YU ISSN 0015—3206
FZKAAA 16 (4) 363 (1984)

ON THE KLEIN-GORDON PROPAGATOR IN SCALAR ELECTRODYNAMICS

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Received 28 March 1984

Revised manuscript received 11 June 1984

UDC 539.12

Original scientific paper

The first non-trivial approximation of the Schwinger-Dyson equation for the Klein-Gordon propagator in scalar electrodynamics with $\lambda(\varphi^* \varphi)^2/4$ self-interaction is investigated. An analysis for different values of space-time dimensions (d), gauge parameter (G), coupling constants (e_0, λ) and the scalar particle masses (m_0, m) is performed. It is shown that finite (trivial) solutions exist in the following two cases:

$$1)\ d = 2,\ \lambda = e_0^2,\ G = -1 \text{ and } 2)\ d = 4,\ \lambda = 3e_0^2,\ G = 3,\ m = m_0 = 0.$$

1. Introduction

In this short paper we report on the main points of our investigation of the Schwinger-Dyson (SD) equation for the spin-0 propagator in scalar electrodynamics — the theory describing interaction of the charged (pseudo) particles with photons. The work on this subject was motivated by a similar investigation[1] in finite spinor electrodynamics of Johnson, Baker and Willey[2]. The possibility of finite scalar electrodynamics was investigated by Fry[3] who concluded that finite scalar electrodynamics probably does not exist. Instead of investigating some general aspects of possible finite scalar electrodynamics we analyse the structure and finitness of a definite non-trivial approximation of the SD equation for scalar particle.

At the beginning, we derive the SD equation for the Klein-Gordon (KG) propagator for an arbitrary number of the space-time dimensions (d). An analysis of the first nontrivial approximation for definite dimensions $d = 2, 3, 4$ is also performed.

2. *The Schwinger-Dyson equation for the Klein-Gordon propagator*

The action for scalar electrodynamics with φ selfinteraction term is

$$S = \int \mathrm{d}x \left[D_\mu^* \varphi^* D^\mu \varphi - m_0^2 \varphi^* \varphi + \frac{\lambda}{4} (\varphi^* \varphi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{J}^\mu A_\mu + \right.$$

$$\left. + \mathcal{J}^* \varphi + \varphi^* \mathcal{J} \right] \tag{1}$$

where $D_\mu = \partial_\mu - \mathrm{i} e_0 A_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\mathcal{J}^\mu, \mathcal{J}^*, \mathcal{J}$ are external currents for electromagnetic (A_μ) and scalar (φ, φ^*) fields. The SD equation can be written in the following variational form[4)]

$$\int D\varphi \, D\varphi^* \, DA_\mu \frac{\delta}{\delta \varphi^* (x)} \mathrm{e}^{\mathrm{i}S} = 0. \tag{2}$$

From (1) and (2), using usual transformations and expression

$$W(\mathcal{J}) = \int D\varphi \, D\varphi^* \, DA_\mu \, \mathrm{e}^{\mathrm{i}S} = \mathrm{e}^{\mathrm{i}Z}, \tag{3}$$

we get

$$\left\{ \left[\partial_\mu - \mathrm{i} e_0 U_\mu (x) - e_0 \frac{\delta}{\delta \mathcal{J}^\mu (x)} \right]^2 + m_0^2 + \mathrm{i}\lambda \Delta (x, x) + \lambda \frac{\delta^2}{\delta \mathcal{J} (x) \, \delta \mathcal{J}^* (x)} \cdot \right.$$

$$\cdot \Delta (x, y) = \delta (x - y) \tag{4}$$

where $U_\mu (x) = \delta Z / \delta \mathcal{J}^\mu (x)$ is the classical electromagnetic field and

$$\Delta (x, y) = \mathrm{i} \frac{\delta^2 Z}{\delta \varphi^* (x) \, \delta \varphi (y)} \tag{5}$$

is the meson (Klein-Gordon) propagator. One also has to introduce the photon propagator

$$D_{\mu\nu} (x, y) = - \frac{\delta^2 Z}{\delta \mathcal{J}^\mu (x) \, \delta \mathcal{J}^\nu (y)} = - \frac{\delta U_\mu (x)}{\delta \mathcal{J}^\nu (y)} \tag{6}$$

and different vertex functions

$$\Gamma^{\mu}(x, y/z) = -\frac{\delta \Delta^{-1}(x, y)}{e_0\, \delta U_\mu(z)}$$

$$\Gamma^{\mu\nu}(x, y/z, t) = \frac{\delta^2 \Delta^{-1}(x, y)}{e_0^2\, \delta U_\mu(z)\, \delta U_\nu(t)} = -\frac{\delta \Gamma^{\nu}(x, y/t)}{e_0\, \delta U_\mu(z)} \tag{7}$$

