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ON THE LARGE-N LIMIT IN A MODEL WITH THE COLLECTIVE VARIABLE

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The large-N limits are compared for complete and reduced effective potentials in a quantum field model with the collective variable based on orthogonal symmetry.

1. Introduction

One dimensional quantum-field models with collective variable based on various symmetries were considered by Jevicki and Sakita[1], Andrić, Jevicki and Levine[2] and Andrić[3]. If the collective variable describing the N-particle state is denoted by ϱ, then the hamiltonian is of the form

$$H = \frac{1}{2} \int \mathrm{d}x \, \partial_x \pi(x) \, \varrho(x) \, \partial_x \pi(x) + V_{eff}(\varrho), \tag{1.1}$$

where the first term is called the kinetic term and the second one is called the effective potential. The effective potential has the form

$$V_{eff}(\varrho) = \frac{1}{2} \left\{ \frac{\lambda^2 \pi^2}{3} \int \varrho(x)^3 \, \mathrm{d}x - \lambda (1 - \lambda) P \int \mathrm{d}x \, \mathrm{d}y \frac{\varrho(y) \, \dot{\varrho}(x)}{x - y} + \right.$$
$$\left. + \frac{(\lambda - 1)^2}{4} \int \frac{\dot{\varrho}(x)^2}{\varrho(x)} \mathrm{d}x \right\} + \int x^2 \varrho(x) \, \mathrm{d}x, \tag{1.2}$$

$\dot{\varrho}$ being the derivative of ϱ and $\lambda = 1/2, 1, 2$ for orthogonal, unitary and symplectic symmetries, respectively.

Under the hypothesis that V_{eff} is the dominant term for large N, we obtain the following minimization problem for a ground state:

$$V_{eff}(\varrho) \to min, \qquad \int \varrho(x)\,dx = N, \; \varrho > 0. \tag{1.3}$$

For $\lambda = 1$, there exists a unique solution of this problem which can be easily written in a closed form:

$$\varrho_0(x) = \begin{cases} \frac{2}{\pi}(N - x^2)^{1/2} & \text{for } |x| < N, \\ 0 & \text{otherwise.} \end{cases}$$

For $\lambda = 1/2$, the functional V_{eff} is strictly convex on an appropriate domain, so that (1.3) has a unique solution ϱ_0 with the following properties[4]:

(1) $\varrho_0(x) > 0$ for $-\infty < x < \infty$,

(2) ϱ_0 is symmetric and has derivatives of all orders,

(3) $\int ([\ddot{\varrho}_0(x)]^2 + \varrho(x)^2)\,dx < \infty$,

(4) $\varrho_0(x)$ vanishes as $C_1 \exp(-C_2 x^2)$ for large $|x|$, where C_1, C_2 are certain positive numbers.

For $\lambda = 2$, there are some difficulties in solving (1.3) and we discuss them in Section 4.

Let $E(N)$ be the value of $V_{eff}(\varrho_0)$ at a solution ϱ_0 of (1.3). For $\lambda = 1$, we easily calculate $E(N) = \omega N^2$, $\omega = 3/4$. For $\lambda = 1/2, 2$, $E(N)$ is unknown. We expect no contributions of the second and the third terms in (1.2) to the value of V_{eff} as N becomes large. Therefore we simplify the minimization problem (1.3) by omitting the two terms mentioned. Again, a unique approximate solution ϱ_{app} can be obtained in a closed form and the corresponding ground-state energy $E_{app}(N) = \omega_1 N^2$, with a certain ω_1, can be easily calculated. Our aim is to prove

$$\frac{E(N)}{E_{app}(N)} = 1 + O\left(\frac{1}{N}\right) \tag{1.4}$$

for $\lambda = 1/2$, i. e. for orthogonal symmetry.

2. Notation

Instead of a nonnegative function ϱ, mentioned in Section 1, we prefer to use its normalized form $u = N^{-1}\varrho$ and formulate all results in terms of u.

