

A SUM RULE FOR A QUARKONIUM SPECTROSCOPY AND ITS PREDICTIONS FOR THE LEVEL-SCHEMES OF THE CHARMONIUM STATES

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This paper introduces a sum rule for the levels of a quarkonium spectroscopy. This rule correlates the masses M_1 , M_2 , M_3 of the three levels described, respectively, by the sets of spectroscopic quantum numbers (n_1, l_1, s_1, J_1) , (n_2, l_2, s_2, J_2) , (n_3, l_3, s_3, J_3) which satisfy the conditions:

$$(i) \quad l_1 = l_2 = l_3, \quad s_1 = s_2 = s_3$$

and

$$(ii) \quad n_3 = (n_2 + l_2 + s_2 + J_2) + 1,$$

$$n_2 = (n_1 + l_1 + s_1 + J_1) + 1.$$

For any three levels belonging to a given spectroscopy, which satisfy the conditions stated above, the following sum rule holds true

$$(n_2 + l_2 + s_2 + J_2) M_2 = \frac{(n_3 + l_3 + s_3 + J_3) M_3 + (n_1 + l_1 + s_1 + J_1) M_1}{2}.$$

This sum rule is found to be valid with a high degree of accuracy when tested with the upsilon states interpreted as the levels of the bottomium spectroscopy. Also, the level-schemes of all the charmonium states so far observed have been envisaged

within the framework of the sum rule concerned. It has been shown that the generally accepted level-schemes for the charmonium states ψ (3.10), ψ (3.69), ψ (3.77), χ (3.41), χ (3.51) and χ (3.55) are in conformity with the same admissible by this sum rule for the states concerned. This fact implies that the predictions of this sum rule regarding the level-schemes of the other charmonium states should be reliable to a great extent. According to this sum rule the states ψ (4.03), ψ (4.16) and ψ (4.42) should be interpreted as the $3^3 S_1$, $2^3 D_1$ and $3^3 D_1$ levels, respectively, of the charmonium spectroscopy of which the $1^1 S_0$, $2^1 S_0$ and $3^1 S_0$ levels should correspond to the states χ (2.98), χ (3.45) and χ (3.60), respectively.

1. Introduction

It is well known that the quarkonium states, composed of a quark Q and its anti-quark \bar{Q} , are interpreted¹⁻⁹⁾ as the levels of the $Q\bar{Q}$ -spectroscopy. For this purpose the standard non-relativistic procedure, discussed in details by Machacek and Tomozawa⁹⁾, is to solve the Schrödinger equation with a suitable choice of an inter-quark potential. The resulting solutions, which represent the levels of the $Q\bar{Q}$ -spectroscopy, are compared with the observed mass spectrum of the $Q\bar{Q}$ -states. It is worth mentioning that in the interpretation problem of quarkonia the following types of inter-quark potentials find frequent use in the literature^{1,2)}

$$(i) \quad V(r) = -\frac{a_s}{r}$$

$$(ii) \quad V(r) = \beta r^2$$

$$(iii) \quad V(r) = -\frac{a_s}{r} + L(r)$$

where $L(r)$, occurring in $V(r)$ of type (iii), represents the long-range part of the $Q\bar{Q}$ -potential concerned. It may be recalled, in this context, that the various forms⁵⁻⁹⁾ for $L(r)$ suggested in the literature include $L(r) = ar$ as proposed by Eichten et al.⁵⁾ and $L(r) = br^2$ as introduced by Krammer and Kraseman⁶⁾ whereas other suggested forms⁷⁻⁹⁾ for $L(r)$ include the logarithmic dependence^{8,9)} on r . All these forms for $L(r)$ are, however, really the outcome of educated guessworks. This gives rise to the uncertainties associated with the long-range part $L(r)$ of type (iii) potential. In spite of this, type (iii) potential appears to be more realistic compared to potentials of type (i) and type (ii) apart from the fact that it is a two-parametric potential and as such mathematically more flexible (and, therefore, more accommodative) than the other two types which are, needless to mention, one-parametric potentials. It is to be noted, however, that the so far accumulated experimental data on quarkonia do not completely justify or reject any one of the three types of potentials referred to above. This in turn implies that the current status of our knowledge regarding the exact nature of the inter-quark potential is far from satisfactory. This is the genesis of the conflicting inter-

pretations which exist upto now in the literature^{1,2)} for the charmonium states occurring above 4 GeV. Also, there are ambiguities in the proposed interpretations^{1,2)} for some χ -states which include $\chi(3.45)$ and $\chi(3.60)$ reported, respectively, by the *SLAC* group¹⁰⁾ and the *DESY* group¹¹⁾. In fact, the interpretation problem is so serious for these charmonium states that it is rightly regarded as an *open question*¹¹⁾. This situation has prompted us to introduce in this paper a sum rule for the levels of a quarkonium spectroscopy and to employ the same for a verification of the proposed interpretations for the so far observed charmonium states.