$$\Gamma(x, y/z, t) = \frac{\delta^2 \Delta^{-1}(x, y)}{\lambda \delta \xi^*(z)\, \delta \xi(t)}$$

where $\xi^*(x) = \delta Z/\delta \mathcal{J}(x)$ and $\xi(x) = \delta Z/\delta \mathcal{J}^*(x)$ are the classical meson fields. Performing some differentiations in equation (3), taking limit $\mathcal{J}^\mu = \mathcal{J} = \mathcal{J}^* = 0$ and applying technique of the Fourier transformations we obtain the corresponding *SD* equation in the momentum space

$$\begin{aligned}\Delta^{-1}(p) = {} & \Delta_0^{-1}(p) + \frac{i\lambda}{(2\pi)^d}\int d^d q\, \Delta(q) - \frac{ie_0^2}{(2\pi)^d}\int d^d q\, D^{\mu}_{\mu}(q) - \\ & - \frac{ie_0^2}{(2\pi)^d}\int d^d q\, D_{\mu\nu}(q-p)(p+q)^{\mu}\Delta(q)\Gamma^{\nu}(q, p) + \\ & + \frac{e_0^4}{(2\pi)^{2d}}\int d^d q\, d^d k D_{\mu\varrho}(k-q) D_{\nu\sigma}(q-p)\, 2g^{\mu\nu}\Delta(k)\Gamma^{\varrho}(k, q)\Delta(q)\Gamma^{\sigma}(q, p) - \\ & - \frac{e_0^4}{(2\pi)^{2d}}\int d^d q\, d^d k D_{\mu\varrho}(k-q) D_{\nu\sigma}(q-p) g^{\mu\nu}\Delta(k)\Gamma^{\varrho\sigma}(k; -p, q-k) - \\ & - \frac{\lambda^2}{(2\pi)^{2d}}\int d^d q\, d^d k \Delta(q)\Delta(k)\Gamma(k; p, q)\Delta(k+q-p).\end{aligned} \tag{8}$$

As a first non-trivial approximation of the *SD* equation (8) we take the lowest terms proportional to the constants e_0^2 and λ and the first approximation of the vertex function and of the photon propagator:

$$\begin{aligned}\Gamma^{\nu}(p, q) &\to \Gamma_0^{\nu}(p, q) = (p+q)^{\nu} \\ D_{\mu\nu}(k) &\to D^0_{\mu\nu}(k) = -\frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + G\frac{k_\mu k_\nu}{k^2}\right).\end{aligned} \tag{9}$$

Hence, our first approximation of the *SD* equation for the *KG* propagator is

$$\begin{aligned}\Delta^{-1}(p) = {} & m_0^2 - p^2 + \frac{i\lambda}{(2\pi)^d}\int d^d q\, \Delta(q) - \frac{ie_0^2}{(2\pi)^d}\int d^d q\, g^{\mu\nu} D^0_{\mu\nu}(q) - \\ & - \frac{ie_0^2}{(2\pi)^d}\int d^d q\, D^0_{\mu\nu}(q-p)(p+q)^{\mu}\Delta(q)(p+q)^{\nu}.\end{aligned} \tag{10}$$

Substituting the expression (9) for the bare photon propagator $D^0_{\mu\nu}(k)$ into the SD equation (10) and performing the Wick rotation ($p_0 \to p_0' = ip_4$, $q_0 \to q_0' = iq_4$, $-p^2 \to x = \vec{p}^{\,2} + p_4^2$, $-q^2 \to y = \vec{q}^{\,2} + q_4^2$) we get the corresponding SD equation in Euclidean momentum space

$$\Delta^{-1}(x) = m_0^2 + x + \frac{e_0^2}{2(2\pi)^d}\int_0^\infty dy\, y^{\frac{d-2}{2}}\Big\{\frac{d+G-1}{y} I_0^0(x,y) - \Big[\frac{\lambda}{e_0^2} + (x-y)^2 \cdot$$
$$\cdot I_4^0(x,y)(G-1) + 2I_2^1(x,y) + (x+y)I_2^0(x,y)\Big]\Delta(y)\Big\} \tag{11}$$

where

$$I_m^n(x,y) = \int \frac{(pq)^n\, d^d\Omega}{(p-q)^m}, \qquad (n = 0, 1, 2, \ldots;\ m = 0, 2, 4 \ldots). \tag{12}$$