Let us define the set K of all functions u on $(-\infty, \infty)$ with the properties $u(x) > 0$, $\int u(x)\,dx = 1$ and

$$|u| = [\int (u(x)^2 + \dot{u}(x)^2)\,dx]^{1/2} < \infty.$$

The set K is convex. For $u \in K$, we consider four functionals

$$I_1(u) = \int u(x)^3\,dx,$$

$$I_2(u) = (\dot{u} \mid Tu),$$

$$I_3(u) = \int \frac{\dot{u}(x)^2}{u(x)}\,dx,$$

$$\langle V|u\rangle = \int x^2\, u(x)\,dx,$$

where $(Tu)(x) = P\int (x-y)^{-1}\, u(y)\,dy$. Let

$$\mathcal{J}_N(u) = \frac{1}{8}\left\{\frac{N^2\pi^2}{3} I_1(u) - N I_2(u) + \frac{1}{4} I_3(u)\right\} + \langle V \mid u\rangle.$$

Then $\mathcal{J}_N(u) = N^{-1}\, V_{eff}(\varrho)$ for $\lambda = 1/2$ and problem (1.3) is equivalent to the problem

$$\mathcal{J}_N(u) \to min, \qquad (2.1)$$

$$u \in K.$$

This problem, which we call the complete problem, has a unique solution $u_{N,cr}$ as described in Section 1. Let us define the reduced functional R_N

$$R_N(u) = \frac{N^2\pi^2}{24} I_1(u) + \langle V \mid u\rangle,$$

the corresponding set L consisting of nonnegative functions u on $(-\infty, \infty)$ for which $\int u(x)\,dx = 1$ and the corresponding reduced problem

$$R_N(u) \to min, \qquad (2.2)$$

$$u \in L.$$

This problem has a unique solution u_N, as described in Section 1, and $R_N(u_N) = \omega_1 N$ with a certain ω_1.

In our proof of (1.4) we have extensively used the Fourier transform of the following function:

$$s(x) = \begin{cases} \sqrt{2/\pi}\,(1 - x^2)^{1/4} & \text{for } |x| < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

To obtain some information about the properties of its Fourier transform, we consider the function

$$w_\mu(x) = \begin{cases} x^\mu \exp(-x) & \text{for } x > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and its Fourier transform $\hat{w}_\mu$ for $\mu > 0$. We have[5]

$$\hat{w}_\mu(p) = \mathrm{i}\,\Gamma(1 + \mu) \exp\left(\mathrm{i}\mu \frac{\pi}{2}\right)(p + \mathrm{i})^{-1-\mu}.$$

The function s behaves like $w_{1/4}(1 \mp x)$ at $x = \pm 1$, respectively. In particular, the difference

$$f(x) = s(x) - \left(\frac{2}{\pi}\right)^{1/2} 2^{1/4}\,[w_{1/4}(1 - x) + w_{1/4}(1 + x)]$$

has the following behaviour:

$$f(x) = (1 - x^2)^{5/4} \times (\text{Taylor series})$$

at $x = \pm 1$, so that $|f| < \infty$. This means that $\hat{s}$ and $\hat{w}_{1/4}$ have the same behaviour for large p. Therefore

$$\lim_{|p|\to\infty} |p|^{5/4}\,\hat{s}(p) \neq 0, \quad |\hat{s}(p)| < \frac{\varrho_1}{1 + p^{5/4}}. \tag{2.4}$$

Similarily for the function $u_1(x) = s(x)^2$, we have

$$\lim_{|p|\to\infty} |p|^{3/2}\,\hat{u}_1(p) \neq 0, \qquad |\hat{u}_1(p)| < \frac{\varrho_2}{1 + |p|^{3/2}}. \tag{2.5}$$

3. *The large — N limit*

Let u_N be the solution of (2.2) and $v_N = \sqrt{u_N}$. Then $v_N(x) = N^{-1/4} s(N^{-1/2} x)$ and

$$\hat{v}_N(p) = N^{1/4}\,\hat{s}(N^{1/2}p) = N^{1/4}\,\hat{v}_1(N^{1/2}p), \tag{3.1}$$

$$\hat{u}(p) = \int \hat{s}(q)\,\hat{s}(N^{1/2}p - q)\,dq = \hat{u}_1(N^{1/2}p). \tag{3.2}$$

