c^2}\right)(a_1+\mathrm{i}a_2) \text{ and } K_2 = (b_1+\mathrm{i}b_2) \tag{6.22}$$

are two arbitrary complex constants of integration. The expressions (6.19), (6.20) and (6.12) describe the geodesic of a particle in the accelerated frame.

Inserting the expressions for $x(t)$, $y(t)$ and $z(t)$ from (6.19), (6.20) and (6.12), respectively, and their derivatives with respect to t in the expression for the line element (6.1) we get

$$ds^2 = -\beta\,[c^2 - (a_1^2 + a_2^2 + a_3^2)]\,dt^2 \tag{6.23}$$

where

$$\beta = \frac{\left(1 + \frac{gz}{c^2}\right)^2}{\left(1 + \frac{gb_3}{c_2}\right)} = \dot{\alpha}. \tag{6.24}$$

From this we get the constant of integration k of equation (6:8). Thus we have

$$\dot{s}^2 = -\frac{(1 + gz/c^2)^4}{(1 + gb_3/c^2)^2}\,[c^2 - (a_1^2 + a_2^2 + a_3^2)]. \tag{6.25}$$

Thus equations (6.19), (6.20) and (6.12) give the time track of the material particle when

$$a_1^2 + a_2^2 + a_3^2 < c^2. \tag{6.26}$$

For

$$a_1^2 + a_2^2 + a_3^2 = c^2 \tag{6.27}$$

$ds = 0$. In this case equations (6.3) — (6.6) are no longer suitable for defining geodesics. Equations (6.19), (6.20) and (6.12), however, retain their significance and offer a means of obtaining null geodesics by a limiting process from ordinary geodesics. Equations (6.9) and (6.10) are the set of modified Lagrangian equations for photon satisfied by $L^2 = \dot{s}^2/c^2$ of Levi — Civita[5]. Thus, with the condition (6.27) we obtain the geodesics for a photon from the equations (6.19), (6.20) and (6.12). As an illustration geodesics for a photon projected from an arbitrary point (b_1, 0, b_3) at right angle to the direction of linear acceleration are given by

$$(1 + gz/c^2)\cosh(gt/c) = (1 + gb_3/c^2) \tag{6.28}$$

$$x(t) = \frac{c^2}{g}(1 + gz/c^2)\sinh(gt/c)\cos(\omega t - \mu) + b_1\cos\omega t \tag{6.29}$$

$$y(t) = -\frac{c^2}{g}(1 + gz/c^2)\sinh(gt/c)\sin(\omega t - \mu) - b_1\sin\omega t. \tag{6.30}$$

These reduce to the null geodesics for a uniform gravitational field when $\omega = 0$. Suppose a light signal is sent in the direction defined by

$$\frac{a_1}{c} = \frac{a_2}{c} = \frac{1}{2}, \text{ and } a_3 = 0 \tag{6.31}$$

in such a field. This photon cannot go beyond a certain region.

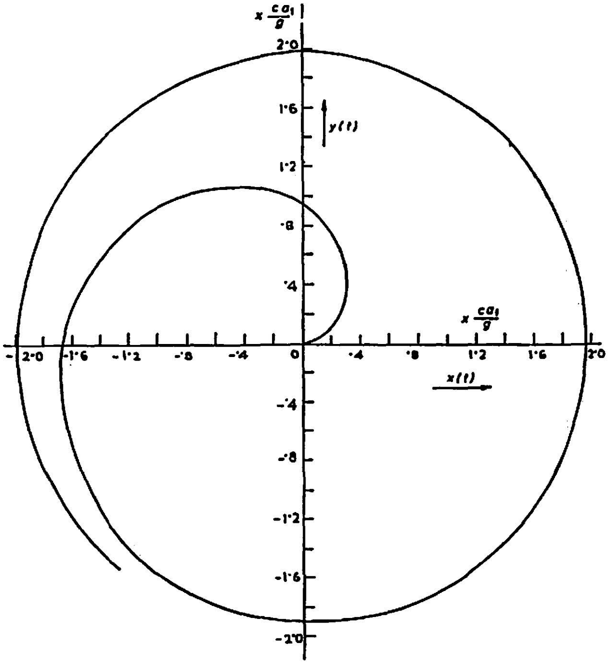

Figure 1. Motion of a particle in the xy — plane projected from the origin. $g = 6 \times 10^3$ cm · s^{-2}, $\omega = 6 \times 10^3$ s^{-1}, $\sigma = 0.5$, time interval $gt/c = 0.1$.