The plan of this paper is as follows. The sum rule for the levels of a quarkonium spectroscopy has been formulated in Sec. 2 by following a semi-empirical method. The validity of this sum rule has been verified in Sec. 3 by taking advantage of the fact that there is no ambiguity or dispute as regards the interpretation¹²⁻¹⁴⁾ of the vector bottomium states $\Upsilon(9.45)$, $\Upsilon(10.10)$, $\Upsilon(10.32)$ and $\Upsilon(10.54)$ as the 1^3S_1 , 2^3S_1 , 3^3S_1 and 4^3S_1 levels, respectively, of bottomium spectroscopy. The sum rule is found to be excellently satisfied for these levels. In Sec. 4 the problem of interpretation of all the charmonium states so far observed has been envisaged within the framework of this sum rule.

2. Formulation of the sum rule

In order to derive the desired sum rule for a quarkonium spectroscopy we consider a $Q\bar{Q}$ -system, Q and \bar{Q} denoting a quark and its anti-quark, respectively. The allowed states of this system admit the interpretation of the levels which constitute the $Q\bar{Q}$ -spectroscopy. For the moment we confine our attention on one of the allowed states of this $Q\bar{Q}$ -system. We describe this state as the $n^{2s+1}L_J$ level of the $Q\bar{Q}$ -spectroscopy where n, l, s and J are the familiar set of quantum numbers which are conventionally used in the specification of spectroscopic levels. The mass $M_{n,l,s,J}$ of this state (i.e. the level) can be formally expressed as

$$M_{n,l,s,J} = M_Q + M_{\bar{Q}} + E_{n,l,s,J} \quad (1)$$

where M_Q and $M_{\bar{Q}}$ represent the free masses of the quarks, Q and \bar{Q} , respectively; $E_{n,l,s,J}$ denoting the interaction energy appropriate for the state under considerations. It is well known that the standard (non-relativistic) procedure⁹⁾ for determining $E_{n,l,s,J}$ occurring in Eq. (1) is to solve the Schrödinger equation which incorporates an inter-quark potential. This evaluation for the interaction energy will be realistic to the extent the chosen inter-quark potential is so. Of the three types of inter-quark potentials, already referred to in Sec. 1, for describing quarkonium spectroscopies the following type

$$V(r) = -\frac{\alpha_s}{r} + L(r) \quad (2)$$

appears to be more realistic compared to others. This is because in this type of potential there are provision for the short-range part (which is Coulomb-like) as

well as the long-part $L(r)$ which is necessary for describing quark confinement. In addition to this, the potential concerned is mathematically more flexible, as it is a two-parametric potential (since it involves two terms), compared to the one-parametric idealistic potentials of types (i) and (ii) mentioned in Sec. 1. For these reasons, we will be guided by this potential in what follows. In this connection it is worth noting that the only unsatisfactory feature regarding the potential of our concern is the uncertainty associated with the exact form of its long-range part $L(r)$. We have already discussed in Sec. 1 the various suggested forms for $L(r)$. None of these forms, however, can be taken too seriously for the reasons pointed out in the earlier section. Nevertheless, the success with Eichten et al.⁵⁾ could predict the mass of the first unbound charmonium state i. e. $\psi(3.77)$ well in advance of its experimental detection with their proposed form for $L(r)$, namely, $L(r) = ar$ clearly reveals that the exact form of $L(r)$ should be close to their suggested form. This implies that the appropriate form for $L(r)$ should be

$$L(r) = ar^b \quad (3)$$

and, further, it is reasonable to assume that

$$0 < b < 2$$

which implies that $L(r)$, given by Eq. (3), will be not far from the form of (r) as suggested by Eichten et al.⁵⁾. With the help of Eq. (3), the $Q\bar{Q}$ -potential described by Eq. (2) can be written more explicitly as

$$V(r) = -\frac{a_s}{r} + ar^b; \quad 0 < b < 2. \quad (4)$$

It is to be noted, however, that the above potential is not completely specified as the value of b has not been uniquely fixed. Our intention here is to avoid any uncertainty associated with the power of the radial dependence of the long-range part of $V(r)$. For this reason we have not assigned a specific value to b and, instead, we have considered only the bounds on its value in order that the long-range term of $V(r)$ does not remain completely unspecified. As b is not uniquely fixed, which in turn implies $V(r)$ is not completely specified, therefore, the standard procedure (discussed earlier) for determining $E_{n,l,s,j}$ occurring in Eq. (1), cannot be followed for this potential. This difficulty can be bypassed by following the procedure outlined below.

