

BHABHA WAVE-EQUATIONS FOR HALF INTEGER SPIN STUDIED FROM THE GEL'FAND-YAGLOM VIEWPOINT

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By deriving the simpler expression $f(x, S) = (-1)^{S-P-1/2}$ for the hermitianizing matrix than the one given by Madavarao et al. it was shown that all Bhabha wave-equations for half integer spin $\geq 3/2$ based on the group $SO(4,1)$ have indefinite charge.

1. Introduction

In an earlier paper by the author¹⁾ it was shown that the matrices L_k , $k = 0, 1, 2, 3$ appearing in the Bhabha wave-equation²⁻³⁾ can be expressed as linear combinations of the basis elements of the Lie algebra B_2 . These matrices were given explicitly in the case of the spin $3/2$ wave equation based on the 16 and 20 dimensional representations of the group $SO(4,1)$. By means of L_0 the hermitianizing matrix A was constructed using the formula⁷⁾

$$A = \frac{1}{3} L_0 \{4(L_0)^2 - 7\}. \quad (1)$$

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It was also shown that the matrix L_0 of the 20 dimensional field has the eigenvalues $3/2$ (four fold), $-3/2$ (four fold), $1/2$ (six fold), $-1/2$ (six fold) and likewise the matrix L_0 of the 16 dimensional field has the eigenvalues $3/2$ (two fold), $-3/2$ (two fold), $1/2$ (six fold) and $-1/2$ (six fold). Using these results in this paper we shall study the charge associated with these equations looking upon them as Gel'fand-Yaglom wave-equations⁸⁻¹¹⁾ and then by using a simplified expression of the hermitianizing matrix we shall be able to study the charge of the Bhabha fields in the general case for any half integer spin.

With every Bhabha wave-equation the quantity ρ can be associated known as the charge density and given by the formula

$$\rho = \Psi^\dagger A L_0 \Psi \tag{2}$$

where $\Psi = (\Psi_1, \Psi_2, \dots \Psi_n)$ is a vector with same number of components as the dimension of the representation on which the wave-equation is based and Ψ^\dagger is its hermitian conjugate. The total charge then is $Q = \int \rho \, du$ where du is the infinitesimal volume element.

2. Charge of the Bhabha fields based on the 16 and 20 dimensional representations of the group $SO(4,1)$

In this section we consider the charge of the Bhabha wave-equation for which the wave function Ψ transforms according to the 16 dimensional representation of the group $SO(4, 1)$. This representation in the notation of Gel'fand and Yaglom breaks into the irreducibles $\tau_1 \sim (1/2, 3/2)$, $\check{\tau}_1 \sim (-1/2, 3/2)$, $\tau_2 \sim (1/2, 5/2)$, $\check{\tau}_2 \sim (-1/2, 5/2)$ interlocking according to the scheme

$$\begin{array}{ccc}
 & \xrightarrow{C^{\tau_1 \check{\tau}_1}} & \\
 (1/2, 3/2) \sim \tau_1 & & \check{\tau}_1 \sim (-1/2, 3/2) \\
 & \xleftarrow{C^{\check{\tau}_1 \tau_1}} & \\
 \uparrow & \begin{array}{cc} C^{\tau_1 \tau_2} & C^{\tau_2 \check{\tau}_1} \\ C^{\tau_1 \check{\tau}_2} & C^{\tau_2 \tau_2} \end{array} & \uparrow \\
 (1/2, 5/2) \sim \tau_2 & & \check{\tau}_2 \sim (-1/2, 5/2) \\
 & \xleftarrow{C^{\check{\tau}_2 \tau_2}} &
 \end{array} \tag{3}$$

and accepts the following canonical basis

$$\{\xi_{lm}\} = \{\xi_{l_1, m_1}^{\tau_1}, \xi_{l_1, m_1}^{\check{\tau}_1}, \xi_{l_1, m_1}^{\tau_2}, \xi_{l_2, m_2}^{\tau_2}, \xi_{l_1, m_1}^{\check{\tau}_2}, \xi_{l_2, m_2}^{\check{\tau}_2}\}$$

$$l_1 = 1/2, \quad m_1 = 1/2, -1/2, \quad l_2 = 3/2, \quad m_2 = 3/2, 1/2, -1/2, -3/2. \tag{4}$$

