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LETTER TO THE EDITOR

QUARK CONFINEMENT POTENTIAL AND REGGE TRAJECTORY

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We derive an approximate analytical formula for the energy levels for the Schrödinger equation (with linear potential) which is valid for all values of the angular momentum. We also discuss the effect on the energy levels of the inclusion of a Coulomb potential.

After the striking discovery$^{1,2)}$ of the J/Ψ' and Ψ' resonances produced in e^+e^- collisions at 3095 and 3685 MeV, respectively, there have been many attempts to explain the spectra of the new particles. One standard model is that of a linearly binding potential$^{3,4,5)}$ for quark confinement. The linear potential model is in good agreement with the data. Some time ago Bose et al.$^{5)}$ discussed quark confinement potential in a paper in which they also derived an analytical expression for the eigenvalues for a linear potential. Though their results for $l \neq 0$ are in reasonable agreement with the exact numerical results$^{6)}$, the zeros of Airy functions predicted by the formula differ considerably from the exact numerical values for large n. In the present note we shall derive a formula for the eigenvalues of the Schrödinger wave function in the framework of the linear potential model. Our formula will be valid for all $l > 0$ and this would easily give the Regge trajectory for the linear potential.

Since the linear potential model has been discussed extensively in the literature we shall not go into details but start right away from the Schrödinger equation with a potential given by $V = kr$. The radial Schrödinger equation in a centre of mass frame is

$$\left[-\frac{1}{m}\frac{d^2}{\mathrm{d}r^2} + V - E + \frac{l(l+1)}{mr^2}\right]\varphi(r) = 0 \tag{1}$$

where $\varphi(r) = rR(r)$ and m is the mass of a constituent (binding particle).

If we take $V(r) = kr$ then (1) becomes

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2} + \left[mE - mkr - \frac{l(l+1)}{r^2}\right]\varphi(r) = 0. \tag{2}$$

For $l = 0$ as is well known, equation (2) is exactly solvable and its eigenvalues are given by

$$E_n = \left(\frac{k^2}{m}\right)^{1/3} a_n \tag{3}$$

where a_n are the negative zeros of the Airy function[7] $Ai(-x)$.

For $l \neq 0$, Eq. (2) has to be solved numerically. An asymptotic solution for $l \neq 0$, however, has been given by Müller[3] who dealt with a generalised potential $V(r) = \sum_{i=-1}^{\infty} M_{i+1} r^i$. But unfortunately Müller's formula fails to reproduce the eigenvalues (3) for the particular case $l = 0$, $M_i = k\,\delta_{i1}$ i. e.

$$V(r) = kr. \tag{4}$$

In fact we have calculated, by Müller's prescription $E_{n,l}$ for the particular case (4); the result is

$$E_{n,l} = \frac{(k^2/m)^{1/3}}{8}\left\{\frac{-[4(n+l)^2 + n(2l+n+1) + (2l+n+2)(n+1)]}{(l+n)^{2/3}}\right\}. \tag{5}$$

It can easily be seen that (5) fails to reproduce (3) for $l = 0$ even if we take the complex zeros of the Airy function for a_n. This is because, as will be shown below, the procedure followed by Müller may not given unique values for $E_{n,l}$.

We derive a formula for the eigenvalues, modifying the procedure suggested by Müller, which will reproduce the result given by (3) for the particular case $l = 0$. Let us start with equation (2). Putting $x = \sqrt{m\,E}\,r$ equation (2) reduces to

$$x^2\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + [x^2 - cx^3 - l(1+l)]\,\varphi(x) = 0 \tag{6}$$

where

$$c = \frac{m\,k}{(mE)^{3/2}}. \tag{7}$$

Taking $\varphi(x) = e^{-\alpha x} x^{l+1} w(x)$, equation (6) reduces to ($\lambda = l + 1$)

$$xw'' + (2\,\lambda - 2\,a\,x)\,w'\,[x(1 + a^2) - cx^2 - 2\,a\lambda]\,w = 0. \tag{8}$$

The substitution

$$x = \beta\,z, \quad \beta = \frac{1}{2a} \tag{9}$$

changes Eq. (8) to

$$z\frac{d^2 w}{dz^2} + (2\lambda - z)\frac{dw}{dz} + [\beta^2 z(1 + a^2) - c\,\beta^3 z^2 - \lambda]\,w = 0. \tag{10}$$

Eq. (10) can be written as

$$z\frac{d^2 w}{dz^2} + (2\lambda - z)\frac{dw}{dz} - aw = [\lambda - a + c\,\beta^3 z^2 - \beta^2 z(1 + a^2)]\,w. \tag{11}$$

Now if we consider λ to be large (in fact we seek a formula for $l > 1$ or $\lambda > 2$) compared with the term on the right hand side of (11), then we can find out the eigenvalues of Eq. (10) by using the following method of perturbation.