7. *Motion along the axis of rotation*

This motion is unaffected by the rotation of the frame and has been investigated extensively[2)], but with little attention on the velocity of light[7)]. Trajectory of a particle projected from an arbitrary point $(0, 0, b_3)$ on the axis of rotation along $+z$ or $-z$ direction is obtained by setting $b_1 = b_2 = 0$ and $a_1 = a_2 = 0$ in (6.12), (6.19) and (6.20). This gives

$$\frac{\dot{z}/c}{(1 + gb_3/c^2)} = -2\frac{[(1-\sigma]\, e^{gt/c} - (1+\sigma)\, e^{-gt/c}]}{[(1-\sigma)\, e^{gt/c} + (1+\sigma)\, e^{-gt/c}]^2} \tag{7.1}$$

$$= -\cosh\lambda \frac{\tanh((gt/c) - \lambda)}{\cosh((gt/c) - \lambda)} \tag{7.2}$$

where

$$\sigma = \tanh \lambda. \tag{7.3}$$

For a material particle $|\sigma| < 1$ and for a photon $|\sigma| = 1$.

(i) *Motion in the forward direction*

A particle projected along $+z$ direction has the maximum displacement at time t given by

$$\frac{gt}{c} = \frac{1}{2} \ln \frac{(1+\sigma)}{(1-\sigma)}. \tag{7.4}$$

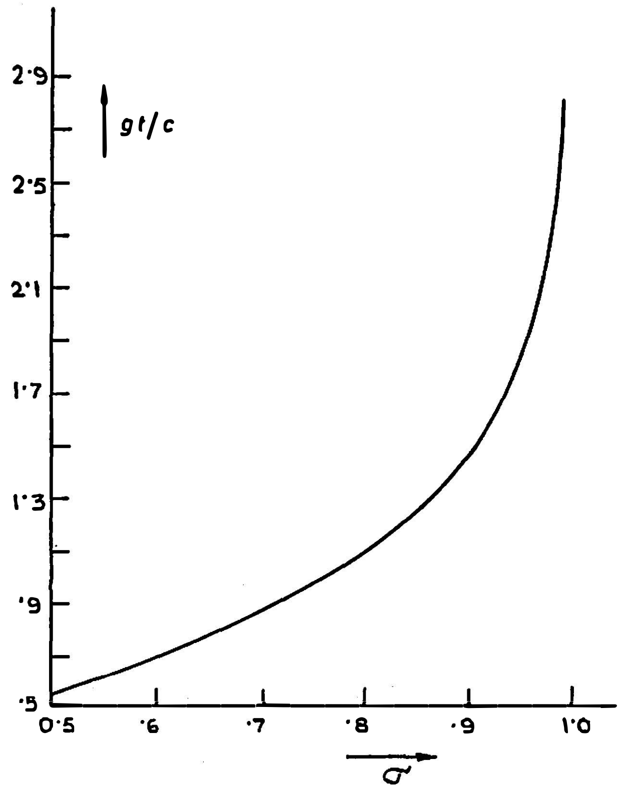

Figure 2. Time required to reach the maximum displacement as a function of σ for a particle projected along $+ z$ axis.

Then it suffers a free-fall and asymptotically reaches the value $z = -c^2/g$. Both the time t to reach the peak and magnitude of the maximum value strongly depends on σ (Fig. 2). The maximum value is given by

$$z_{max} = \left[\frac{(1 + gb_3/c^2)}{\sqrt{1-\sigma^2}} - 1\right] \frac{c^2}{g}. \tag{7.5}$$