We have to prove $\mathcal{J}_N(u_{N,c})/R_N(u_N) \to 1$ as $N \to \infty$. Because of $\mathcal{J}_N(u_{N,c}) < \mathcal{J}_N(u)$, for every $u \in K$, $u \neq u_{N,c}$, one would try to prove $\mathcal{J}_N(u_N)/R_N(u_N) \to 1$. However, $I_3(u_N) = \infty$, which follows from (2.4), (3.1) and

$$I_3(u_N) = 4\int \dot{v}_N(x)^2 \,\mathrm{d}x = 4\int |\hat{v}_N(p)|^2 p^2 \,\mathrm{d}p = 4N^{-1}\int |\hat{v}_1(p)|^2 p^2 \,\mathrm{d}p = \infty.$$

We therefore have to change slightly this simple idea for proving (1.4). Let $u_N(\varepsilon)$ be a regularization of u_N and let us consider

$$1 < \frac{\mathcal{J}_N(u_{N,c})}{R_N(u_N)} = \frac{R_N(u_N(\varepsilon))}{R_N(u_N)} \cdot \frac{\mathcal{J}_N(u_{N,c})}{R_N(u_N(\varepsilon))} < \frac{R_N(u_N(\varepsilon))}{R_N(u_N)} \cdot \frac{\mathcal{J}_N(u_N(\varepsilon))}{R_N(u_N(\varepsilon))}.$$

we intend to select a regularization $u_N(\varepsilon)$ with ε depending on N such that

$$\frac{R_N(u_N(\varepsilon))}{R_N(u_N)} = 1 + O\left(\frac{1}{N}\right), \tag{3.3}$$

$$\frac{\mathcal{J}_N(u_N(\varepsilon))}{R_N(u_N(\varepsilon))} = 1 + O\left(\frac{1}{N}\right). \tag{3.4}$$

Let us define the function

$$\hat{\varphi}(-p) = \hat{\varphi}(p) = \begin{cases} 1, & 0 \leqslant p < 1 \\ -4 + 12p - 9p^2 - 2p^3, & 1 \leqslant p < 2 \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

As $\dot{\hat{\varphi}}$ and $\ddot{\hat{\varphi}}$ exist as bounded functions with a compact support, the Fourier transform φ of $\hat{\varphi}$ has the following properties; φ is a symmetric, real function having derivatives of any order, $x^2\varphi(x)$ is bounded uniformly in x and integrable. The regularization of u_N is defined by $u_N(\varepsilon) = v_N(\varepsilon)^2$, where $v_N(\varepsilon, x) = N^{-1/4} \cdot s(\varepsilon, N^{-1/2}x)$, and $s(\varepsilon)$ is the convolution of $\varphi(\varepsilon, x) = \varepsilon^{-1}\varphi(x/\varepsilon)$ and s, i. e. $s(\varepsilon) = \varphi(\varepsilon) * s$. The properties of the function φ mentioned above imply

$$\begin{gathered} |s(\varepsilon, x)| < C_1(0) \\ \int |s(\varepsilon, x)|^n \,\mathrm{d}x < C_2(n), \quad n = 1, 2, \ldots, \end{gathered} \tag{3.6}$$

where $C_1(0)$, $C_2(n)$ do not depend on ε.

Now we easily obtain

$$\hat{v}_N(\varepsilon, p) = N^{1/4}\hat{\varphi}(\varepsilon N^{1/2} p)\,\hat{s}(N^{1/2} p) = N^{1/4}\hat{v}_1(\varepsilon, N^{1/2} p), \tag{3.7}$$

$$\hat{u}_N(\varepsilon, p) = \int \hat{\varphi}(\varepsilon q)\, \hat{\varphi}((N^{1/2} p - q)\,\varepsilon)\, \hat{s}(q)\, \hat{s}(N^{1/2} p - q)\, \mathrm{d}q =$$