With respect to this basis L_0 has the blocks

$$L_0^{1/2} = \begin{matrix} & \begin{matrix} \tau_1 & \dot{\tau}_1 & \tau_2 & \dot{\tau}_2 \end{matrix} \\ \begin{matrix} \tau_1 \\ \dot{\tau}_1 \\ \tau_2 \\ \dot{\tau}_2 \end{matrix} & \begin{bmatrix} 0 & C^{\tau_1 \dot{\tau}_1} & C^{\tau_1 \tau_2} & 0 \\ C^{\dot{\tau}_1 \tau_1} & 0 & 0 & C^{\dot{\tau}_1 \dot{\tau}_2} \\ C^{\tau_2 \tau_1} & 0 & 0 & C^{\tau_2 \dot{\tau}_2} \\ 0 & C^{\dot{\tau}_2 \tau_1} & C^{\dot{\tau}_2 \tau_2} & 0 \end{bmatrix} \end{matrix}, \quad L_0^{3/2} = \begin{matrix} & \tau_2 & \dot{\tau}_2 \\ \tau_2 & 0 & 2C^{\tau_2 \dot{\tau}_2} \\ \dot{\tau}_2 & 2C^{\dot{\tau}_2 \tau_2} & 0 \end{matrix}. \quad (5)$$

Since the matrix L_0 of every Bhabha wave equation is diagonalizable this implies that the minimal equation for L_0 includes each eigenvalue only once with multiplicity one. For the 16 dimensional Bhabha wave-equation L_0 has the eigenvalues $\pm 1/2$ and $\pm 3/2$ with the multiplicities given in Section 1 and the minimal equation is of the form

$$m(L_0) = \{(L_0)^2 - (1/2)^2\} \{(L_0)^2 - (3/2)^2\} = 0. \quad (6)$$

This translated into the minimal equation of the two blocks of L_0 gives rise to the following possible cases

- (i) $m(L_0^{1/2}) = (L_0^{1/2})^2 - (1/2)^2 = 0, \quad m(L_0^{3/2}) = (L_0^{3/2})^2 - (3/2)^2 = 0,$
- (ii) $m(L_0^{1/2}) = \{(L_0^{1/2})^2 - (3/2)^2\} \{(L_0^{1/2})^2 - (1/2)^2\} = 0, \quad m(L_0^{3/2}) = (L_0^{3/2})^2 - (3/2)^2 = 0,$
- (iii) $m(L_0^{1/2}) = (L_0^{1/2})^2 - (3/2)^2 = 0, \quad m(L_0^{3/2}) = (L_0^{3/2})^2 - (1/2)^2 = 0,$
- (iv) $m(L_0^{1/2}) = \{(L_0^{1/2})^2 - (3/2)^2\} \{(L_0^{1/2})^2 - (1/2)^2\} = 0, \quad m(L_0^{3/2}) = (L_0^{3/2})^2 - (1/2)^2 = 0.$

Cases (i), (ii) and (iii) have to be discarded because they require that the eigenvalues of L_0 appear with incorrect multiplicities. Case (iv) is the right one. (We recall that an eigenvalue λ corresponding to the block L_0^l is a $2l + 1$ fold eigenvalue.) Thus the eigenvalue $\pm 3/2$ and $\pm 1/2$ appear simultaneously as eigenvalues of the block $L_0^{1/2}$ and $\pm 1/2$ as the only kind of eigenvalues of the block $L_0^{3/2}$. Since L_0 is a diagonalizable matrix then there is a frame not the same as $\{\xi_{lm}\}$ in which it acquires a diagonal form and hence

$$L_0^{1/2} = \text{diag} \{3/2, -3/2, 1/2, -1/2\}, \quad L_0^{3/2} = \text{diag} \{1/2, -1/2\}. \quad (7)$$

Using (1) we find that the hermitianizing matrix corresponding to (7) has the diagonal blocks

$$A_{1/2} = \text{diag} \{1, -1, -1, 1\}, \quad A_{3/2} = \text{diag} \{-1, 1\}. \quad (8)$$

From (2) we find the following individual charge densities $\rho_{1/2}^1 = \rho_{1/2}^2 = 3/2$, $\rho_{1/2}^3 = \rho_{1/2}^4 = -1/2$ for the state $l = 1/2$ and $\rho_{3/2}^1 = \rho_{3/2}^2 = -1/2$ for the state $l = 3/2$, respectively. These densities give an indefinite charge for the 16 dimensional Bhabha field. This is so because the charge densities do not have all the same sign.