With arbitrary initial value a_3, the velocity of the particle increases and reaches a maximum at time given by

$$\sinh (gt/c - \lambda) = -1. \tag{7.6}$$

The velocity then decreases passes through zero and the particle starts free-fall. With further increasing time the velocity increases in magnitude reaches the same maximum value at time given by the solution of

$$\sinh (gt/c - \lambda) = +1 \tag{7.7}$$

(Fig. 3). The magnitude of the maximum value is

$$\frac{\dot{z}_{max}/c}{\sqrt{1 + gb_3/c^2}} = \frac{1}{2} \frac{1}{\sqrt{1 - \sigma^2}}. \tag{7.8}$$

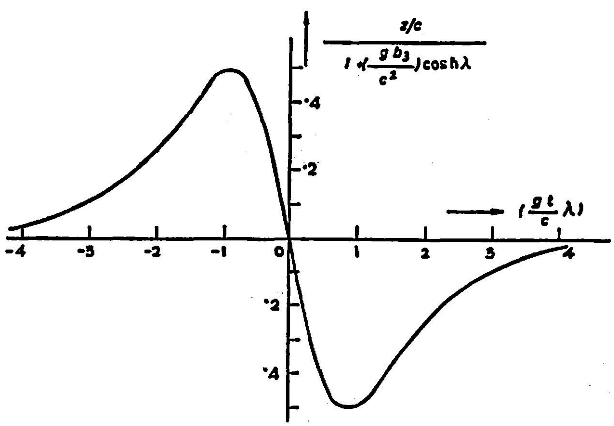

Figure 3. Velocity of a particle projected along z-axis as a function of $(gt/c - \lambda)$ showing time reversal symmetry.

For a photon projected along $+z$, the null geodesics are determined by the condition

$$a_1 = a_2 = 0, \qquad \sigma = 1, \tag{7.9}$$

which are seen to be

$$z_{ph}(t) = \frac{c^2}{g}[(1 + gb_3/c^2)\, e^{gt/c} - 1]. \tag{7.10}$$

From (7.10)

$$\dot{z}_{ph}^2 = c^2 \left(1 + \frac{gz}{c^2}\right)^2 \tag{7.11}$$

as can also be seen from (6.17). A photon projected along $+z$ never suffers a *free-fall* whereas a material particle with its initial velocity however close to c will always show a free-fall after reaching a maximum height. (Fig. 4).

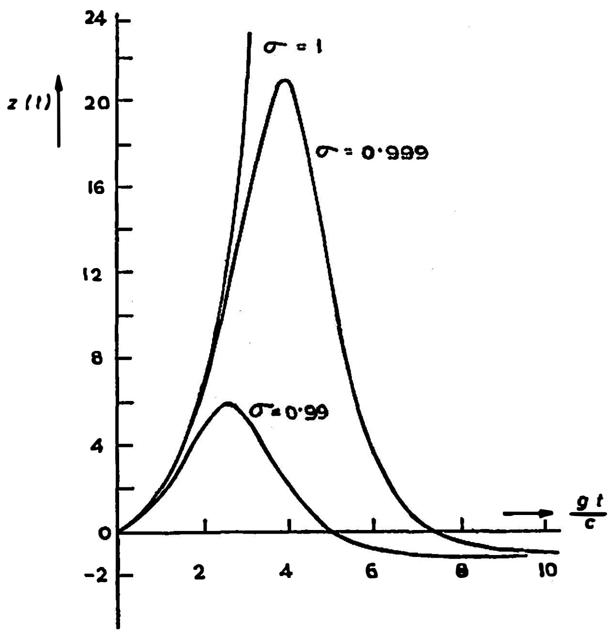

Figure 4. Motion of a material particle and a photon projected along $+z$ axis from the origin.

(ii) *Motion in the backward direction*

It can be easily seen that for t tending to infinity z_{ph} (also z) reaches a limiting value $-c^2/g$; and the velocity of photon decays exponentially. It will be seen that (7.11) is a particular case of more general result (8.15).