For the moment we confine our attention to the Coulomb and the harmonic oscillator potentials. It is well known from quantum mechanics that the energy-level spectrum for the Coulomb and the same for the harmonic oscillator potential are given by the following relations, respectively

$$E_n = An^{-2} \text{ for } V(r) = -\frac{a_s}{r} \quad (5)$$

and,

$$E_n = Bn + C \text{ for } V(r) = \beta r^2 \quad (6)$$

where A , B and C are familiar constants and n represents principal quantum number. It is evident from Eq. (5) and (6) that

$$E_n \sim n^{-2} \text{ for } V(r) = -\frac{a_s}{r} \quad (5a)$$

$$E_n \sim n \text{ for } V(r) = \beta r^2. \quad (6a)$$

At this point we note that as we go from the r^{-1} -potential (i. e. Coulomb potential) to the r^2 -potential (i. e. harmonic oscillator potential), the n -dependence of E_n changes from n^{-2} to n . Now, we shift our attention to the n -dependence of E_n for $V(r)$ described by Eq. (4). For this potential we note that

$$V(r) \approx -\frac{a_s}{r} \text{ for sufficiently small } r$$

and correspondingly we have

$$E_n \sim n^{-2}.$$

Also, for the same potential we note that

$$V(r) \approx ar^b \text{ with } 0 < b < 2 \text{ for sufficiently large } r.$$

In this case $E_n \sim n^x$ where $x < 1$ since $b < 2$. We have seen above that as the power of radial dependence of $V(r)$ increases the power of n -dependence of E_n also increases. From what have been said so far it is clear that the bounds on the power of n -dependence of E_n corresponding to $V(r)$ described by Eq. (4) should be

$$-2 < \text{power of } n\text{-dependence of } E_n < +1.$$

Therefore, E_n appropriate for $V(r)$ concerned can be expressed as

$$E_n = Dn^x + F; \quad -2 < x < +1 \quad (7)$$

where D and F are two constants. We may note here that the value of x , occurring in Eq. (7), depends on the value chosen for b , the power of r appearing in the long-range part of $V(r)$ described by Eq. (4). In this connection it is worth noting that we have not specified the value of b , therefore, we are entitled to make a choice for the value of x . It may be emphasized here that in the conventional procedure $V(r)$ is completely specified and as such there is no scope for any choice of the power of n -dependence of E_n . Stated differently, the power of n -dependence of E_n is uniquely dictated by the (completely specified) potential. In sharp contrast to this, x appearing in Eq. (7) is not uniquely fixed as b occurring in Eq. (4) has not been completely specified.

However, we have seen above that x must lie in between -2 and $+1$ for $0 < b < 2$. The choice of the value for x appearing in (7) is dictated by the following considerations. It is a familiar fact in all types of well known spectroscopies that the analytical expressions for the energy-levels involve the principal quantum number (n) with an integral power only. The only exception to this is seen in alkali spectra where the power of n appears to deviate from an integral value due to *screening* effect. The absence of such effects in quarkonia as well as the reasons discussed above indicate that we are justified in choosing an integral value for x appearing in Eq. (7). It is equally true, however, that the choice of an integral value of x is mathematically not aesthetic. Needless to mention, we are definitely not entitled to throw away by hands the non-integral values which are admissible for x . For the reasons mentioned above and also because we are interested in a semi-empirical method of derivation for the desired sum rule, we will restrict ourselves to the considerations of integral values which are consistent with the condition $-2 < x < +1$. Such a procedure receives a strong empirical support. This point will be evident at a later stage (in Sec. 3). Clearly, the only integral value allowed for x which is consistent with the condition $-2 < x < +1$ is -1 as the value $x = 0$ is inadmissible as for this value n -dependence of E_n vanishes as evident from Eq. (7). Obviously, the chosen value $x = -1$ fixes uniquely the value of b which, however, does not concern us for further development of our formulation. The value $x = -1$ reduces Eq. (7) to the following form