$$\hat{u}_1(\varepsilon, N^{1/2} p).$$

Therefore

$$I_1(u_N(\varepsilon)) = N^{-1} \int s(\varepsilon, x)^6\, \mathrm{d}x,$$

$$I_2(u_N(\varepsilon)) = - \int |p|\, |\hat{u}_N(\varepsilon, p)|^2\, \mathrm{d}p = - N^{-1} \int |p|\, |\hat{u}_1(\varepsilon, p)|^2\, \mathrm{d}p, \tag{3.8}$$

$$I_3(u_N(\varepsilon)) = 4 \int v_N(\varepsilon, x)^2\, \mathrm{d}x = 4 \int p^2\, |\hat{v}_N(\varepsilon, p)|^2\, \mathrm{d}p =$$

$$= N^{-1} \int p^2\, |\hat{v}_1(\varepsilon, p)|^2\, \mathrm{d}p = N^{-1} \int p^2\, |\hat{s}(p)|^2\, |\hat{\varphi}(\varepsilon p)|^2\, \mathrm{d}p,$$

$$\langle V | u_N(\varepsilon) \rangle = - 2\pi\, \hat{\hat{u}}_N(\varepsilon, 0) = - 2\pi N\, \hat{\hat{u}}_1(\varepsilon, 0).$$

Proof of (3.3). We have $R_N(u_N(\varepsilon)) - R_N(u_N) = (\pi^2/24)\, N \cdot \int [\, s(\varepsilon, x)^6 - - s(x)^6]\, \mathrm{d}x - 2\pi N\, [\hat{\hat{u}}_1(\varepsilon, 0) - \hat{\hat{u}}_1(0)]$. Let us consider these two terms separately:

$$\left| \int [s(\varepsilon, x)^6 - s(x)^6]\, \mathrm{d}x \right| < \left| \int [s(\varepsilon, x)^2 - s(x)^2]\, \mathrm{d}x \right| \Big[\sum_{k=0}^{2}$$

$$\int s(\varepsilon, x)^{4-2k}\, s(x)^{2k}\, \mathrm{d}x \Big].$$

The second factor is bounded by a number C_1 uniformly in ε as follows from (3.6). Then (2.4) implies

$$\left| \int [s(\varepsilon, x)^6 - s(x)^6]\, \mathrm{d}x \right| < C_1 \int |\hat{\varphi}(\varepsilon p)^2 - 1|\, |\hat{s}(p)|^2\, \mathrm{d}p <$$

$$2 C_1\, \varrho_1^2 \int_{1/\varepsilon}^{\infty} p^{-5/2}\, \mathrm{d}p = \frac{4}{3}\, \varepsilon^{3/2}\, \varrho_1\, C_1\, C_2.$$

Next,

$$|\hat{\hat{u}}_1(\varepsilon, 0) - \hat{\hat{u}}_1(0)| = \left| \int \Big\{ \hat{s}(q)\, \hat{\hat{s}}(-q)\, [\hat{\varphi}(\varepsilon q)^2 - 1] + 2\varepsilon\, \hat{s}(q)\, \dot{\hat{s}}(-q) \cdot \right.$$

$$\left. \hat{\varphi}(\varepsilon q)\, \hat{\varphi}(-\varepsilon q) + \varepsilon^2\, |\hat{s}(q)|^2\, \hat{\varphi}(\varepsilon q)\, \hat{\varphi}(-\varepsilon q) \Big\}\, \mathrm{d}q. \right.$$

The last two terms in this expression tend towards zero at least as ε. To estimate the first term, let us mention first that the function $x \mapsto x^2 s(x)$ has the same singularities at $x = \pm 1$ as the function s. Therefore its Fourier transform behaves in the same way for large p as the transform of the function s. Hence $|\hat{\ddot{s}}(p)| = O(|p|^{-5/4})$. This implies

$$|\hat{\ddot{u}}_1(\varepsilon, 0) - \hat{\ddot{u}}_1(0)| < C_3\, \varepsilon.$$

We obtained

$$\left| \frac{R_N(u_N(\varepsilon))}{R_N(u_N)} - 1 \right| < \varepsilon\, C_4.$$

The asymptotic expression (3.3) follows if $\varepsilon(N)$ tends towards zero at least as $1/N$.