8. *Velocity of particle and photon*

The expression for the line element (6.1) in terms of cylindrical coordinates (ϱ, Θ, z) is

$$ds^2 = d\varrho^2 + \varrho^2 d\Theta^2 + dz^2 + 2\omega\, \varrho^2\, d\Theta\, dt - \left[\left(1 + \frac{gz}{c^2}\right)^2 - \frac{\omega^2 \varrho^2}{c^2}\right] c^2 dt^2 \quad (8.1)$$

where (ϱ, Θ, z) are related to (x, y, z) by

$$x + iy = \varrho\, e^{i\Theta},\ z = z. \quad (8.2)$$

From (6.18), (6.21) and (6.22) we obtain

$$\varrho e^{i\Theta} = [a(t)(a_1 + ia_2) + (b_1 + ib_2)]\, e^{-i\omega t}. \qquad (8.3)$$

Without any loss of generality we can choose the xy-coordinates such that $a_2 = 0$. With this simplification we get,

$$\dot{\varrho} = \beta a_1 (a\, a_1 + b_1)/\varrho \qquad (8.4)$$

$$\dot{\Theta} = -\beta \frac{a_1 b_2}{\varrho^2} - \omega. \qquad (8.5)$$

To define the coordinate velocity we require the spatial distance covered by a particle in a certain interval of time. The time interval is to be recorded by synchronized clocks placed along the trajectory of the particle. The spatial distance $d\sigma$ corresponding to a four dimensional interval ds is given by

$$d\sigma^2 = \gamma_{ij}\, dx^i\, dx^j \qquad (8.6)$$

where

$$\gamma_{ij} = \left(g_{ij} - \frac{h_i h_j}{h_4}\right). \qquad (8.7)$$

If dt' is the time interval recorded by synchronized clocks for covering the spatial distance $d\sigma$, then the square of the velocity in the space with the metric tensor γ_{ij} is given by

$$dv^2 = \left(\frac{d\sigma}{dt'}\right)^2 = v_j\, v^j \qquad (8.8)$$

where

$$v^j = \frac{dx^j}{dt'} \qquad (8.9)$$

and

$$v_j = \gamma_{ij} \frac{dx^i}{dt'}. \qquad (8.10)$$

Now we consider a particle leaving a point (x^i) at the moment t and arriving at the infinitesimally close point ($x^i + dx^i$) at the moment $t + dt$; the moments t and $t + dt$ being recorded by the clocks stationed at x^i and $x^i + dx^i$, respectively. Then, for definition of velocity we require the actual time elapsed during this course of motion which is not dt. The actual time interval will be the difference between

$(t + dt)$ and the moment $\left(t - \frac{h_i}{ch_4} dx^i\right)$ which is simultaneous at the $(x^i + dx^i)$ with the moment t at the point x^i, as shown by Landau and Lifshitz[8]. Such a synchronization is possible along any open curve. Thus

$$dt' = \mathrm{d}t + \frac{1}{c}\frac{h_i}{h_4}\mathrm{d}x^i = \frac{h_4\,\mathrm{d}x^4 + h_i\,\mathrm{d}x^i}{c\,h_4} \tag{8.11}$$

where $x^4 = ct$. The expression for the contravariant component of velocity is

$$v_i = \frac{\dot{x}^i}{1 + \frac{1}{c}\frac{h_j}{h_4}\dot{x}^j}. \tag{8.12}$$

The explicit expression for v^2 in (8.8) is

$$v^2 = \gamma_{ij}\frac{\dot{x}^i\,\dot{x}^j}{\left(h_4 + \frac{1}{c}h_k\,\dot{x}^k\right)^2}. \tag{8.13}$$

The four dimensional line element in terms of velocity is

$$\mathrm{d}s^2 = \mathrm{d}\sigma^2 + c^2 h_4\left(\frac{h_4\,\mathrm{d}x^4 + h_i\,\mathrm{d}x^i}{ch^4}\right)^2 = \left(\frac{h_4'\mathrm{d}x^4 + h_i\,\mathrm{d}x^i}{ch_4}\right)^2 (v^2 + c^2 h_4). \tag{8.14}$$

For photon $ds = 0$, and the velocity of light is given by

$$v_{photon} = c\sqrt{-h_4}. \tag{8.15}$$

From (8.14) we get

$$\dot{s}^2 = \left(\frac{h_4 + \frac{1}{c}h_i\,\dot{x}^i}{h_4}\right)^2 (v^2 + c^2 h_4). \tag{8.16}$$

