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FROM RELATIVE ENTROPY TO ENTROPY

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We give an intrinsic definition of the entropy of a state over an algebra and show that it has the general properties of von Neumann's entropy.

1. Introduction

The contents of information of a normal state ω over $\mathscr{B}(\mathscr{H})$, the bounded operators on a Hilbert space $\mathscr{H}$, is measured by von Neumann's entropy (for further details see Refs. 1—3):

$$S_{\mathscr{B}(\mathscr{H})}(\omega) = -\operatorname{Tr}\omega\ln\omega.$$

Here the density matrix corresponding to ω is also denoted by ω and Tr is the trace in $\mathscr{H}$. The usefulness of this expression is limited by the fact that in quantum statistics one mainly deals with reducible representations of the algebra of observables which is thus only a subalgebra of $\mathscr{B}(\mathscr{H})$. For instance, in the G. N. S.-representation of ω this state is represented by a vector of $\mathscr{H}$ and the above expression gives $S = 0$. This simply means that this vector contains maximal information on all of $\mathscr{B}(\mathscr{H})$. Thus one needs an intrinsic definition of $S_{\mathscr{A}}(\omega)$ referring only to $\mathscr{A}$ and ω and not to a representation of $\mathscr{A}$. In the following we shall give such a definition and show that it enjoys the important properties of von Neumann's entropy[5,6]. We shall proceed via the relative entropy for which Araki[4] gave an intrinsic definition. We shall generalize this slightly to the relative entropy of linear subspaces of $\mathscr{A}$ and show that it is monotonic. This will be the key feature from which all other properties follow. The entropy is intrinsically defined as the convex combination of the relative entropies of ω relative to the components of its extremal decomposition.

This paper is written in memoriam Jurko Glaser. As it does not contain any essentially new result it is only a modest tribute to this man of great scientific discipline. However, we shall bring considerable technical simplifications in the proofs of the properties of the entropies. They arise from a monotonicity inequality which might seem to go in the wrong direction. We feel that V. Glaser might have enjoyed this kind of surprise.

2. *The entropies and their properties*

Throught this paper we shall consider states over a von Neumann algebra $\mathscr{A}$ which has the U. H. F.-property $\mathscr{A} = \{\bigcup \mathscr{A}_l\}''$ where $\mathscr{A}_l$ are finite-dimensional subalgebras. Most of the results can be extended to more general von Neumann algebras but for the applications to physics U. H. F.-algebras seem general enough.

Definition (1)

Let ω and φ be normal positive linear functionals over a U. H. F.-algebra $\mathscr{A}$ such that $\varphi(A^* A) = 0 \Rightarrow \omega(A^* A) = 0$. Then

$$\varphi(A \Delta_{\mathscr{A}}(\omega \mid \varphi) B) = \omega(B A) \quad \forall A, B \in \mathscr{A}$$

defines a positive, densely defined quadratic form in $\mathscr{H}_\varphi$, the Hilbert space of the G. N. S.-representation of $\mathscr{A}$ based on φ. Since it is closable it corresponds to a unique positive operator, the relative modular operator, also denoted by $\Delta_{\mathscr{A}}(\omega \mid \varphi)$. The relative modular operator with respect to a linear subspace $\mathscr{B} \in \mathscr{A}$, $\mathbf{1} \in \mathscr{B}$ is defined by

$$\Delta_{\mathscr{B}}(\omega \mid \varphi) = P_{\mathscr{B}} \Delta_{\mathscr{A}}(\omega \mid \varphi) P_{\mathscr{B}}$$

where $P_{\mathscr{B}}$ is the projector onto the subspace of $\mathscr{H}_\varphi$ generated by $\mathscr{B}$, i. e. $\overline{\mathscr{B}\Phi}$.

Remarks (2)

a) For the proof of closability, see Ref. 1, Eq. (2.5.33).

b) If $\mathscr{B}$ is a subalgebra of $\mathscr{A}$ then $\Delta_{\mathscr{B}}(\omega \mid \varphi)$ is in $P_{\mathscr{B}}\mathscr{H}_\varphi$ just the relative modular operator constructed with the restrictions of ω and φ to $\mathscr{B}$. The U. H. F.-property guarantees that there is a sequence $\mathscr{B}_l$ of finite-dimensional subalgebras such that $P_{\mathscr{B}_l} \uparrow \mathbf{1}$.

c) (1) is equivalent to $\varphi(A^* \Delta A) = \omega(AA^*)\ \forall$ A $\in \mathscr{A}$.

d) Δ belongs only in the abelian case to $\mathscr{A}$, so that $\varphi(\Delta)$ is strictly speaking not defined. We mean, of course $\langle \Phi \mid \Delta \mid \Phi \rangle$ where $|\Phi\rangle$ is the vector associated to φ in the G. N. S.-construction. For simplicity we shall sometimes use this notation.