When the right hand side is zero the solution of w is $M(a, 2\lambda, z)$ and the eigenfunctions are obtained by insisting that the function φ in (6) be square integrable. Here the condition is satisfied if $M(a, 2\lambda, z)$ becomes a polynomial, which happens when $a = -n$, $n = 0, 1, 2, \ldots$

Now to get a solution of (10), we expand w in terms of eigenfunctions of equation (11), with right hand side equal to zero, i. e. we write

$$w(z) = \sum_{m=-\infty}^{\infty} A_m\,\varphi_{n+m}(z) \tag{12}$$

where $\varphi_n(z)$ are normalized eigenfunctions defined by

$$\varphi_n(z) = e^{-\frac{z}{2}}\,x^{\lambda - \frac{1}{2}}\sqrt{\frac{\Gamma(2\,\lambda + n)}{n}}\,M(-n, 2\,\lambda, z). \tag{13}$$

If we put (12) into (11) we get (with $a \doteq \lambda$ for $c \doteq 0$)

$$\sum_m (a - \lambda + m)\cdot A_m\,\varphi_{n+m} = [c\,\beta^3 z^2 - \beta^2 z(1 + a^2)]\sum_m A_m\,\varphi_{n+m}. \tag{14}$$

The first approximation $w = \varphi_n$ leaves uncompensated terms equal to

$$L_n = \sum_m \{[a - \lambda + m) - (c\,\beta^3 z^2 - \beta^2 z\,(1 + \alpha^2)]\}\, A_m\, \varphi_{n+m}. \tag{15}$$

To the first order the eigenvalues are obtained by setting equal to zero the sum of the terms in φ_n in (15). For this, we must find out the contributions of $ZM\,(a, b, z)$ and $Z^2M\,(a, b, z)$ to $M\,(a, b, z)$ using the following recurrence relations for the Kummer's function.

$$ZM\,(a, b, z) = (b - 2a)\,M\,(a, b, z) + aM\,(a + 1, b, z) + (a - b)\,M\,(a - 1, b, z) \ldots \tag{16}$$

and

$$\begin{aligned} z^2M\,(a, b, z) &= a\,(a + 1)\,M\,(a + 2, b, z) - a\,(4a - 2b + 2)\,M\,(a + 1, b, z) + \\ &+ [(2a - b)^2 - (b - a)\,(a - 1) - a\,(b - a - 1)]\,M\,(a, b, z) + (b - a)\,(b - \\ &- a + 1)\,M\,(a - 2, b, z) + (b - a)\,(4a - 2b - 2)\,M\,(a - 1, b, z). \end{aligned} \tag{17}$$

We get

$$c\,\beta^3\,X = (\lambda + n)\,[2\,\beta^2\,(1 + \alpha^2) - 1] \tag{18}$$

where

$$X = [(2\,\lambda + 2n)^2 + (2\,\lambda + n)\,(n + 1) + n\,(2\,\lambda + n - 1)] \quad n = 0, 1, 2, \ldots \tag{19}$$

Using the value of c given by (7) we obtain

$$E^3_{n,l} = k_0^3 \frac{\beta^6}{\left(2\beta^2 - \frac{1}{2}\right)^2} \left[\frac{4\,(\lambda + n)^2 + (2\lambda + n)\,(n + 1) + n\,(2\lambda + n - 1)}{\lambda + n}\right]^2 \qquad n = 0, 1, 2, \ldots \tag{20}$$

where $k_0 = \left(\frac{k^2}{m}\right)^{1/3}$.

Now β is chosen in such a way that $E_{n,l}$ is a minimum, the condition for which is $\frac{\partial E}{\partial \beta} = 0$. From equation (20) we obtain $\beta = \frac{\sqrt{3}}{2}$.

For $l = 0$, Eq. (20) reduces to

$$E_{n,0} = k_0 \left(\frac{\beta^3}{2\beta^2 - \frac{1}{2}}\right)^{2/3} \left[\frac{3}{2}\,(4n)\right]^{2/3} \qquad n = 1, 2, 3, \ldots \tag{21}$$

The values of $E_{n,0}/k_0$ are given in Table 1, and are compared with a_n and the values obtained by Bose et al.[5].

TABLE 1.

n	E/k_0 (from (21))	a_n	E/k_0 (obtained in Ref. 5)
1	2.43	2.34	2.36
2	4.03	4.10	4.17
3	5.38	5.52	5.86
4	6.55	6.79	7.28
5	7.62	7.94	8.59

a_n is the nth zero of the Airy function $Ai\,(-x)$.