$$E_n = \frac{D}{n} + F. \quad (8)$$

Guided by the form of Eq. (8) along with a little reflection, we may try to express $E_{n,l,s,J}$ in the following form

$$E_{n,l,s,J} = \frac{D}{n + l + s + J + F}. \quad (9)$$

One important remark is on order here. In the non-relativistic procedure one cannot expect to find both s and J in the expression for the energy. However, we have repeatedly emphasized that we are not following the conventional procedure and as such the occurrence of both s and J in Eq. (9) is not unreasonable. In fact, it is indeed supported by the empirical facts. We shall return to this point afterwards (in Sec. 3). For the moment, we proceed by noting that although Eq. (9) reduces to Eq. (8) when $l = s = J = 0$, as needed, but it is too much simplistic as it implies that the quantum numbers l and s are completely separable. It may be emphasized that due to the spin-orbit coupling it is not expected, in general, that l and s can be completely separated. Therefore, the performance of Eq. (9) is not expected to be satisfactory. To test the validity of Eq. (9) we recast Eq. (1) in the following form making use of the former equation.

$$M_{n,l,s,J} = \frac{D}{(n + l + s + J) + F} + M_Q + M_{\bar{Q}}. \quad (10)$$

Now we consider the $3^3 S_1$, $1^3 D_1$ and $1^3 P_2$ levels of the charmonium spectroscopy. For all these three levels the quantity $(n + l + s + J)$ has the same value

i. e. 5 apart from the same value for M_Q and $M_{\bar{Q}}$ (which are fixed for a given spectroscopy). This means that Eq. (1) demands that the masses of these levels must be identical contrary to the experimental fact. Needless to mention, the observed masses of the charmonium states which may be interpreted as the $3^3 S_1$, $1^3 D_1$ and $1^3 P_2$ levels of the charmonium spectroscopy are different from each other. Also, according to Eq. (10) the masses of the $1^3 S_1$ and $1^3 P_0$ levels of the charmonium spectroscopy must have the same value — a conclusion strongly negated by their observed masses. From what have been said above it is clear that Eq. (10) is not at all satisfactory for a quarkonium spectroscopy. A similar remark, therefore, is also true for Eq. (9) which differs from Eq. (10) only by the quantity $(M_Q + M_{\bar{Q}})$ which is a constant for a given spectroscopy. From this we come to the conclusion that the genesis of the deficiency of Eq. (9) lies in our implicit assumption that the quantities D and F occurring in the equation concerned are constants. Therefore, we modify Eq. (9) as

$$E_{n,l,s,J} = \frac{D(l,s)}{(n+l+s+J)} + F(l,s) \quad (11)$$

where the quantities $D(l,s)$ and $F(l,s)$ are considered as the functions of the quantum numbers l and s . Obviously, the forms of the functions $D(l,s)$ and $F(l,s)$ should be such that $D(0,0) = D$ and $F(0,0) = F$, D and F being the quantities which appear in Eq. (9). This is because Eq. (8) must be recovered from Eq. (11) when $l = s = J = 0$. The most important point regarding Eq. (11) is that it suggests, unlike Eq. (9), that the quantum numbers l and s are not completely separable as argued by us earlier.

To see the justification for Eq. (11) we may proceed as follows. By making use of this equation, we can recast Eq. (1) as

$$M_{n,l,s,J} = \frac{D(l,s)}{n+l+s+J} + F(l,s) + M_Q + M_{\bar{Q}}. \quad (12)$$

For a given spectroscopy M_Q and $M_{\bar{Q}}$ are constant quantities. Further, if we restrict ourselves to the levels having the same l and s and belonging to a given spectroscopy, then, for such levels $D(l,s)$ and $F(l,s)$ may be treated as constant quantities and, as a consequence, Eq. (12) predicts a linear variation of quarkonium mass $M_{n,l,s,J}$ with the quantity $(n+l+s+J)^{-1}$. In order to check the validity of this prediction we have plotted in Fig. 1 the observed masses of the quarkonia which can be interpreted as the levels having the same l and s and belonging to a given spectroscopy. For convenience we have used in Fig. 1 the notation $M_{Q\bar{Q}}$ instead of $M_{n,l,s,J}$. The above mentioned prediction turns out to be very accurate as reflected in Υ -curve of Fig. 1. We are here referring only to the Υ -curve as for the upsilons there are no disagreement in the literature as regards their level-schemes. If we accept the interpretation proposed in the literature^{1,2)} for the $\chi(3.41)$, $\chi(3.51)$, $\chi(3.55)$ as the $1^3 P_0$, $1^3 P_1$, $1^3 P_2$ levels, respectively, of the charmonium spectroscopy, then, for these triplet states the prediction of Eq. (12) is found to be justified as evident from the χ (triplet)-curve given in Fig. 1. As there are serious disagreements in the literature concerning the interpretation of the charmonium states $\psi(4.03)$, $\psi(4.16)$ and $\psi(4.42)$ and since the available interpretations for the states

