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THE OPERATOR METHOD AND EIGENVALUES OF QUARTIC AND SEXTIC ANHARMONIC OSCILLATORS

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Using the operator method proposed by Feranchuk and Komarov, we have calculated the eigenvalues of quartic and sextic anharmonic oscillator for all ranges of λ and n. Approximate analytical formulas up to fourth order are given for the particular cases $n = 0$ and $n = 1$.

1. *Introduction*

Recently there has been a renewed interest in the problem of anharmonic oscillators of the type $\frac{1}{2} x^2 + \lambda x^{2m}$. There is an extensive literature on anharmonic oscillators[1-3], but until recently numerical results for all ranges of λ and n were not calculated for anharmonic oscillators for which $m > 2$. Very recently Feranchuk and Komarov[4] have found an approximate analytical formula for the anharmonic oscillator of the type given by $V = \frac{x^2}{2} + \lambda x^4$ using the operator method. We used this technique to calculate the eigenvalues of the sextic anharmonic oscillators for all values of n and λ. Since large λ region is the really interesting one, we used the operator method technique to calculate eigenvalues of the anharmonic oscillator of the type λx^4 in the large λ region, taking into account second and fourth order corrections. The results are in excellent agreement with the numerical calculations of Hioe et al.[5] and Biswas et al.[3]. Encouraged by this we derived an approximate analytical formula for the eigenvalues of the sextic anharmonic oscillator. We checked, wherever possible, our results with those of Hioe et al.[6].

2. *Sextic anharmonic oscillator*

Since the operator method for λx^4 was given in detail by Feranchuk and Komarov[4] we start here straight away with the Hamiltonian for the sextic anharmonic oscillator. (We take $\hbar = c = 1$ and also mass = 1.)

We have

$$H = \frac{1}{2}(p^2 + x^2) + \lambda x^6. \tag{1}$$

x and p can be written in terms of the creation and annihilation operators in the following way

$$x = \frac{1}{\sqrt{2\omega}}(a + a^+) \qquad p = \mathrm{i}\sqrt{\frac{\omega}{2}}(a^+ - a) \tag{2}$$

where ω is a sort of scale parameter to be determined later on. From (1) and (2) we obtain

$$H = \frac{\omega}{4}[1 - (a^+)^2 + 2a^+ a - a^2] + \frac{1}{4\omega}[1 + (a^+)^2 + 2a^+ a + a^2] +$$
$$+ \frac{\lambda}{8\omega^3}[3 + 6(a^+)^2 + 12a^+ a + 6a^2 + (a^+)^4 + 4(a^+)^3 a + 6(a^+) a^2 +$$
$$+ 4a^+ a^3 + a^4][1 + a^2 + (a^+)^2 + 2a^+ a]. \tag{3}$$

We write H in the following way

$$H = H_0 + H \tag{4}$$

where H_0 contains all the terms of (3) that commute with $a^+ a$ viz.

$$H_0 = \frac{\omega}{4} + \frac{1}{4\omega} + \frac{3\lambda}{8\omega^3} + a^+ a\left[\frac{\omega}{2} + \frac{1}{2\omega} + \frac{12\lambda}{8\omega^3} + \frac{6\lambda}{8\omega^3}\right] + \left[\frac{12\lambda}{\omega^3}(a^+)^2 a^2 +\right.$$
$$+ \frac{4\lambda}{8\omega^3}(a^+)^3 a^3 + \frac{6\lambda}{8\omega^3} a^2 (a^+)^2 + \frac{24\lambda}{8\omega^3}(a^+ a)^2 + \frac{12\lambda}{8\omega^3}(a^+)^2 a^2 a^+ a +$$
$$\left. + \frac{4\lambda}{8\omega^3} a^+ a^3 (a^+)^2\right]. \tag{5}$$

To the lowest order eigenvalues are given by $\langle n|H_0|n\rangle$. Using the results $\langle n|a^+a|n\rangle = n$ and $[a, a^+] = 1$ we can easily calculate $\langle n|H_0|n\rangle$ from (5). We have

$$E_n^{(0)} = \langle n|H_0|n\rangle = \frac{\omega}{4} + \frac{1}{4\omega} + \frac{3\lambda}{8\omega^3} + n\left(\frac{\omega}{2} + \frac{1}{2\omega} + \frac{18\lambda}{8\omega^3}\right) +$$

$$+ \frac{12\lambda}{8\omega^3}(n^2 - n) + \frac{4\lambda}{8\omega^3}(n^3 - 3n^2 + 2n) + \frac{6\lambda}{8\omega^3}(2 + 3n + n^2) + \frac{24\lambda}{8\omega^3}n^2 +$$

$$+ \frac{12\lambda}{8\omega^3}(n^3 - n^2) + \frac{4\lambda}{8\omega^3}n(n + 1)(n + 2) \tag{6}$$

where ω is to be determined from the condition $\frac{\partial E_n^0}{\partial \omega_n} = 0$ which is the necessary condition to make $E_n^0(\omega_n)$ a minimum. From (6) and the condition $\frac{\partial E_n}{\partial \omega} = 0$, ω is given by the following equation

$$\omega^4(2n + 1) - \omega^2(2n + 1) - \frac{3\lambda}{2}(20n^3 + 30n^2 + 40n + 15) = 0. \tag{7}$$

