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ION CYCLOTRON ABSORPTION BY MIXING TWO MONOCHROMATIC ELECTROMAGNETIC WAVES IN A UNIFORM COLLISIONLESS MAGNETICALLY CONFINED PLASMA

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When two monochromatic electromagnetic (*em*) waves mix in a uniform magnetized plasma, a longitudinal wave is excited in the plasma. We derive an expression for the energy absorbed by the ions in the case the frequency difference of those two *em* waves is tuned to the ion cyclotron frequency. For time much longer than the period gyration of an ion, it is shown that mixing geometries producing small values of the wave number of the excited electrostatic wave are more effective in heating up the plasma.

1. Introduction

In a previous publication[1], the motion of a charged particle of charge e and mass M was considered in the field of two monochromatic electromagnetic waves of frequencies ω_1 and ω_2 and wave vectors $\vec{k}_1$ and $\vec{k}_2$, respectively. It was found that the particle will effectively experience a longitudinal field $\vec{E}_s$ along the direction $\vec{k}_1 - \vec{k}_2$ of the form $\vec{E}_s = \vec{E}_0 \sin(\vec{k}_0 \cdot \vec{x} - \omega_0 t)$, where $\vec{E}_0 = (eE_{10}E_{20}/ /2M\omega_1\omega_2)\vec{k}_0$, $\omega_0 = \omega_1 - \omega_2$ and E_{10}, E_{20} denote the electric field intensities of the two waves. Next, the energy transferred from the wave to the electrons of the plasma was calculated for the case of two waves of the same frequency and intensity propagating in opposite direction in a uniform unmagnetized plasma.

The purpose of this paper is to analytically examine the mixing of two monochromatic electromagnetic waves in a spatially homogeneous magnetically confined plasma for the case in which their frequency difference $\omega_0 = \omega_1 - \omega_2$ is tuned to the ion gyrofrequency $\omega_c = eB_0/Mc$.

In addition to the previously mentioned longitudinal field $\vec{E}_s$, each plasma ion experiences a self-consistent electrostatic field $\vec{E}$, the form of which is briefly given in Section 2. Then we consider the case of a low density plasma in which the self-consistent electrostatic field can become negligible in comparison with the external field $\vec{E}_s$. Next, the Vlasov equation is solved by the method of characteristics and an expression for the ion distribution function is found.

Section 3 is devoted to the calculation of the energy transferred from the longitudinal field $\vec{E}_s$ to the ions of the plasma. We finalize with a brief discussion of the results.

2. *Wave mixing in a magnetized plasma*

In this treatment, we consider a uniform fully ionized plasma of electrons and protons, immersed in a stationary magnetic field of intensity $\vec{B}_0$ directed along the z-axis of a cartesian system coordinates. At some instant of time, two monochromatic electromagnetic waves with frequencies ω_1 and ω_2 much larger then the electron plasma frequency, wave vectors $\vec{k}_1$ and $\vec{k}_2$ and electric field intensities $\vec{E}_{10}$ and $\vec{E}_{20}$ polarized along an axis perpendicular to the plane of their direction of propagation vectors $\vec{n}_1 = \frac{\vec{k}_1}{|\vec{k}_1|}$ and $\vec{n}_2 = \frac{\vec{k}_2}{|\vec{k}_2|}$, correspondingly, are shot in the plasma in such a way that the vector $\vec{k}_0 = \vec{k}_1 - \vec{k}_2$ is at right angles with the magnetic field $\vec{B}_0$. Prior to this instant, the electron and ion velocities are assumed to be distributed according to the Maxwellian law

$$F_e = n_0 \pi^{-\frac{3}{2}} a_e^{-3} \exp(-v^2/a_e^2)$$

$$F_i = n_0 \pi^{-\frac{3}{2}} a_i^{-3} \exp(-v^2/a_i^2),$$

where n_0 is the particle density, $a_e^2 = 2k\,T_e/m$, $a_i^2 = 2k\,T_i/M$ and the electron and ion masses and temperatures are given by m, M and T_e, T_i, respectively. For simplicity we are restricted by the temperature relation $T_e \gg T_i$, which is usually encountered in practice.