$\chi(3.45)$ and $\chi(3.60)$ are not free from ambiguity, therefore, for these states we have used the level-schemes admissible by the sum rule we are going to derive. For the sake of completeness we have also shown in Fig. 1 the $\psi(n^3 S_1)$ —, $\psi(n^3 D_1)$ —, and $\chi(\text{singlet})$ -curves. In passing we may note that the Υ - and $\chi(\text{triplet})$ -curves of Fig. 1 clearly reveal that Eq. (12) is indeed justified.

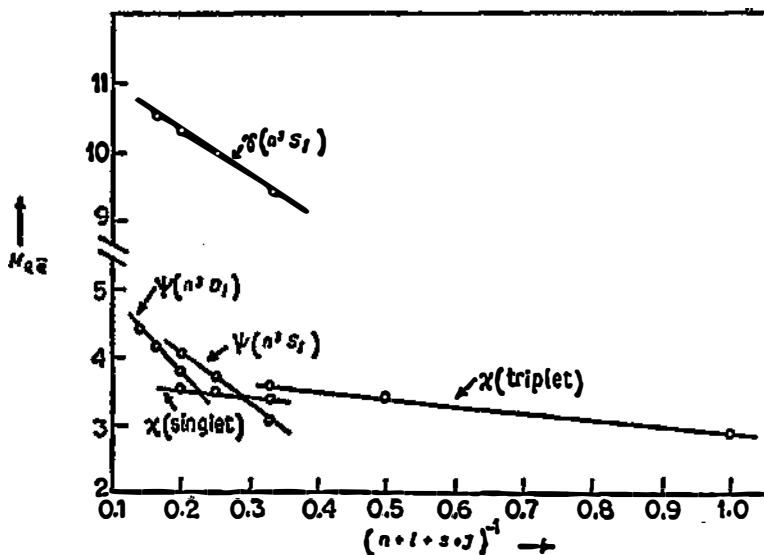


Fig. 1. Variation of $M_{Q\bar{Q}}$ with $(n + l + s + J)^{-1}$.

To proceed to our goal of deriving the desired sum rule, we rewrite Eq. (12) in the following form

$$M_{n,l,s,J} = \frac{D(l,s)}{X} + F(l,s) + M_Q + M_{\bar{Q}} \quad (12a)$$

where $X = n + l + s + J$. For our convenience, we give Eq. (12a) the form indicated below.

$$XM_{n,l,s,J} = X[F(l,s) + M_Q + M_{\bar{Q}}] + D(l,s). \quad (13)$$

Now we consider three levels belonging to a given spectroscopy (for which M_Q and $M_{\bar{Q}}$ are fixed) and described by the three sets of quantum numbers (n_1, l_1, s_1, J) , (n_2, l_2, s_2, J_2) and (n_3, l_3, s_3, J_3) . For notational simplicity we will denote the mass of the level specified by the quantum numbers (n_1, l_1, s_1, J_1) by M_1 instead of M_{n_1, l_1, s_1, J_1} , and similarly for the other two levels. For these levels Eq. (13) takes the following forms l_1 , respectively

$$X_1 M_1 = X_1 [F(l_1, s_1) + M_Q + M_{\bar{Q}}] + D(l_1, s_1) \quad (14a)$$

$$X_2 M_2 = X_2 [F(l_2, s_2) + M_Q + M_{\bar{Q}}] + D(l_2, s_2) \quad (14b)$$

$$X_3 M_3 = X_3 [F(l_3, s_3) + M_Q + M_{\bar{Q}}] + D(l_3, s_3) \quad (14c)$$

where $X_1 = n_1 + s_1 + J_1$, and so on. If the three levels under considerations are so chosen that their quantum numbers satisfy the conditions

$$(i) \quad l_1 = l_2 = l_3 \text{ and } s_1 = s_2 = s_3$$

and

$$(ii) \quad X_3 = X_2 + 1, X_2 = X_1 + 1,$$

then, condition (i) implies $F(l_1, s_1) = F(l_2, s_2) = F(l_3, s_3)$ and $D(l_1, s_1) = D(l_2, s_2) = D(l_3, s_3)$.