Let us now consider the Bhabha wave-equation for which the wave function Ψ transforms according to the 20 dimensional representation of the group $SO(4,1)$. This representation brakes into the irreducibles

$$\tau_1 \sim (3/2, 5/2), \dot{\tau}_1 \sim (-3/2, 5/2), \tau_2 \sim (1/2, 5/2), \dot{\tau}_2 \sim (-1/2, 5/2) \tag{9}$$

interlocking according to the scheme

$$\tau_1 \sim (3/2, 5/2) \begin{array}{c} \xrightarrow{C^{\tau_1 \tau_2}} \\ \xleftarrow{C^{\tau_2 \tau_1}} \end{array} \tau_2 \sim (1/2, 5/2) \begin{array}{c} \xrightarrow{C^{\tau_2 \dot{\tau}_2}} \\ \xleftarrow{C^{\dot{\tau}_2 \tau_2}} \end{array} \dot{\tau}_2 \sim (-1/2, 5/2) \begin{array}{c} \xrightarrow{C^{\dot{\tau}_2 \dot{\tau}_1}} \\ \xleftarrow{C^{\dot{\tau}_1 \dot{\tau}_2}} \end{array} \dot{\tau}_1 \sim (-3/2, 5/2) \tag{10}$$

and accepts the following canonical basis

$$\{\xi_{lm}^{\dot{\tau}}\} = \{\xi_{12,m_2}^{\tau_1}, \xi_{12,m_2}^{\dot{\tau}_1}, \xi_{11,m_1}^{\tau_2}, \xi_{12,m_2}^{\dot{\tau}_2}, \xi_{11,m_1}^{\dot{\tau}_2}, \xi_{12,m_2}^{\tau_1}\} \tag{11}$$

$$l_1 = 1/2, \quad m_1 = 1/2, -1/2, \quad l_2 = 3/2, \quad m_2 = 3/2, 1/2, -1/2, -3/2.$$

With respect to this basis L_0 has the blocks

$$L_0^{3/2} = \begin{array}{c} \tau_1 \\ \tau_2 \\ \dot{\tau}_2 \\ \tau_1 \end{array} \left[\begin{array}{c|c|c|c} \tau_1 & \tau_2 & \dot{\tau}_2 & \dot{\tau}_1 \\ \hline 0 & \sqrt{3}C^{\tau_1 \tau_2} & 0 & 0 \\ \hline \sqrt{3}C^{\tau_2 \tau_1} & 0 & 2C^{\tau_2 \dot{\tau}_2} & 0 \\ \hline 0 & 2C^{\dot{\tau}_2 \tau_2} & 0 & \sqrt{3}C^{\dot{\tau}_2 \dot{\tau}_1} \\ \hline 0 & 0 & \sqrt{3}C^{\dot{\tau}_1 \tau_2} & 0 \end{array} \right], \quad L_0^{1/2} = \begin{array}{c} \tau_2 \\ \dot{\tau}_2 \end{array} \begin{pmatrix} \tau_2 & \dot{\tau}_2 \\ 0 & C^{\tau_2 \dot{\tau}_2} \\ \dot{\tau}_2 & C^{\dot{\tau}_2 \tau_2} \\ & 0 \end{pmatrix}. \tag{12}$$

Its eigenvalues are $\pm 3/2$ and $\pm 1/2$ with the multiplicities given in Section 1. Since L_0 is diagonalizable it has the following minimal equation

$$m(L_0) = \{(L_0)^2 - (1/2)^2\} \{(L_0)^2 - (3/2)^2\} = 0. \tag{13}$$

This expressed in terms of the minimal equations of the two blocks gives rise to the following possible cases

$$(a) \quad m(L_0^{3/2}) = \{(L_0^{3/2})^2 - (1/2)^2\} = 0, \quad m(L_0^{1/2}) = \{(L_0^{1/2})^2 - (3/2)^2\} = 0,$$

$$(b) \ m(L_0^{3/2}) = \{(L_0^{3/2})^2 - (3/2)^2\} \{(L_0^{3/2})^2 - (1/2)^2\} = 0, \quad m(L_0^{1/2}) = \{(L_0^{1/2})^2 - (3/2)^2\} = 0,$$

$$(c) \ m(L_0^{3/2}) = \{(L_0^{3/2})^2 - (3/2)^2\} = 0, \quad m(L_0^{1/2}) = \{(L_0^{1/2})^2 - (1/2)^2\} = 0,$$

$$(d) \ m(L_0^{3/2}) = \{(L_0^{3/2})^2 - (3/2)^2\} \{(L_0^{3/2})^2 - (1/2)^2\} = 0, \quad m(L_0^{1/2}) = \{(L_0^{1/2})^2 - (1/2)^2\} = 0.$$