In the field of these two waves, an ion will experience a longitudinal field $\vec{E}_s$ along the direction $\vec{k}_0 = \vec{k}_1 - \vec{k}_2$ (taken parallel to the x-axis) of the form

$$\vec{E}_s = \vec{E}_0 \sin(k_0 x - \omega_0 t), \tag{1}$$

where

$$\vec{E}_0 = (e\,E_{10}\,E_{20}/2M\,\omega_1\,\omega_2)\,\vec{k}_0 \tag{2}$$

and

$$\omega_0 = \omega_1 - \omega_2. \tag{3}$$

In addition to this field, each individual plasma ion will experience a self-consistent electrostatic field $\vec{E}(\vec{x}, t)$ which satisfies the system of Vlasov-Poisson integrodifferential equations. Linearizing $(E_0/\sqrt{4\pi\,n_0\,k\,T_i} < 1)$ and solving these equations for the case in which the frequency $\omega_0 = \omega_1 - \omega_2$ resonates with the ion cyclotron frequency

$$\omega_c = e\,B_0/M\,c \tag{4}$$

(where c is the speed of light), we find that

$$\vec{E}(\vec{x}, t) = -\vec{E}_0 \sin(k_0\,x - \omega_c\,t) + \vec{E}_0 \sin(k_0\,x - \omega_\beta\,t) + $$

$$+ \vec{E}_0\,[(\omega_\beta - \omega_c)/\omega_\beta] \cos k_0\,x \sin \omega_\beta\,t \tag{5}$$

with

$$\omega_\beta^2 = \omega_c^2\,[1 + (2\omega_{pi}^2/\omega_c^2)\,(2\omega_c^2/k_0^2\,a_i^2)\,I_1\,(k_0^2\,a_i^2/2\omega_c^2)\exp(-k_0^2\,a_i^2/2\omega_c^2)]. \tag{6}$$

Here ω_{pi} is the ion plasma frequency, I_1 is the modified Bessel's function of the first order and ω_β is the frequency of a Bernstein mode[2].

When the plasma density is low $(\omega_{pi}^2 \ll \omega_c^2)$, it is possible, by adjusting the directions of $\vec{k}_1$ and $\vec{k}_2$, to bring ω_β very close to the ion cyclotron frequency. This way, the amplitude of $\vec{E}(\vec{x}, t)$ becomes very small, as a matter of fact, negligible compared with E_0.

Integrating the Vlasov equation along the orbits of the particles, we obtain

$$f_i\,(x, \vec{v}, t) = \text{constant}, \tag{7}$$

where f_i is the ion distribution function.

These orbits are given by the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\vec{x}\,(t) = \vec{v}\,(t) \quad (v \ll c), \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\vec{v}\,(t) = (e\,E_0/M) \sin(k_0\,x\,(t) - \omega_0\,t)\,\vec{e}_x + \omega_c\,\vec{v}\,(t) \times \vec{e}_z. \tag{9}$$

The vectors $\vec{e}_x$ and $\vec{e}_z$ are the unit vectors along the x and z axes, correspondingly.

For a sufficiently strong confining magnetic field $\vec{B}_0$, the electric field $\vec{E}_0$ can be considered as a small perturbation to the magnetic field. The orbits of the particles in the unperturbed field $\vec{B}_0$ are best expressed in cylindrical coordinates in the velocity space; i. e. $v_x = v_\perp \cos\varphi$, $v_y = v_\perp \sin\varphi$, $v_z = v_{||}$. In terms of these variables, we have that

$$x(t) = x_0 + (v_\perp/\omega_c)\sin\varphi + (v_\perp/\omega_c)\sin(\omega_c t - \varphi). \tag{10}$$

Because of (10), equation (9) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{v}(t) = (e E_0/M)\sin[k_0 x_0 + (k_0 v_\perp/\omega_c)\sin\varphi + (k_0 v_\perp/\omega_c)\sin(\omega_c t - \varphi) - \omega_0 t]\,\vec{e}_x + \omega_c \vec{v}(t)\times\vec{e}_z. \tag{11}$$