Therefore, from Eq. (14a) — (14c) we get

$$\frac{X_2 M_2 - X_1 M_1}{X_3 M_3 - X_2 M_2} = \frac{X_2 - X_1}{X_3 - X_2}. \quad (15)$$

Finally, taking advantage of condition (ii) i. e. $X_3 = X_2 + 1$ and $X_2 = X_1 + 1$, Eq. (15) can be recast in the form of the following sum rule

$$X_2 M_2 = \frac{X_3 M_3 + X_1 M_1}{2} \quad (16)$$

where $X_1 = n_1 + l_1 + s_1 + J_1$, etc. We repeat to emphasize that this sum rule correlates three levels of a given spectroscopy if the conditions (i) and (ii), mentioned, are satisfied for these levels. It is interesting to note that condition (ii) i. e. $X_3 = X_2 + 1$ and $X_2 = X_1 + 1$ can be satisfied in two alternative ways. If the levels are so chosen that $n_3 = n_2 + 1$, $n_2 = n_1 + 1$, $l_3 = s_3 + J_3 = l_2 + s_2 + J_2 = l_1 + s_1 + J_1$, then, we have $X_3 = X_2 + 1$ and $X_2 = X_1 + 1$ as $X_1 = n_1 + l_1 + s_1 + J_1$, and so on. Alternatively, if for the levels we have $n_3 = n_2 = n_1$, $l_3 + s_3 + J_3 = l_2 + s_2 + J_2 + 1$, $l_2 + s_2 + J_2 = l_1 + s_1 + J_1 + 1$, then also $X_3 = X_2 + 1$ and $X_2 = X_1 + 1$. Needless to mention, the levels which satisfy the sum rule must have the same l and s as evident from condition (i) which demands that $l_1 = l_2 = l_3$ and $s_1 = s_2 = s_3$.

3. Verification of the sum rule

In this section we want to examine the validity of the sum rule introduced in Sec. 2. In order to do this we can exploit the undisputed level-schemes $1^3 S_1$, $2^3 S_1$, $3^3 S_1$, and $4^3 S_1$ assigned to the vector bottomium states^{1-3,14,15}

Υ (9.45), Υ (10.01), Υ (10.32) and Υ (10.54), respectively. In this context we may recall that the application of the sum rule to the levels belonging to a given spectroscopy necessitates the selection of a set of three levels for which conditions (i) and (ii), discussed in Sec. 2, are satisfied. Condition (i) requires that the three levels chosen must have the same l and s whereas condition (ii) demands that $X_3 = X_2 + 1$, $X_2 = X_1 + 1$ where X_1 , X_2 and X_3 represent the sum of the quantum numbers (n, l, s, J) for the levels i. e. $X_1 = n_1 + l_1 + s_1 + J_1$, etc. Now, out of the four levels $1^3 S_1$, $2^3 S_1$, $3^3 S_1$ and $4^3 S_1$ of the bottomium spectroscopy if we choose the set of three levels $1^3 S_1$, $2^3 S_1$ and $3^3 S_1$, then, it is easy to check that condition (i) is manifestly satisfied as evident from their level-schemes. Further, as the multiplicity of these levels is 3 which in turn implies that $s = 1$ for these levels, therefore, for the levels $1^3 S_1$, $2^3 S_1$ and $3^3 S_1$ we have $X_1 = n_1 + l_1 + s_1 + J_1 = 1 + 0 + 1 + 1 = 3$, $X_2 = 2 + 0 + 1 + 1 = 4$ and $X_3 = 3 + 0 + 1 + 1 = 5$. Clearly, then, condition (ii) is also satisfied for the three levels concerned. Obviously, the levels having $X_1 = 3$, $X_2 = 4$ and $X_3 = 5$ correspond, respectively, to the masses $M_1 = 9.45$ GeV, $M_2 = 10.01$ GeV and $M_3 = 10.32$ GeV. This fact enables one to easily evaluate the left hand and the right hand sides of Eq. (16) which describes the sum rule. These values, shown in the last two columns of Table 1, are very close to each other justifying the validity of the sum rule. It is a simple matter to check that out of the four levels $1^3 S_1$, $2^3 S_1$, $3^3 S_1$ and $4^3 S_1$ of the bottomium spectroscopy if we select the following set of three levels $2^3 S_1$, $3^3 S_1$ and $4^3 S_1$, then, also conditions (i) and (ii) are found to be satisfied for the chosen levels. The masses corresponding to these levels as well as their respective values for $X = n + l + s + J$ have been indicated in Table 1. For these levels the values of the left hand and right hand sides of Eq. (16) have also been given in the last two columns of the same table. A look into this table will reveal the validity of the sum rule to a great extent. Encouraged by this, we go over to the next section to examine the validity of the level-schemes, proposed in the literature, for the charmonium states in the light of the sum rule under investigations. The impressive performance of this rule clearly justifies our choice for an integral value for x occurring in Eq. (7) as well as verifies the validity of the simultaneous occurrence of s and J in Eq. (9).