Using properties

$$I_m^n(x,y) = -\frac{1}{2} I_{m-2}^{n-1}(x,y) + \frac{x+y}{2} I_m^{n-1}(x,y),$$
$$(x-y)^2 I_4^0(x,y) + (d-3)(x+y) I_2^0(x,y) = (d-2) I_0^0(x,y), \tag{13}$$

all integrals $I_m^n(x,y)$ in the last SD equation can be reduced to the integrals $I_2^0(x,y)$ and $I_0^0(x,y)$ giving us the following equation

$$\Delta^{-1}(x) = m_0^2 + x + \frac{e_0^2}{2(2\pi)^d}\int_0^\infty dy\, y^{\frac{d-2}{2}}\Big\{\frac{d+G-1}{y} I_0^0(x,y) - \Big[\frac{\lambda}{e_0^2} +$$
$$+ (d-2)(G-1) - 1\Big] I_0^0(x,y) - ((d-3)(G-1)+2)(x+y) I_2^0(x,y)]\Delta(y)\Big\}. \tag{14}$$

Since the general solution of the last equation has ultraviolet (UV) divergences we are interested in the existence of some special finite (without UV divergences before applying the renormalisation procedure) solutions looking for the appropriate conditions on parameters e_0, λ, d and G.

Let Λ be an UV cutoff. Taking for asymptotic behaviour ($y \to \infty$) of the meson propagator and integral $I_2^0(x,y)$:

$$\Delta(y) \to \Delta_0(y) = (m_0^2 + y)^{-1}, \qquad I_2^0(x,y) \to \frac{1}{y} \tag{15}$$

we get

$$\Delta^{-1}(x) = g_d\left(\mathrm{d} - 1 - \frac{\lambda}{e_0^2}\right)\int^{\Lambda} y^{\frac{d-4}{2}}\,\mathrm{d}y - g_d\left\{[(3-d)\,G + d - 1]\,x - \right.$$

$$\left. - m_0^2\left(\frac{\lambda}{e_0^2} + G\right)\right\}\int^{\Lambda} y^{\frac{d-6}{2}}\,\mathrm{d}y + \text{finite part} \tag{16}$$

where $g_d = \dfrac{e_0^2}{(4\pi)^{\frac{d}{2}}\,\Gamma\left(\frac{d}{2}\right)}$. From (16) we obtain the necessary and sufficient conditions for finiteness:

$$\begin{aligned} &d = 2, \quad \lambda = e_0^2, \\ &d = 3, \quad \lambda = 2e_0^2, \\ &d = 4, \quad \lambda = 3e_0^2, \quad G = 3, \quad m_0 = 0. \end{aligned} \tag{17}$$

It is also useful to discuss equation (14) for dimensions $d = 2, 3, 4$ separately. For $d = 2$ and condition $\lambda = e_0^2$ we have

$$\Delta^{-1}(x) = m_0^2 + x + \frac{e_0^2}{2\,(2\pi)^2}(G+1)\int_0^{\infty}\mathrm{d}y\left[\frac{1}{y}I_0^0(x,y) - (x+y)\,I_2^0(x,y)\,\Delta(y)\right]. \tag{18}$$

Substituting expressions (d = 2)

$$I_0^0(x,y) = 2\pi, \qquad I_2^0(x,y) = \frac{2\pi}{x-y}\left[\Theta(x-y) - \Theta(y-x)\right] \tag{19}$$

into equation (18) we obtain

$$\Delta^{-1}(x) = m_0^2 + x + \frac{e_0^2}{4\pi}(G+1)\left\{\int_0^x \mathrm{d}y\left[\frac{1}{y} - \frac{x+y}{x-y}\Delta(y)\right] + \int_x^{\infty}\mathrm{d}y\left[\frac{1}{y} + \right.\right.$$

$$\left.\left. + \frac{x+y}{x-y}\Delta(y)\right]\right\}. \tag{20}$$

The last equation is free of the *UV* divergence, but the presence of $(x-y)^{-1}$ in the integrand give us, like in spinor electrodynamics[5], divergence (an infrared divergence) in its solution which we remove choosing gauge $G = -1$. It leads to the trivial solution

$$\Delta(x) = \Delta_0(x) = (m_0^2 + x)^{-1}. \tag{21}$$

For $d = 3$, $\lambda = 2e_0^2$ and

$$I_0^0(x, y) = 4\pi, \quad I_2^0(x, y) = \frac{\pi}{\sqrt{xy}} \ln \left(\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}} \right)^2 \tag{22}$$

we get

$$\Delta^{-1}(x) = m_0^2 + x + \left(\frac{e_0}{2\pi} \right)^2 \int_0^\infty \frac{dy}{\sqrt{y}} \left\{ G + 2 - \left[yG + \right. \right.$$

$$\left. \left. + \sqrt{\frac{y}{x}} \frac{x+y}{2} \ln \left(\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}} \right)^2 \right] \Delta(y) \right\}. \tag{23}$$

An integration of equation (23) will also give an infrared divergence which cannot be removed by any choice of the gauge parameter G. Hence the first approximation of the SD equation in 3-dimensional case cannot be made (infrared) finite.