Assuming a constant amplitude electric field $\vec{E}_0$, the solution of equation (11) for the case $\omega_0 \to \omega_c$ may be written in the form:

$$v_x(t) = v_\perp\cos(\omega_c t - \varphi) - a J_0(k_0 v_\perp/\omega_c)\sin(\omega_c t - \Phi_0) + a J_2(k_0 v_\perp/\omega_c)\sin(\omega_c t + \Phi_2), \tag{12}$$

$$v_y(t) = -v_\perp\sin(\omega_c t - \varphi) - a J_0(k_0 v_\perp/\omega_c)\cos(\omega_c t - \Phi_0) + a J_2(k_0 v_\perp/\omega_c)\cos(\omega_c t + \Phi_2), \tag{13}$$

$$v_z(t) = v_{||}. \tag{14}$$

Here J_n denotes the Bessel's function of order n and

$$\Phi_n = k_0 x_0 + (k_0 v_\perp/\omega_c)\sin\varphi - n\varphi; \tag{15}$$

$$a = e E_0 t/2M. \tag{16}$$

Besides the externally applied electric field $\vec{E}_0$, each plasma ion, on account of its collisions with electrons (ion-ion collisions being neglected), is acted upon by a collisional drag force equivalent to a critical electric field

$$E_c = (4\pi\,\delta\, e^3 n_0 \ln\Lambda)/k T_i, \tag{17}$$

where $\ln\Lambda$ is the Coulomb logarithm and the coefficient δ is constant and equal to 0.21 for a Maxwellian plasma[3].

If the external electric field is smaller than E_c, there exists internal friction which causes the plasma temperature to rise with time.

If, on the other hand, the external field is much larger than E_c (as the present treatment requires), collisions become non-important and free motion of the plasma ions in the electric field E_0 takes place. In a way, the electric field frees the plasma of collisions. This is because the particles (ions) acquire speed between collisions, the collision cross section decreases and, for sufficiently rapid acceleration, the ions simply gain energy and run away. The absence of internal friction leaves the temperature T_i unchanged and having assumed that at the initial moment $t = 0$ the plasma is Maxwellian, the ion distribution function (Eq. (7)) may be written as follows:

$$f_i(x_0, v, t) = n_0 \pi^{-3/2} a_i^{-3} \exp\{[-(v \pm a(J_0 + J_2))^2 \pm \\ \pm 2a v (J_0 + J_2)(1 \mp (v_\perp/v)\sin\Phi_1 + 4a^2 J_0 J_2 \cos^2\Phi_1]/a_i^2\} \tag{18}$$

with $J_0(k_0 v_\perp/\omega_c)$, $J_2(k_0 v_\perp/\omega_c) \equiv J_0, J_2$, respectively.

The estimation of the energy which is absorbed by the ions of the plasma is attempted in the next section.

3. *Energy transferred to the ions*

The energy transferred to the plasma ions, averaged over a wavelength of the electrostatic field, may be expressed as follows:

$$U(t) = (m k_0/4\pi)\int_0^{2\pi} d\varphi \int_{-\pi/k_0}^{\pi/k_0} dx_0 \int_{-\infty}^{\infty} dv_{\parallel} \int_0^{\infty} v_\perp f_i(x_0, v, t)\, v^2(t)\, dv_\perp \tag{19}$$

where $v^2(t) = v_x^2(t) + v_y^2(t) + v_z^2(t)$.

As f_i is given by (18), equation (19) is rewritten in the form

$$U(t) = (n_0 m k_0/4\pi^{5/2} a_i^3) \int_0^{2\pi} d\varphi \int_{-\pi/k_0}^{\pi/k_0} dx_0 \int_{-\infty}^{\infty} dv_{\parallel} \int_0^{\infty} v_\perp \exp\{([-(v \pm a(J_0 + J_2))^2 \pm \\ \pm 2a v (J_0 + J_2)(1 \mp (v_\perp/v)\sin\Phi_1) + 4a^2 J_0 J_2 \cos^2\Phi_1]/a_i^2\}\, v^2(t)\, dv_\perp. \tag{20}$$

The evaluation of this integral, as it stands, is very complicated. Fortunately, a particularly simple situation exists when $a/a_i \gg 1$. Under this condition, the electric field of resonant frequency $\omega_0 = \omega_c$ catches ions whose velocities lie on a plane perpendicular to the magnetic field, i. e. $v \sim v_\perp \sim \pm a(J_0 + J_2)$ and $\sin\Phi_1 \sim \pm 1$. The appearance in the plasma of this group of fast ions whose velocities far exceed the ion thermal speed, allows us to simplify the problem by neglecting the term containing $\cos^2\Phi_1$.