Of these cases (a), (b) and (c) must be discarded because they require that the eigenvalues of L_0 appear with incorrect multiplicities. Case (d) is the right one and thus the eigenvalues $\pm 3/2$ and $\pm 1/2$ appear simultaneously as eigenvalues of the block $L_0^{3/2}$ and $\pm 1/2$ as the only kind of eigenvalues of the block $L_0^{1/2}$. L_0 in the diagonalizing frame has the blocks

$$L_0^{3/2} = \text{diag} \{3/2, -3/2, 1/2, -1/2\}, \quad L_0^{1/2} = \text{diag} \{1/2, -1/2\} \quad (14)$$

and the hermitianizing matrix A the blocks

$$A_{3/2} = \text{diag} \{1, -1, -1, 1\}, \quad A_{1/2} = \text{diag} \{-1, 1\}. \quad (15)$$

For the charge densities we find $\rho_{3/2}^1 = \rho_{3/2}^2 = 3/2$, $\rho_{3/2}^3 = \rho_{3/2}^4 = -1/2$ for the state $l = 3/2$ and $\rho_{1/2}^1 = \rho_{1/2}^2 = -1/2$ for the state $l = 1/2$. These densities give again an indefinite charge for the 20 dimensional Bhabha field.

Working in the same way as above we have studied the 56, 64 and 40 dimensional representations of the group $SO(4, 1)$, which are suitable for the description of the spin 5/2 Bhabha fields and also the 120, 160, 140 and 80 dimensional representations of $SO(4, 1)$ which are suitable for the description of the spin 7/2 fields and we have found that the charge of all these fields is again indefinite.

3. General case

We consider now in this section the general case for any half integer spin S . In the case of the Bhabha fields for any spin S the matrix L_0 is diagonalizable and has eigenvalues $\pm S, \pm(S-1), \pm(S-2) \dots$ with multiplicities > 1 . The matrix L_0 then in the diagonalizing frame has the form

$$L_0 = \text{diag} \{S, -S, (S-1), -(S-1), (S-2), -(S-2), \dots\}. \quad (16)$$

It can be shown that the corresponding hermitianizing matrix A can be written also in a diagonal form with elements on the main diagonal given by the formula

$$f(x, S) = (-1)^{S-P-1/2} \quad (17)$$

where S is the maximum value of the spin described by the field, x is a variable taking values $S, S-1, \dots, -(S-1), -S$ and P is another variable related to

x by the formula $x = P + 1/2$. Using (17) we find that A has the following eigenvalues:

$$\text{for } x = S, 1; \text{ for } x = -S, -1; \text{ for } x = S - 1, -1;$$

$$\text{for } x = -S + 1, 1; \text{ for } x = S - 2, 1; \text{ for } x = -S + 2, -1,$$

and so on. Consequently

$$A = \text{diag} \{1, -1, -1, 1, 1, -1, \dots\} \quad (18)$$

and hence

$$AL_0 = \text{diag} \{S, S, -(S - 1), -(S - 1), (S - 2), (S - 2), \dots\}. \quad (19)$$

Using (2) we find for the charge density

$$\rho = S(\Psi_1^* \Psi_1 + \Psi_2^* \Psi_2) - (S - 1)(\Psi_3^* \Psi_3 + \Psi_4^* \Psi_4) + (S - 2)(\Psi_5^* \Psi_5 + \Psi_6^* \Psi_6) + \dots \quad (20)$$

which gives an indefinite charge for every $S > 3/2$. For $S = 1/2$ the charge is definite.

4. Conclusions

In this paper we have seen that all the Bhabha wave-equations for half integer spin based on the group $SO(4,1)$ have indefinite charge. The only exception is the Dirac wave-equation.

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BHABHINE VALNE JEDNADŽBE ZA POLUCIJELI SPIN STUDIRANE
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Autor izvodi za hermitizirajuću matricu izraz $f(x, S) = (-1)^{S-P-1/2}$, koji je jednostavniji od izraza Madavaraoa i suradnika. Na taj način pokazuje se da sve Bhabha valne jednačbe za polucijeli spin $> 3/2$, koje se zasnivaju na grupi $SO(4,1)$ nemaju definitivan predznak naboja.