Using (8.5) we get for the line element (8.1)

$$h_4 + \frac{1}{c}h_2\dot{\Theta} = -k_1\left(1 + \frac{gz}{c^2}\right)^2 \tag{8.17}$$

where

$$k_1 = 1 + \frac{\omega\, a_1\, b_2/c_2}{1 + \frac{gb_3}{c_2}}. \tag{8.18}$$

A comparison with (6.25) shows that

$$v^2 = c^2 (-h_4) + \eta \, h_4^2 \tag{8.19}$$

where

$$\eta = - \frac{[c^2 - (a_1^2 + a_3^2)]}{\left[\left(1 + \frac{g b_3}{c^2}\right)^2 + \frac{a_1 b_2}{c^2}\right]} . \tag{8.20}$$

For the line element with $h_i = 0$, the contravariant component of velocity from (8.12) is

$$v_i = \dot{x}^i . \tag{8.21}$$

In this case $\dot{x}^i$ itself is easily seen to be a quadratic function of h_4 of the type (8.19), a result already obtained by direct calculation in (6.15). Thus velocity of a particle and a photon vanishes with h_4.

9. *The event horizon*

The h_4 term in the line element (8.1) vanishes when

$$\left(1 + \frac{gz}{c^2}\right)^2 = \frac{\omega^2 \varrho^2}{c^2} = \frac{\omega^2}{c^2}(x^2 + y^2). \tag{9.1}$$

The region of space-time accessible to the accelerated observer is limited by this condition. Equation (9.1) represents a conical hypersurface with its vertex at $z = -c^2/g$ and this becomes a cylinder in 4-dimensional space, the axis of the cylinder being the time axis. When the frame is subjected to only constant linear acceleration the hyper-plane $z = -c^2/g$ defines the event horizon.

The region of the background Minkowski space accessible to the accelerated observer corresponding to the condition (9.1) is obtained from the transformation connecting the coordinates of the accelerated observer (x, y, z, t) and that of an inertial observer (X, Y, Z, T). The explicit form of this transformation is obtained from the matrix (4.2). With constant ω and with the origin executing a hyperbola motion these are

$$\begin{aligned} X &= x \cos \omega t - y \sin \omega t \\ Y &= x \sin \omega t + y \cos \omega t \\ Z &= z \cosh \frac{gt}{c} + \frac{c^2}{g}\left(\cosh \frac{gt}{c} - 1\right) \\ T &= \frac{z}{c} \sinh \frac{gt}{c} + \frac{c^2}{g} \sinh \frac{gt}{c} . \end{aligned} \tag{9.2}$$

Thus portion of Minkowski space to which the observations of accelerated observer is limited is given by the condition

$$\left(1+\frac{gZ}{c^2}\right)^2-\frac{g^2\,T^2}{c^2}=\frac{\omega^2}{c^2}(X^2+Y^2). \tag{9.3}$$

The boundary of the space-time region of the accelerated observer is the characteristic of the curvilinear coordinate system used by him. The presence of terms proportional to the derivative of coordinates with real positive coefficient in the equations of motion (6.9) and (6.10) indicate that these coordinates will attain a limiting value after sufficient long interval of time. Observation with synchronized clocks along the trajectory of the particle makes its velocity a quadratic function of h_4 with its constant term zero. The compatibility condition on metric coefficients demands that available space-time region is bounded and the boundary is defined by $h_4 = 0$. Thus to the accelerated observer all motion will cease on the event horizon.

The time to reach the event horizon is obtained by solving the equation $h_4 = 0$ for t. For a particle projected from the origin it gives

$$\sinh\frac{gt}{c}=\frac{g}{\omega\,a_4}.$$

For values of g, ω, σ given in Fig. 1 as illustration and with $a_1 = 0.5\ \mathrm{cm}\cdot\mathrm{s}^{-1}$

$$t \approx 83.9 \text{ days}.$$

Though independent of a_3, t depends on the linear acceleration g along z-axis, as is expected. Limiting value of ϱ is reached in this case around $gt = 4.8$, as can be seen from Fig. 1. This corresponds to $t \approx 277.7$ days. Thus long before attaining the limiting values of coordinates, the motion will cease on event horizon.

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Izveden je linijski element akceleriranog opažača pretpostavljajući linearnost transformacije između njegovih koordinata u nekom danom vremenu i Galilejevih koordinata.