The expansion of $\exp[-2a v_\perp (J_0 + J_2)\sin\Phi_1/a_i^2]$ in a series of modified Bessel functions and the integration of (20) over x_0, φ and $v_{\parallel}$ yields

$$U(t) = (n_0 k T_i/a_i)\int_0^{\infty} [(2A + (v_\perp/a_i)) I_0(a_1) - 2B I_1(a_1)] \exp(-(v_\perp^2 + \\ + a^2 (J_0 + J_2)^2)/a_i^2)\, dv_\perp, \tag{21}$$

where

$$\alpha_1 = 2a\, v_\perp (J_0 + J_2)/a_i^2; \tag{22}$$

$$A = (v_\perp/a_i)\,[(v_\perp^2 + a^2 (J_0 + J_2)^2)/a_i^2]; \tag{23}$$

$$B = (v_\perp/a_i)\,\alpha_1. \tag{24}$$

Since a substantial contribution to the integral of (21) comes from particles whose velocities lie close to $\pm\, a (J_0 + J_2)$, the use of the asymptotic form of $I_0(\alpha_1)$ and $I_1(\alpha_1)$[4] leads to the following expression:

$$U(t) = (n_0\, k\, T_i/a_i)\Big[\int_{N(a(J_0+J_2))} [(2(A - B) + (v_\perp/a_i) + 2B\,\sigma_1)/(2\pi\,\alpha_1)^{1/2}] \cdot$$

$$\cdot \exp(-(v_\perp - a(J_0 + J_2))^2/a_i^2)\,\mathrm{d}v_\perp +$$

$$+ e^{-i\pi/2}\int_{N(-a(J_0+J_2))} [(2(A + B) + (v_\perp/a_i) + 2B\,\sigma_1^0)/(2\pi\,\alpha_1)^{1/2}] \cdot$$

$$\exp(-(v_\perp + a(J_0 + J_2))^2/a_i^2)\,\mathrm{d}v_\perp\Big], \tag{25}$$

where σ_1, $\sigma^0 \cong 3/8\alpha_1$ and $N(u)$ denotes some suitable neighborhood of u to be specified below.

If λ is a root of either of the equations

$$v_\perp = \pm\, a\,(J_0(k_0 v_\perp/\omega_c) + J_2(k_0 v_\perp/\omega_c)), \tag{26}$$

we may write $v_\perp = (\omega_c/k_0)\,\lambda + \Delta v_\perp$, $(\Delta v_\perp \ll (\omega_c/k_0)\,\lambda)$. By expanding $J_0(k_0 v_\perp/\omega_c)$ and $J_2(k_0 v_\perp/\omega_c)$ in Taylor series around λ and retaining only first order terms with respect to $w = (k_0/\omega_c)\,\Delta v_\perp$, we have

$$v_\perp \pm a\,(J_0(k_0 v_\perp/\omega_c) + J_2(k_0 v_\perp/\omega_c)) \simeq (\omega_c/k_0)\,\beta_\mp\, w \tag{27}$$

where

$$\beta_\mp = 1 \mp 2J_2(\lambda)/(J_0(\lambda) + J_2(\lambda)). \tag{28}$$

Since $v_\perp$ is a positive quantity, we must have

$$J_0(\lambda) + J_2(\lambda) > 0;\; J_0(\lambda) + J_2(\lambda) < 0 \tag{29,30}$$

for β_+ and β_-, respectively.

By using the expanded form of $J_0(k_0 v_\perp/\omega_c)$ and $J_2(k_0 v_\perp/\omega_c)$ to obtain new expressions for a_1, A, B from (22), (23), (24) and then substituting them into (25) and taking (27) into account, we obtain

$$\eta = \frac{U(t)}{U(0)} = \frac{4\pi n_0 k T_i}{E_0^2} \left[H(J_0(\lambda) + J_2(\lambda)) \left(\frac{11}{8} \operatorname{erf} y_0^+ - \frac{y_0^+}{\sqrt{\pi}\, e^{(y_0^+)^2}} \right) \Big/ \beta_+ + \right.$$

$$\left. + H(-J_0(\lambda) - J_2(\lambda)) \left(\frac{11}{8} \operatorname{erf} y_0^- - \frac{y_0^-}{\sqrt{\pi}\, e^{(y_0^-)^2}} \right) \Big/ \beta_- \right]. \qquad (31)$$

Here H is the Heaviside function, $\eta = U(t)/U(0)$ is the energy absorption coefficient, $U(0) = \frac{E_0^2}{4\pi}$, $y_0^\pm = (\omega_c/k_0 a_i)(\beta_\pm) w_0$ and w_0 is the upper limit of w. The value of w_0, which determines the domains of integration in (25), is somewhat arbitrarily chosen and, to a good approximation, it can be taken equal to unity. This value of w_0 would not be proper only where second order terms with respect to w could be effective in the expansion of the sum $J_0 + J_2$. This happens only when the derivative of this sum vanishes, i. e. when $u J_1'(u) = J_1(u)$, where $u = k_0 v_\perp/\omega_c$, but these values are different from λ and their neighborhood have been excluded as non substantially contributing to the value of the integral (25).