Properties of the relative modular operator (3)

a) $\Delta_{\mathscr{A}}(\omega \mid \varphi) \geqslant 0$

b) $\varphi(\Delta_{\mathscr{A}}(\omega \mid \varphi)) = \omega(1)$

c) $\Delta_{\mathscr{A}}(\sum \lambda_i \omega_i \mid \varphi) = \sum \lambda_i \Delta_{\mathscr{A}}(\omega_i \mid \varphi)$.

d) If $\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2$, $\omega = \omega_1 \otimes \omega_2$, $\varphi = \varphi_1 \otimes \varphi_2$ then

$$\Delta_{\mathscr{A}_1 \otimes \mathscr{A}_2}(\omega_1 \otimes \omega_2 \mid \varphi_1 \otimes \varphi_2) = \Delta_{\mathscr{A}_1}(\omega_1 \mid \varphi_1) \otimes \Delta_{\mathscr{A}_2}(\omega_2 \mid \varphi_2).$$

e) $\Delta_{\mathscr{A}}(\alpha\omega \mid \beta\varphi) = \frac{\alpha}{\beta}\Delta_{\mathscr{A}}(\omega \mid \varphi) \quad \forall\, \alpha, \beta \in \mathbf{R}^+$.

Proof

These properties follow directly from (1). For d) one has to remember that for a product state $\mathscr{H}_\varphi$ is a tensor product.

Definition (4)

The relative entropy of two normal positive linear functionals ω and φ over a U. H. F. -algebra $\mathscr{A}$ is defined by

$$S_{\mathscr{A}}(\omega \mid \varphi) = \lim_{\varepsilon \downarrow 0} - \langle \Phi \mid \ln \frac{\varepsilon + \Delta_{\mathscr{A}}(\omega \mid \varphi)}{1 + \varepsilon \Delta_{\mathscr{A}}(\omega \mid \varphi)} \mid \Phi \rangle$$

where $\Phi \in \mathscr{H}_\varphi$ is the vector corresponding to φ.

Remarks (5)

a) Let $\{\omega_i\}$ and $\{\varphi_i\} \in \mathbf{R}^n_+$ define faithful states over the algebra C^n.

Then

$$\Delta_{\mathscr{A}}(\omega \mid \varphi) = \left\{\frac{\omega_i}{\varphi_i}\right\}$$

and

$$S_{\mathscr{A}}(\omega \mid \varphi) = \sum_i \varphi_i (\ln \varphi_i - \ln \omega_i).$$

Similarly one can show[1] that for $\mathscr{A} = \mathscr{B}(C^n)$ $S_{\mathscr{A}}(\omega \mid \varphi) = \mathrm{Tr}\, \varphi (\ln \varphi - \ln \omega)$ if the density matrices are denoted by the same letter as the states.

b) Since $\varepsilon \leqslant \frac{\varepsilon + x}{1 + \varepsilon x} \leqslant \frac{1}{\varepsilon}$ for $x \in \mathbf{R}^+$ the expectation value is finite but the limit may be $+\infty$.

Properties of the relative entropy (6)

a) Monotonicity: $S_{\mathscr{B}}(\omega \mid \varphi) \leqslant S_{\mathscr{A}}(\omega \mid \varphi)$ if $\mathscr{B} \subset \mathscr{A}$.

b) Additivity: If ω_i and φ_i are states (i. e. $\omega_i(1) = \varphi_i(1) = 1$) then

$$S_{\mathscr{A}_1 \times \mathscr{A}_2}(\omega_1 \otimes \omega_2 \mid \varphi_1 \otimes \varphi_2) = S_{\mathscr{A}_1}(w_1 \mid \varphi_1) + S_{\mathscr{A}_2}(\omega_2 \mid \varphi_2).$$

c) Positivity: If ω and φ are states then $S_{\mathscr{A}}(\omega \mid \varphi) \geqslant 0$ with $= 0$ iff $\omega = \varphi$.

d) Convexity: $(\omega, \varphi) \to S_{\mathscr{A}}(\omega \mid \varphi)$ is convex.

e) Semicontinuity: $(\omega \mid \varphi) \to S_{\mathscr{A}}(\omega \mid \varphi)$ is weakly lower semicontinuous.

f) $S_{\mathscr{A}}(a\,\omega \mid \beta\,\varphi) = \beta\, S_{\mathscr{A}}(\omega \mid \varphi) - \beta\,\varphi(1) \ln \frac{a}{\beta} \quad \forall\, a, \beta \in \mathbf{R}^+$.