Proof. of (3.4). We have

$$\frac{\mathcal{J}_N(u_N(\varepsilon))}{R_N(u_N(\varepsilon))} = 1 - \frac{1}{8} \cdot \frac{NI_2(u_N(\varepsilon))}{R_N(u_N(\varepsilon))} + \frac{1}{32} \cdot \frac{I_3(u_N(\varepsilon))}{R_N(u_N(\varepsilon))}.$$

Because of (3.3), there exists a positive number C_1 such that $R_N(u_N(\varepsilon)) > C_1^{-1} N$ (for N large enough). Then (3.8) gives

$$a(N) = \frac{NI_2(u_N(\varepsilon))}{R_N(u_N(\varepsilon))} < C_1 N^{-1} \int |p|\, |\hat{u}_1(\varepsilon, p)|^2\, \mathrm{d}p.$$

Let us estimate the integral uniformly in ε. From (3.7) and (2.4) we have

$$|\hat{u}_1(\varepsilon, p)| < C_2 \int \frac{1}{1 + |q|^\sigma} \cdot \frac{1}{1 + |p - q|^\sigma}\, \mathrm{d}q =$$

$$C_2 \int \frac{1}{1 + \left|\frac{p - q}{2}\right|^\sigma} \cdot \frac{1}{1 + \left|\frac{p + q}{2}\right|^\sigma}\, \mathrm{d}q,$$

where $\sigma = 1 + 1/4$. Now it is easy to obtain an upper bound on $\hat{u}_1$:

$$(1 + |p|)^\sigma\, |\hat{u}_1(\varepsilon, p)| < C_3.$$

Hence $a(N) < C_1 C_3 N^{-1} \int |p|\, (1 + |p|)^{-5/2}\, \mathrm{d}p = O(1/N)$.

Again using (3.3), (3.8) and (2.4) we have

$$b(N) = \frac{I_3(u_N(\varepsilon))}{R_N(u_N(\varepsilon))} < 4 N^{-2} C_1 \int p^2\, |\hat{s}(p)|^2 |\hat{\varphi}(\varepsilon p)|^2\, \mathrm{d}p <$$

$$4N^{-2} \varrho_1^2 C_1 \int \frac{\hat{\varphi}(\varepsilon p)^2}{[1 + |p|]^{1/2}}\, \mathrm{d}p < 8\varepsilon^{-1/2} N^{-2} \varrho_1^2 C_1 \int_0^2 \frac{\mathrm{d}p}{\sqrt{\varepsilon + p}} < \frac{C_2}{\varepsilon N^2}.$$

Hence $b(N) = O(1/N)$ if $N\varepsilon$ is bounded from bellow. This condition on $\varepsilon(N)$ is compatible with the condition on $\varepsilon(N)$ implying (3.3). We conclude that the chosen regularization $u_N(\varepsilon)$ with $\varepsilon(N) = 1/N$ gives (3.3) and (3.4).

4. Conclusion

It would be interesting to see whether the asymptotic behaviour (1.4) holds for the symplectic case as the more important one. However, in the case of symplectic symmetry, the functional (1.2) can have undesired features. For a class of potentials, the minimization problem (1.3) can have no solution, as has been shown in Ref 4. In this case the kinetic term is essential in studying problem (1.3) irrespectively of its small order in $1/N$. If an approximation of the form

$$\int \varrho(x) \left[\int \varrho(y)\, \chi(x-y)\, dy\right]^2 dx$$

of the kinetic term is known, where χ depends on N, then the asymptotic behaviour (1.4) can be obtained even for symplectic case.

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LIMES ZA VELIKI N U MODELU S KOLEKTIVNOM VARIJABLOM

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Uspoređeni su limesi za veliki N potpunog i reduciranog efektivnog potencijala u jednodimenzionalnom modelu teorije polja s kolektivnim varijablom i ortogonalnom simetrijom.