Eq. (31) proves easily that η increases with $y_0^\pm$. Therefore, for given plasma parameters and a certain mixing geometry the energy absorption increases with the magnetic field intensity. In turn, for a specified value of B_0 and considering that $k_0 \simeq 2k \sin(\vartheta/2)$ ($k = |\vec{k}_1| \simeq |\vec{k}_2|$, ϑ is the angle formed by the vectors $\vec{k}_1$ and $\vec{k}_2$), mixing geometries producing long wavelengths (i. e. small values of k and ϑ) give large values of $y_\pm^0$ and consequently of η. Limiting ourselves to cases in which the externally applied electric field E_0 is much larger than the critical electric field E_c given by Eq. (17) we ignore internal friction. Therefore the plasma temperature during the absorption process remains constant. In the absence of collisions in the plasma part of ions absorbing energy from the electric field, are accelerated to relatively high velocities. As a consequence the ion distribution function becomes non-isotropic and there appears a group of runing away ions forming a *tail* in a direction perpendicular to the magnetic field B_0 with velocities close to that which satisfies Eq. (25).

When a plasma of 1 eV temperature is considered immersed in a magnetic field of about 5 T, the ions of mass $M = 1.67 \times 10^{-24}$ g and charge $e = 1.6 \times \times 10^{-19}$ C rotate at $\omega_c = 5 \times 10^8\ \mathrm{s}^{-1}$. If the two mixing waves of power P each are focused down to a diameter d, then $E_{10} = E_{20} = \frac{4}{d}\left(\frac{P}{c}\right)^{1/2}$. Choosing ω_1, $\omega_2 \sim 9 \times 10^{29}\ \mathrm{s}^{-1}$ and directing the wave vectors $\vec{k}_1$ and $\vec{k}_2$ in such a way as to form an angle of around 52° for $P \simeq 142$ MW and $d \approx 25\ \mu$, a $k_0 = 8.9 \times 10^3\ \mathrm{cm}^{-1}$ is produced and the amplitude of the longitudinal wave E_0 excited in the plasma is approximately equal to $360 \frac{V}{\mathrm{cm}}$. At this value of E_0 for $t = 10^{-6}$ s (t being the smaller of the duration times of the two waves), Eq. (16) gives $a = 1.8 \times 10^8 \frac{\mathrm{cm}}{s}$.

Under these circumstances, Eq. (26) with the minus sign may be written in the form

$$\left(\frac{k_0 v_\perp}{\omega_c}\right)^2 = -6.425 \times 10^3 J_1 \left(\frac{k_0 v_\perp}{\omega_c}\right)$$

which has a solution $\frac{k_0 v_\perp}{\omega_c} \approx 30.6$. In addition we have $J_0(30.6) = -0.0285$, $J_2(30.6) = 0.019$ and $\beta_- = 5$. The quantity y_0^- is then easily found to equal approximately 0.2. Finally, for $\eta_0 \approx 1.4 \times 10^{11}$ cm^{-3} the value η is found to be 0.0788, i. e. 7.88%. Since the characteristic dimension of the mixing volume parallel to the vector $\vec{k}_0$ can be roughly written as $2d \sin \frac{\vartheta}{2} \approx 20\ \mu$ which is equivalent to $2\lambda_0$ ($\lambda_0 \sim 10\ \mu$), the effective value of η is approximately 15.7% over a distance of two wavelengths. This, allowing always for a small error, practically satisfies our initial assumption that the amplitude E_0 of the longitudinal wave remains constant in time and space during the absorption process.

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IONSKA CIKLOTRONSKA APSORPCIJA NASTALA MIJEŠANJEM DVAJU MONOKROMATSKIH VALOVA U UNIFORMNOJ BEZKOLIZIONOJ MAGNETSKOJ PLAZMI

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Ispitano je rezonantno zagrijavanje magnetizirane i bezkolizione plazme na ciklotronskoj frekvenciji pomoću dva elektromagnetska vala. Izračunat je i diskutiran odgovarajući koeficijent apsorpcije.