Proof

a) As the vector corresponding to φ is contained in $P_{\mathscr{B}} \mathscr{H}_{\varphi}$ it suffices to observe that

$$P_{\mathscr{B}} \left(\ln \frac{\varepsilon + P_{\mathscr{B}} \Delta P_{\mathscr{B}}}{1 + \varepsilon P_{\mathscr{B}} \Delta P_{\mathscr{B}}} - \ln \frac{\varepsilon + \Delta}{1 + \varepsilon \Delta} \right) P_{\mathscr{B}} = \int\limits_{\varepsilon}^{1/\varepsilon} \mathrm{d}a\, P_{\mathscr{B}} [(a + \Delta)^{-1} -$$
$$- (P_{\mathscr{B}} (a + \Delta) P_{\mathscr{B}})^{-1}]\, P_{\mathscr{B}} = \int\limits_{\varepsilon}^{1/\varepsilon} \mathrm{d}\, a\, P_{\mathscr{B}} (a + \Delta)^{-1} (1 - P_{\mathscr{B}}) [(1 -$$
$$- P_{\mathscr{B}})(a + \Delta)^{-1} (1 - P_{\mathscr{B}})]^{-1} (1 - P_{\mathscr{B}})(a + \Delta)^{-1} P_{\mathscr{B}} \geqslant 0. \tag{7}$$

Here we used the decomposition

$$PA^{-1}P = P(PAP)^{-1}P + PA^{-1}(1 - P)[(1 - P)A^{-1}(1 - P)]^{-1}(1 - P)A^{-1}P$$

of the inverse of an operator.

b) Follows from (3d) as $\langle \Phi_1 \otimes \Phi_2 \mid \ln(\Delta_1 \otimes \Delta_2) \mid \Phi_1 \otimes \Phi_2 \rangle = \langle \Phi_1 \mid \ln \Delta_1 \mid \Phi_1 \rangle \cdot$ $\cdot \langle \Phi_2 \mid 1 \mid \Phi_2 \rangle + \langle \Phi_1 \mid 1 \mid \Phi_1 \rangle \langle \Phi_2 \mid \ln \Delta_2 \mid \Phi_2 \rangle$.

c) Positivity follows from monotonicity since for the smallest algebra $\mathscr{B}_0 = \{a\, \mathbf{1}\}$ we have $P_{\mathscr{B}_0} \Delta P_{\mathscr{B}_0} = P_{\mathscr{B}_0} = |\Phi\rangle \langle \Phi|$ and

$$- \lim_{\varepsilon \downarrow 0} \langle \Phi \mid \ln \frac{\varepsilon + |\Phi\rangle \langle \Phi|}{1 + \varepsilon |\Phi\rangle \langle \Phi|} |\Phi\rangle = 0.$$

Thus $S_{\mathscr{A}}(\omega \mid \varphi) \geqslant S_{a1}(\omega \mid \varphi) = 0$. According to Eq. (7) equality holds only if for almost all a

$$P_{\mathscr{B}_0} (a + \Delta)^{-1} (1 - P_{\mathscr{B}_0}) [(1 - P_{\mathscr{B}_0})(a + \Delta)^{-1} (1 - P_{\mathscr{B}_0})]^{-1} (1 - P_{\mathscr{B}_0})(a +$$
$$+ \Delta)^{-1} P_{\mathscr{B}_0} = 0 \Rightarrow P_{\mathscr{B}_0} (a + \Delta)^{-1} (1 - P_{\mathscr{B}_0}) = 0.$$