TABLE 1

$b\bar{b}$ -state	Level-scheme $n^{2s+1} l_J$	Mass M (GeV)	Value of $X =$ $n + l + s + J$	Value of L.H.S. of (16)	Value of R.H.S. of (16)
Υ (9.45) Υ (10.01) Υ (10.32)	$1^3 S_1$ $2^3 S_1$ $3^3 S_1$	$M_1 = 9.45$ $M_2 = 10.01$ $M_3 = 10.32$	$X_1 = 3$ $X_2 = 4$ $X_3 = 5$	40.04	39.97
Υ (10.01) Υ (10.32) Υ (10.54)	$2^3 S_1$ $3^3 S_1$ $4^3 S_1$	$M_1 = 10.01$ $M_2 = 10.32$ $M_3 = 10.54$	$X_1 = 4$ $X_2 = 5$ $X_3 = 6$	51.60	51.64

The sum rule and the $b\bar{b}$ -spectroscopy.

4. The sum rule and the level-schemes of the charmonium states

As mentioned above, our motivation in this section is to investigate the level-schemes of the charmonium states admissible by the sum rule. With this end in view we proceed as follows. We have seen that the sum rule correlates three levels, belonging to a given spectroscopy, which satisfy conditions (i) and (ii) discussed in the earlier sections. To illustrate the procedure to be followed in this section we consider, for example, three charmonium states $\psi(3.10)$, $\psi(3.69)$ and $\psi(3.77)$ and assume, for the moment, that the level-schemes for these states are 1^3S_1 , 2^3S_1 and 3^3S_1 , respectively. For these three states we have ($M_1 = 3.10$ GeV, $X_1 = 3$), ($M_2 = 3.69$ GeV, $X_2 = 4$) and ($M_3 = 3.77$ GeV, $X_3 = 5$). It is a simple matter to check that Eq. (16) i. e. the sum rule is not satisfied for the assumed level-schemes for the states under considerations. This in turn implies that our assumption regarding the level-schemes for at least any one of them is not correct. By repeating this procedure for all the ψ -states at our disposal, it can be easily checked that the sum rule is best satisfied if, and only if, we assign the level-schemes 1^3S_1 , 2^3S_1 , 3^3S_1 to the states $\psi(3.10)$, $\psi(3.69)$, $\psi(4.03)$, respectively. Similar procedures have been followed for assigning the level-schemes to other charmonium states including the most recently discovered¹⁴⁾ $\chi(2.98)$ as well as $\chi(3.45)$ and $\chi(3.60)$ reported, respectively, by the *SLAC* group¹⁰⁾ and the *DESY* group¹¹⁾. These level-schemes have been shown in the second column of Table 2. In this context it may be recalled that there are no dispute in the literature as regards the level-schemes proposed for the charmonium states $\psi(3.10)$, $\psi(3.69)$ and $\psi(3.77)$ which are interpreted^{1,2)} as the 1^3S_1 , 2^3S_1 and 1^3D_1 levels, respectively, of the charmonium spectroscopy. Also, there is no divergence of opinions about the interpretation of $\chi(3.41)$, $\chi(3.51)$ and $\chi(3.55)$ as the 1^3P_0 , 1^3P_1 and 1^3P_2 levels of the same spectroscopy. It is indeed gratifying to note that these level-schemes of the charmonium states mentioned above are in conformity with the level-schemes admissible by the sum rule for the charmonium states concerned. This point becomes immediately transparent from a glance at Table 2. However, the verdicts of the sum rule on the interpretation of the other charmonium states are especially interesting and important as well. Reasons are not difficult to trace. On several occasions we have referred to the conflicting assignments, available in the literature, for the level-schemes of the charmonium states $\psi(4.03)$, $\psi(4.16)$, $\psi(4.42)$ in particular. These assignments have been discussed by Wiik and Wolf¹⁾ and also by Schöpper²⁾. In this context we may also note that there is mounting evidence in support of the proposed interpretation of $\chi(3.45)$, discovered by the *SLAC* group¹⁰⁾, as the 2^1S_1 level^{1,2)} of the charmonium spectroscopy. This interpretation also receives justification from the sum rule which is manifested in the level-scheme shown in Table 2 for $\chi(3.45)$ admissible by this sum rule. The situation, however, is entirely different for $\chi(3.60)$ reported by the *DESY* group¹¹⁾. The probable interpretations of $\chi(3.60)$, as discussed by Bartel et al.¹¹⁾, are open to criticisms and, in fact, one of the suggestions that it might be η_c i. e. 1^1S_1 state has been ruled out after the very recent discovery¹⁴⁾ of the state $\chi(2.98)$ which is being currently interpreted¹⁴⁾ as the η_c . It is highly interesting to note that the sum rule also justifies the interpretation of $\chi(2.98)$ as the 1^1S_0 state. This becomes evident from a look into Table 2 which also reveals that the sum rule demands that $\chi(3.60)$ should be interpreted as the 3^1S_0 state.