The SD equation (14) for $d = 4$, $\lambda = 3e_0^2$, $G = 3$ and $m_0 = 0$ takes the form

$$\Delta^{-1}(x) = x + 6 \left(\frac{e_0}{4\pi} \right)^2 \int_0^\infty dy\, y \left(\frac{1}{y} - \Delta(y) \right). \tag{24}$$

We can write this equation in the form

$$\Delta^{-1}(x) = x + m^2 \tag{25}$$

where

$$m^2 = 6 \left(\frac{e_0}{4\pi} \right)^2 \int_0^\infty dy\, (1 - y\Delta(y)) \tag{26}$$

may be dynamically generated mass of spin-0 particle. If we substitute (25) in (26), we obtain

$$m^2 = 6 \left(\frac{e_0}{4\pi} \right)^2 m^2 \ln \frac{m^2 + \Lambda}{m^2} \tag{27}$$

what is consistent only for $m = 0$ (a second solution relating m to Λ is infinite). Hence it follows that the first approximation of the SD equation in 4-dimensional case leads to finite solution which is equal to the massless bare KG propagator.

3. Conclusion

We analysed the first non-trivial approximation of the unrenormalised SD equation for the KG propagator in scalar electrodynamics extended by φ^4 theory.

This approximation contains an infinite number of the standard Feynman diagrams which, in a definite form, take into account representatives of all orders of the usual perturbation expansion. It follows that finite solution can be reached only for appropriate values of the parameters in the theory. Such parameters are: space-time dimension (d), electromagnetic coupling constant (e_0), self-interaction coupling constant (λ), gauge parameter (G) and mass parameters (m_0, m). We found that only trivial solutions, in the form of the bare KG propagator, realize the possible finite solutions. It means that in such cases electromagnetic and self-interaction effects completely compensate each other. So, we get the finite (trivial) solution $\Delta(x) = \Delta_0(x) = (m_0^2 + x)^{-1}$ in the following two cases 1) $d = 2$, $\lambda = e_0^2$, $G = -1$, $m = m_0 < +\infty$ and 2) $d = 4$, $\lambda = 3e_0^2$, $G = 3$, $m = m_0 = 0$.

To avoid possible confusion on the gauge covariance of the approximated SD equation (10), we have to make some comments. Any approximation is gauge covariant if it contains all necessary diagrams so that calculated physical predictions do not depend on the gauge parameter G. As it is well known, when we have such gauge covariant approximation, we may take gauge parameter the way we like and ordinary we take it to simplify calculations. In the Johnson-Baker-Willey[2] approach to finite spinor electrodynamics one chooses G in such a way that it cancels some of the ultraviolet divergences. We also do it in this paper but our approximation (10) is not completely gauge covariant. It is gauge covariant at the lowest order ($e_0^2 = (d-1)\lambda$) while higher orders only exhibit effect of cancelation of divergences. To obtain finite solution at the higher (e_0^{2n}) order which is gauge covariant up to e_0^{2n} order, we have to include all relevant diagrams till e_0^{2n} order and make expansions: $\lambda = (d-1)^{-1} e_0^2 + \ldots + a_n e_0^{2n}$, $G = G_0 + G_1 e_0^2 + \ldots + G_{n-1} e_0^{2(n-1)}$ (see also Ref. 3)). By tuning the parameters a_i and G_i, we can obtain finite solution for the KG propagator. If we want to use finite solution in calculation of the physical quantities, we have to take solution of the SD equation at the order which is gauge covariant. In our approach we have non-trivial approximations with finite solutions which are gauge covariant in the lower orders, but also demonstrate some cancelations of divergences in all higher orders.

The first approximation of the SD equation in scalar electrodynamics is also dependent on the formalism which we use. As it is known, scalar electrodynamics can also be well described in terms of the Duffin-Kemmer formalism[6]. The first approximation of the SD equation in the KG formalism is inequivalent to the first approximation in the Duffin-Kemmer formalism[7].

Acknowledgment

We thank RZN of Serbia for financial support.

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O KLEIN-GORDONOVOM PROPAGATORU U SKALARNOJ ELEKTRODINAMICI

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Originalni naučni rad

Ispitivana je prva netrivijalna aproksimacija Schwinger-Dysonove jednačine za Klein-Gordonov propagator u skalarnoj elektrodinamici sa $\lambda(\varphi^* \varphi)^2/4$ samointerakcijom. Analiza je izvršena za razne vrednosti prostorno-vremenskih dimenzija (d), kalibracionog parametra (G), konstanti vezivanja (e_0, λ) i masa skalarne čestice (m_0, m). Pokazano je da postoje konačna (trivijalna) rešenja u sledeća dva slučaja: 1) $d = 2, \lambda = e_0^2, \; G = -1$ i 2) $d = 4, \lambda = 3e_0^2, \; G = 3, \; m = m_0 = 0$.