Thus either $\mathscr{A} = \mathscr{B}_0 = a\, \mathbf{1}$ in which case $\omega = \varphi$ since on $a\, \mathbf{1}$ all states are equal. Otherwise $(a + \Delta)^{-1}$ and therefore Δ have to commute with $P_{\mathscr{B}_0}$: Hence $\Delta \mid \Phi\rangle = \Delta P_{\mathscr{B}_0} \mid \Phi\rangle = c \mid \Phi\rangle$ and $\langle \Phi \mid \ln \Delta \mid \Phi \rangle = 0$ implies $c = 1$. Then $\omega(A) = \langle \Phi \mid A \Delta \Phi \rangle = \langle \Phi \mid A \mid \Phi \rangle = \Phi(A)$. Thus $S_{\mathscr{A}}(\omega \mid \varphi) = 0 \Rightarrow \omega = \varphi$.

d) Consider the following functionals over $\mathscr{A} \otimes \mathscr{B}(C^2)$ (i. e. matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in \mathscr{A}$)

$$W\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \left\langle \begin{matrix} \sqrt{\lambda}\,\Omega_1 \\ \sqrt{1-\lambda}\,\Omega_2 \end{matrix} \middle| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \begin{matrix} \sqrt{\lambda}\,\Omega_1 \\ \sqrt{1-\lambda}\,\Omega_2 \end{matrix} \right\rangle$$

$$F\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \left\langle \begin{matrix} \sqrt{\lambda}\,\Phi_1 \\ \sqrt{1-\lambda}\,\Phi_2 \end{matrix} \middle| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \begin{matrix} \sqrt{\lambda}\,\Phi_1 \\ \sqrt{1-\lambda}\,\Phi_2 \end{matrix} \right\rangle$$

where Ω_i (resp. Φ_i) are the cyclic vectors associated with ω_i (resp. φ_i). Restricting to the subalgebra $\mathscr{A} \otimes \mathbf{1}$ (i. e. matrices of the form $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$), we have

$$W_{|\mathscr{A}\otimes 1} = \lambda\,\omega_1 + (1-\lambda)\,\omega_2, \quad F_{|\mathscr{A}\otimes 1} = \lambda\,\varphi_1 + (1-\lambda)\,\varphi_2$$

and thus

$$S_{\mathscr{A}\otimes 1}(W \mid F) = S_{\mathscr{A}}(\lambda\,\omega_1 + (1-\lambda)\,\omega_2 \mid \lambda\,\varphi_1 + (1-\lambda)\,\varphi_2).$$

Restricting to the subalgebra $\mathscr{A} \oplus \mathscr{A}$ (i. e. of the form $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$),we infer from

$$\left\langle \begin{matrix} \sqrt{\lambda}\,\Phi_1 \\ \sqrt{1-\lambda}\,\Phi_2 \end{matrix} \middle| \begin{pmatrix} A^* & 0 \\ 0 & B^* \end{pmatrix} \Delta_{\mathscr{A}\oplus\mathscr{A}}(W \mid F) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \begin{matrix} \sqrt{\lambda}\,\Phi_1 \\ \sqrt{1-\lambda}\,\Phi_2 \end{matrix} \right\rangle =$$

$$= \left\langle \begin{matrix} \sqrt{\lambda}\,\Omega_1 \\ \sqrt{1-\lambda}\,\Omega_2 \end{matrix} \middle| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} A^* & 0 \\ 0 & B \end{pmatrix} \middle| \begin{matrix} \sqrt{\lambda}\,\Omega_1 \\ \sqrt{1-\lambda}\,\Omega_2 \end{matrix} \right\rangle$$

that

$$\Delta_{\mathscr{A}\oplus\mathscr{A}}(W \mid F) = \begin{pmatrix} \Delta_{\mathscr{A}}(\omega_1 \mid \varphi_1) & 0 \\ 0 & \Delta_{\mathscr{A}}(\omega_2 \mid \varphi_2) \end{pmatrix}$$

and thus

$$F(-\ln \Delta_{\mathscr{A}\oplus\mathscr{A}}(W \mid F)) = -\lambda\,\varphi_1(\ln \Delta_{\mathscr{A}}(\omega_1 \mid \varphi_1)) - (1-\lambda)\,\varphi_2(\ln \Delta_{\mathscr{A}}(\omega_2 \mid \varphi_2)) =$$

$$= \lambda\, S_{\mathscr{A}}(\omega_1 \mid \varphi_1) + (1-\lambda)\, S_{\mathscr{A}}(\omega_2 \mid \varphi_2).$$

Convexity now follows from monotonicity since $\mathscr{A} \oplus \mathscr{A} \supset \mathscr{A} \otimes \mathbf{1}$.