TABLE 2.

$c\bar{c}$ -state	Level-scheme $n^{2s+1}l_J$	Mass M (GeV)	Value of $X =$ $n + l + s + J$	Value of L.H.S. of (16)	Value of R.H.S. of (16)
ψ (3.10) ψ (3.69) ψ (4.0)	1^3S_1 2^3S_1 3^3S_1	$M_1 = 3.10$ $M_2 = 3.69$ $M_3 = 4.03$	$X_1 = 3$ $X_2 = 4$ $X_3 = 5$	14.76	14.72
ψ (3.77) ψ (4.16) ψ (4.42)	1^3D_1 2^3D_1 3^3D_1	$M_1 = 3.77$ $M_2 = 4.16$ $M_3 = 4.42$	$X_1 = 5$ $X_2 = 6$ $X_3 = 7$	24.96	24.89
χ (3.41) χ (3.51) χ (3.55)	1^3P_0 1^3P_1 1^3P_2	$M_1 = 3.41$ $M_2 = 3.51$ $M_3 = 3.55$	$X_1 = 3$ $X_2 = 4$ $X_3 = 5$	14.04	13.99
χ (2.98) χ (3.45) χ (3.60)	1^1S_0 2^1S_0 3^1S_0	$M_1 = 2.98$ $M_2 = 3.45$ $M_3 = 3.60$	$X_1 = 1$ $X_2 = 2$ $X_3 = 3$	6.90	6.89

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5. Conclusions

In the earlier section we have seen that the proposed sum rule reproduces the generally accepted level-scheme assignments for the charmonium states ψ (3.10), ψ (3.69), ψ (3.77), χ (3.41), χ (3.51) and χ (3.55). It is, therefore, logical to expect that its interpretation of the states ψ (4.03), ψ (4.16), ψ (4.42), χ (2.98), χ (3.45) and χ (3.60) as the 3^3S_1 , 2^3D_1 , 3^3D_1 , 1^1S_0 , 2^1S_0 and 3^1S_0 levels, respectively, of the charmonium spectroscopy will be reliable. This expectation receives an excellent support from the extremely accurate linear variation, reflected in the curves given in Fig. 1, of the observed masses of the quarkonia with their corresponding values of $(n + k + s + J)^{-1}$. We conclude this paper with the remark that the sum rule, introduced in this paper, will also be useful in $t\bar{t}$ -spectroscopy which, however, remains to be explored.

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PRAVILO SUME U SPEKTROSKOPIJI KVARKONIUMA I PREDVIĐANJE ŠEME NIVOVA ZA STANJA ŠARMONIUMA

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U članku se uvodi pravilo sume za nivoe u spektroskopiji kvarkoniuma. Ova suma korelira mase triju nivoa. Nalazi se da je valjanost sume vrlo dobra ako se ona testira na stanjima upsilona interpretiranim kao nivoi spektroskopije botoniuma.