The general formula for $E_{n,l}$ for $n \geqslant 1$ may be written as

$$E_{n,l} = \frac{3}{4} k_0 \left[\frac{4(\lambda + n)^2 + (2 + n)(n + 1) + n(2 + n - 1)}{\lambda + n} \right]^{2/3} . \tag{22}$$

In the second order approximation Eq. (18) gets modified into

$$\lambda + n = \frac{c\,\beta^3 X}{[2\beta^2 (1 + a^2) - 1]} + E_n^{(2)} \tag{23}$$

where:

$$\begin{aligned} E_n^{(2)} = &- \frac{n+1}{(2\lambda + n)} \frac{1}{E_{n+1}^0 - E_n^0} [c\,\beta^3 (2\lambda + n)(-4n - 4\lambda - 2) - \\ &\qquad - \beta^2 (1 + a^2)(-n - 2\lambda)]^2 \\ &- \frac{(n+2)(n+1)}{(2\lambda + n + 1)(2\lambda + n)} \frac{1}{E_{n+2}^0 - E_n^0} [c\,\beta^3 (2\lambda + n)(2\lambda + n + 1)]^2 \\ &+ \frac{2\lambda + n - 1}{n} \frac{1}{E_n^0 - E_{n-1}^0} [c\,\beta^3 n(-4n - 4\lambda + 2) + n\,\beta^2 (1 + a^2)]^2 \\ &+ \frac{(2\lambda + n - 1)(2\lambda + n - 2)}{n(n-1)} \frac{1}{E_n^0 - E_{n-2}^0} [c\,\beta^3 (-n)(-n + 1)]^2 \end{aligned} \tag{24}$$

where $E_n^0 = \lambda + n$.

We present the values of $E_{n,l}$ for $n = 2 - 5$ and $l = 1, 2, 3$ (taking always formula (23)), in order to compare our results with those obtained hitherto by numerical methods. Table 2 shows our result with that of Gunion and Willey[3] given within parentheses, with $k_0 = 0.2743$.

TABLE 2.

n	$l = 1$	$l = 2$	$l = 3$
2	1.36 (1.34)	1.57 (1.54)	1.75 (1.73)
3	1.72 (1.70)	1.92 (1.89)	2.10 (2.05)
4	2.04 (2.03)	2.23 (2.20)	2.41 (2.35)
5	2.32 (2.33)	2.52 (2.49)	2.69 (2.64)

If we modify our potential so as to include a Coulomb part $\left(\text{e. g. } V(r) = kr - \frac{\alpha_s}{r}\right)$ then we get (α_s assumed to be the same as in Ref. 8)

$$E_{n,l} = \frac{3k_0}{4} \left[\frac{4(\lambda+n)^2 + (2\lambda+n)(n+1) + n(2\lambda+n-1)}{\lambda + n + \alpha_s \beta/(2\beta^2 - 1/2)\sqrt{mE}} \right]^{2/3}. \qquad (25)$$

The energy shift owing to the presence of the Coulomb term is 0.2 GeV if we take m corresponding to 1.2 GeV.

We must, however, warn that formula (25) is valid only if we take the Coulomb term as a perturbation term on the linear potential. This is where we basically differ from Müller who treated the potentials other than the Coulomb one as perturbation terms. In fact, if we take $k_0 = 0$ and $\beta = \frac{i}{2}$ we get back the well known formula for a Coulomb potential from equation (25).

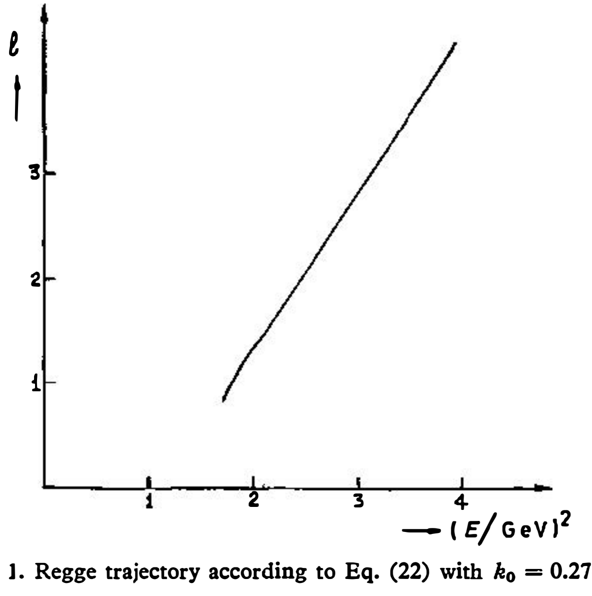

Fig. 1. Regge trajectory according to Eq. (22) with $k_0 = 0.2743$.

From (22) one can obtain the Regge trajectory i. e. l as a function of E^2. We just give the result for $n = 1$

$$l = -2.75 + 1.34\,E^{3/2} \pm \sqrt{1.56 - 2.01\,E^{3/2} + 1.79\,E^{3}}. \qquad (26)$$

In Fig. 1 we plot l against E^2. It is seen that though Eq. (26) is nonlinear in E^2, the trajectory is approximately linear for $l < 3$. Thus, we obtained an analytical formula for the energy eigenvalues which agree within about 2% with the previous numerical results.

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POTENCIJAL ZA SUŽANJSTVO KVARKA I REGGEOVA TRAJEKTORIJA

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Izvedena je aproksimativna analitička formula za dobivanje energetskih nivoa u Schrödingerovoj jednadžbi s linearnim potencijalom i proizvoljnom vrijednošću kutne količine gibanja.