e) Using the inequality $P(PA^{-1}P)^{-1}P < PAP$ which follows from the decomposition used in the proof of a) we see

$$\langle \Phi \mid \ln(\varepsilon + P_{\mathscr{B}}\Delta P_{\mathscr{B}})(1 + \varepsilon P_{\mathscr{B}}\Delta P_{\mathscr{B}})^{-1} - \ln(\varepsilon + \Delta)(1 + \varepsilon\Delta)^{-1} \mid \Phi \rangle <$$

$$< \int_{\varepsilon}^{\varepsilon^{-1}} da \langle \Phi \mid (a + \Delta)^{-1}(1 - P_{\mathscr{B}})(a + \Delta)(1 - P_{\mathscr{B}})(a + \Delta)^{-1} \mid \Phi \rangle.$$

For fixed a and $\mathscr{B} \uparrow \mathscr{A}$ this is the expectation value of a decreasing sequence of bounded operators with infimum zero. A non-zero infimum i would satisfy $P_{\mathscr{B}}(a + \Delta)\, i\, (a + \Delta)\, P_{\mathscr{B}} = 0$ for all $\mathscr{B} \uparrow \mathscr{A}$ which contradicts $P_{\mathscr{B}} \uparrow \mathbf{1}$. Thus by Vigier's theorem the operators converge weakly to zero and therefore also the expectation value. Thus $S_{\mathscr{A}} = \sup\limits_{i} \sup\limits_{\varepsilon} S_{\mathscr{B}_i}(\varepsilon)$ where $\mathscr{B}_i$ is a dense set of finite-dimensional subalgebras. For those $S_{\mathscr{B}_i}(\varepsilon)$ is continuous and $S_{\mathscr{A}}$ as sup over continuous functionals is lower semicontinuous.

f) Follows from (3e).

Remarks (8)

ad a) One might think that $\Delta \geqslant P \Delta P$ and this would imply $-\ln \Delta \leqslant - \ln P\Delta P \Rightarrow S_{\mathscr{A}} \leqslant S_{\mathscr{B}}$. We leave it to the reader to find the flaw in this argument, the correct inequality goes the other way. For density matrices it implies $S(\omega \mid \varphi) \geqslant S(\omega_1 \mid \varphi_1)$ if ω and φ act in $\mathscr{H}_1 \otimes \mathscr{H}_2$ and ω_1 (resp. $\varphi_1) = \mathrm{Tr}_{\mathscr{H}_2}(\omega)$ (resp. $\mathrm{Tr}_{\mathscr{H}_2}(\varphi)$).

ad b) Together with c) it implies $S_{\mathscr{A}_1 \otimes \mathscr{A}_2}(\omega_1 \otimes \nu \mid \varphi_1 \otimes \nu) = S_{\mathscr{A}_1}(\omega_1 \mid \varphi_1)$.

ad c) $S_{\mathscr{A}}(\omega \mid \varphi) = 0 \Leftrightarrow \omega = \varphi$ shows roughly speaking that $S_{\mathscr{A}}(\omega \mid \varphi)$ is not strictly convex in both directions but only in one.

Definition (9)

The entropy of a normal state ω over a U. H. F.-algebra $\mathscr{A}$ is defined dy

$$S_{\mathscr{A}}(\omega) = \sup_{\substack{\sum \lambda_i \omega_i = \omega \\ i}} \sum_i \lambda_i S_{\mathscr{A}}(\omega \mid \omega_i)$$

where $\lambda_i > 0$, $\sum \lambda_i = 1$, and ω, ω_i are considered as states over $\mathscr{A}$.

Remark (10)

For $\mathscr{A} = C^n$, $\omega = \{\omega_j\}$ the entropy becomes

$$S_{C^n}(\omega) = \sup_{\sum \lambda_i \omega_{ij} = \omega_j} \sum_j \sum_i \lambda_i \omega_{ij} (\ln \omega_{ij} - \ln \omega_j) = - \sum_j \omega_j \ln \omega_j$$

by letting ω_{ij} approach δ_{ij} and λ_i approach ω_i. Similarly for $\mathscr{A} = \mathscr{B}(C^n)$ the sup of $\mathrm{Tr} \sum \lambda_i \omega_i (\ln \omega_i - \ln \omega)$ is obtained by decomposing ω into pure states. Then it becomes $-\mathrm{Tr}\, \omega \ln \omega$. In contradistinction to the relative entropy it remains finite for $n < \infty$ even if ω is not faithful. In the continuous classical case $\mathscr{A} = C([0, 1])$ the relative entropy

$$\int_0^1 dx\, \omega_i(x) (\ln \omega_i(x) - \ln \omega(x))$$

approaches ∞ if ω_i becomes pure (i. e. a δ-function). Thus $S_{\mathscr{A}}(\omega) = +\infty$.