Equations (6) and (7) together give the eigenvalues for all ranges of λ and n, to the lowest order. The second order correction (the first order and all odd order corrections will vanish trivially) is given by

$$E_n^{(2)} = \langle n|H'|E_n^{(0)} - H_n^{(0)}|^{-1} H'|n\rangle. \tag{8}$$

We calculated $E_n^{(2)}$ and also the fourth order corrections for $n = 0$. We have also calculated the eigenvalues for λx^4 anharmonic oscillator for which the lowest order eigenvalue formula is given in Ref. 4. The second and fourth order terms for λx^4 are given below

$$E_0^{(2)} = -3\lambda^2/4\omega_0^2(2\omega_0 + 21\lambda) \tag{9}$$

$$E_0^{(4)} = \frac{3\lambda^4}{32\omega_0^4(2\omega_0 + 21\lambda)^2}\left[\frac{3}{2\omega_0 + 21\lambda} - \frac{105}{2\omega_0 + 33\lambda} - \right.$$

$$\left. - \frac{640}{2\omega_0 + 27\lambda} \frac{192}{2\omega_0 + 15\lambda}\right] \tag{10}$$

where

$$\omega_0 = [(1 + \sqrt{90\lambda + 1})/2]^{1/2}. \tag{11}$$

(Note that there is a misprint in the formula for $E_n^{(2)}$ in Ref. 4).

Also

$$E_1^{(2)} = \frac{-15\lambda^2}{4\omega_1^2(2\omega_1 + 31\lambda)} \tag{12}$$

$$E_1^{(4)} = \frac{60\lambda^4}{256\omega_1^2(2\omega_1 + 31\lambda)^2}\left[\frac{30}{2\omega + 31\lambda} - \frac{378}{2\omega_1 + 43\lambda} - \frac{1792}{2\omega_1 + 37\lambda} - \frac{640}{2\omega_1 + 25\lambda}\right] \tag{13}$$

where

$$\omega_1 = [(1 + \sqrt{210\lambda + 1})/2]^{1/2}. \tag{14}$$

In Table 1 the eigenvalues for λx^4 anharmonic oscillator are given for $n = 0, 1, 2$ and values of λ ranging from 0.001 to 20000. The explicit expressions for second and fourth order corrections for λx^6 anharmonic oscillator are not given here as these are extremely lengthy but explicit expressions can be obtained from authors on request. Table 2 contains the eigenvalues for λx^6 anharmonic oscillator.

TABLE 1

λ	E_0	E_0^*	E_1	E_1^*
0.001	0.500746	0.500747	1.503729	1.503728
0.1	0.559142	0.559146	1.769486	1.76950
1.0	0.803747	0.803771	2.737699	2.737889
10.0	1.505054	1.504970	5.321063	5.32161
2000.0	3.931352	3.93093	14.057741	14.05923
8000.0	13.364396	13.36691	47.88544	47.89077
20000.0	18.13918	18.13723	64.98131	64.98668

The star denotes the eigenvalues given by Hioe and Montroll[5].

TABLE 2

λ	E_0	E_0^*	$E_1^\dagger$	E_1^*	$E_2^\dagger$	E_2^*
0.001	0.501829	0.501825	1.51251	1.51248	2.54347	2.54307
0.01	0.515192	0.515443	1.59589	1.59544	2.79984	2.79382
0.1	0.587209	0.586945	1.95139	1.95042	3.71012	3.69082
1.0	0.806851	0.804966	2.87755	2.87467	5.80729	5.77197
10.0	1.28805	1.28232	4.76283	4.75605	9.86747	9.80691
100.0	2.20538	2.19334	8.26644	8.25314	17.2859	17.1803
200.0	2.60882	2.59418	9.79758	9.78157	20.5144	20.3891
8000.0	6.48991	6.45138	24.4711	24.4297	51.3750	51.0619
20000.0	8.15443	8.10586	30.7562	30.7039	64.5820	64.1884

* — The eigenvalues given by Hioe et al.[6].

† — results obtained without second and higher order corrections.

3. *Conclusion*

The operator method was found to be an easy and straightforward way to derive approximate analytical formulas for the eigenvalues of anharmonic oscillator. To obtain better numerical accuracy one has to compute higher order corrections, but even to the lowest order the results obtained are within about 1% accuracy for all ranges of n and λ. The computer time needed to calculate eigenvalues for any n and λ is negligible compared with that required in standard numerical methods.

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METODA OPERATORA I VLASTITE VRIJEDNOSTI ANHARMONIČKOG OSCILATORA ČETVRTOG I ŠESTOG REDA

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Koristeći metodu operatora, koju su predložili Feranchuk i Komarov, izračunate su vlastite vrijednosti anharmoničkog oscilatora četvrtog i šestog reda za cijelo područje vrijednosti λ i n. Dana je približna analitička formula za slučajeve $n = 0$ i $n = 1$.