Properties of the entropy (11)

a) Positivity: $S_{\mathcal{A}}(\omega) \geqslant 0, = 0$ iff ω is pure.

b) Strong subadditivity:

$$S_{\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3}(\omega) + S_{\mathcal{A}_2}(\omega_{|\mathcal{A}_2}) \leqslant S_{\mathcal{A}_1 \otimes \mathcal{A}_2}(\omega_{|\mathcal{A}_1 \otimes \mathcal{A}_2}) + S_{\mathcal{A}_2 \otimes \mathcal{A}_3}(\omega_{|\mathcal{A}_2 \otimes \mathcal{A}_3}).$$

c) Strong concavity:

$$\omega \to S_{\mathcal{A}_1 \otimes \mathcal{A}_2}(\omega) - S_{\mathcal{A}_1}(\omega_{|\mathcal{A}_1})$$

is concave.

d) Semicontinuity: $\omega \to S_{\mathcal{A}}(\omega)$ is weakly lower semicontinuous.

Proof

a) $S_{\mathcal{A}}(\omega) \geqslant 0$ since $S_{\mathcal{A}}(\omega \mid \omega_i) \geqslant 0$. If ω is pure then $\omega_i = \omega$ and $S_{\mathcal{A}}(\omega \mid \omega) = 0$. If one of the $\omega_i \neq \omega$ then $S_{\mathcal{A}}(\omega \mid \omega_i) > 0$ according to (6c).

b) We shall assume that all entropies are finite so that they can be approximated within ε by entropies of finite-dimensional algebras $\bar{\mathcal{A}}$. Let $\omega_{|\bar{\mathcal{A}}_1 \otimes \bar{\mathcal{A}}_2}$ be given by the density matrix ϱ and $\omega_{|\bar{\mathcal{A}}_1}$ by ϱ_1. Then

$$S_{\bar{\mathcal{A}}_1}(\omega_{|\mathcal{A}_1}) - S_{\bar{\mathcal{A}}_1 \otimes \bar{\mathcal{A}}_2}(\omega_{|\bar{\mathcal{A}}_1 \otimes \bar{\mathcal{A}}_2}) = \mathrm{Tr}\, \varrho \ln \varrho - \mathrm{Tr}_1\, \varrho_1 \ln \varrho_1 = \mathrm{Tr}\, \varrho (\ln \varrho -$$

$$- \ln \varrho_1 \otimes \mathbf{1}) = S_{\bar{\mathcal{A}}_1 \otimes \bar{\mathcal{A}}_2}(\omega_{|\mathcal{A}_1} \otimes \mathrm{Tr}_2 \mid \omega).$$

Here Tr_2 is the trace in $\mathscr{H}_2$ (which is not a state, $\mathrm{Tr}\, \mathbf{1} = \dim \mathscr{H}_2$). Thus strong subadditivity is equivalent to

$$S_{\bar{\mathcal{A}}_1 \otimes \bar{\mathcal{A}}_2 \otimes \bar{\mathcal{A}}_3}(\mathrm{Tr}_1 \otimes \omega_{|\bar{\mathcal{A}}_2 \otimes \bar{\mathcal{A}}_3} \mid \omega) \geqslant S_{\bar{\mathcal{A}}_1 \otimes \bar{\mathcal{A}}_2}(\mathrm{Tr}_1 \otimes \omega_{|\bar{\mathcal{A}}_2} \mid \omega_{\bar{\mathcal{A}}_2 \otimes \bar{\mathcal{A}}_3})$$

which follows from monotonicity.

c) Here again we may approximate by finite-dimensional algebras $\bar{\mathcal{A}}_i$. Since

$$S_{\bar{\mathcal{A}}_1}(\omega_{|\mathcal{A}_1}) - S_{\bar{\mathcal{A}}_1 \otimes \bar{\mathcal{A}}_2}(\omega) = S_{\bar{\mathcal{A}}_1 \otimes \bar{\mathcal{A}}_2}(\omega_{|\bar{\mathcal{A}}_1} \otimes \mathrm{Tr}_2 \mid \omega)$$

concavity follows from the joint convexity of the relative entropy.

d) A state ω_i in the decomposition of ω can be written $\omega_i(a) = \omega(aa'_i/\omega(a'_i)) \equiv$ $\equiv \omega_{a'_i}(a)$ where $a'_i \in \mathcal{A}'$, $0 \leqslant a'_i < 1$, $\sum\limits_i a'_1 = 1$, $\omega(a'_i) \neq 0$ as $|\Omega\rangle$ is cyclic for $\mathcal{A}$ and thus separating for $\mathcal{A}'$. Thus $\sup\limits_{\Sigma \lambda_i \omega_i = \omega}$ can be replaced by

$$\sup_{\substack{0 \leqslant a' \in \mathcal{A}' \\ \Sigma a' = 1}} \sum \omega(a'_i)\, S_{\mathcal{A}}(\omega \mid \omega_{a'_i}) = S_{\mathcal{A}}(\omega)$$

such that the sup is over a set independent of ω. Now $S_{\mathscr{A}}(\omega \,|\, \omega_{a_i}')$ is weakly lower semicontinuous by (6e). $S_{\mathscr{A}}(\omega)$ as sup over such functionals is therefore lower semicontinuous too.

Remarks (12)

1) $S_{\mathscr{B}}(\omega)$ is not monotonic in $\mathscr{B} \subset \mathscr{A}$.

$$\bar{S}_{\mathscr{B}} \equiv \sup_{\Sigma \lambda_i \omega_{i|\mathscr{A}} = \omega_{|\mathscr{A}}} \sum_i \lambda_i S(\omega_{|\mathscr{B}} \,|\, \omega_{i_{\mathscr{B}}})$$

would be monotonic but not subadditive. Note that in $S_{\mathscr{B}}(\omega)$ the sup is taken over $\sum \lambda_i \omega_{i_{|\mathscr{B}}} = \omega_{|\mathscr{B}}$ such that $\bar{S} < S$.

2) b) and c) tell us that though

$$\bar{\bar{S}}_{\mathscr{A}_1} \equiv S_{\mathscr{A}_1 \otimes \mathscr{A}_2} - S_{\mathscr{A}_2}$$

is not necessarily positive it is concave and subadditive. Note that for $\mathscr{A}_2 = \alpha \mathbf{1}$ $\bar{\bar{S}} = S$ and therefore our proof implies concavity and subadditivity of $S_{\mathscr{A}}(\omega)$.

3) Because of $S_{\mathscr{A}_1}(\omega_{|\mathscr{A}_1}) + S_{\mathscr{A}_2}(\omega_{|\mathscr{A}_2}) - S_{\mathscr{A}_1 \otimes \mathscr{A}_2}(\omega) = S_{\mathscr{A}_1 \otimes \mathscr{A}_2}(\omega_{|\mathscr{A}_1} \otimes \omega_{|\mathscr{A}_2} \,|\, \omega)$ the strong subadditivity of $S(\omega)$ is equivalent to the following subadditivity of the relative entropy

$$S_{\mathscr{A}_1 \otimes \mathscr{A}_2 \otimes \mathscr{A}_3}(\omega_{|\mathscr{A}_1} \otimes \omega_{|\mathscr{A}_2} \otimes \omega_{|\mathscr{A}_3} \,|\, \omega) < S_{\mathscr{A}_1 \otimes \mathscr{A}_2 \otimes \mathscr{A}_3}(\omega_{|\mathscr{A}_1 \otimes \mathscr{A}_2} \otimes \omega_{|\mathscr{A}_3} \,|\, \omega) +$$
$$+ S_{\mathscr{A}_1 \otimes \mathscr{A}_2 \otimes \mathscr{A}_3}(\omega_{|\mathscr{A}_1} \otimes \omega_{|\mathscr{A}_2 \otimes \mathscr{A}_3} \,|\, \omega).$$

Note added in proof. Dr. D. Petz has kindly informed us that some of the results of this paper are contained in D. Petz, Properties of Quantum Entropy, preprint, Budapest 1984.

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OD RELATIVNE ENTROPIJE DO ENTROPIJE

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Originalni znanstveni rad

Ovdje dajemo intrinsičnu definiciju entropije stanja nad algebrom i pokazujemo da ona ima opća svojstva von Neumannove entropije.