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ASPECTS OF QUANTUM FIELD THEORIES AT FINITE TEMPERATURES

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After general remarks on recent developments concerning quantum field theory at finite temperature, we discuss various aspects of supersymmetric theories in comparison with the case of ordinary symmetries: Ward-Takahashi identities, Goldstone modes, order parameters, breaking of supersymmetry.

1. Introduction

Our friend and colleague Jurko Glaser left us last year, a very sad loss in our Theory Division at CERN and a great loss also to theoretical physics, where he was one of the best experts in the foundations of quantum field theory. When asked to contribute to this commemorative volume, I selected a topic related to a new line of development of quantum field theory, which began to grow in importance in the early seventies and has recently become a very active field of research. The domain in question concerns the thermodynamics of relativistic quantum field theories, i. e., the systematic study of relativistic quantum fields at finite temperature.

To a large extent, the present interest in these problems originated with the realization that modern field theories of the fundamental particles and interactions have remarkable implications for early cosmology, in particular phase transitions which correspond to changes of the symmetry breaking patterns of the fields as the expanding Universe cooled down from the very high temperatures believed to have briefly prevailed according to the hot big bang model[1]. There followed an intense activity at the interface between particle physics and early cosmology. Most papers made use of the finite temperature formalisms published in 1974 by Bernard, Dolan, Jackiw and Weinberg[2]. These methods, based on the Matsubara imaginary time formalism, are adequate for the calculation of static properties. As already mentioned by Dolan and Jackiw, their application to real-time properties presents difficulties, in the sense that without proper care, they lead to inadmissible singularities of type $[\delta(k^2 - m^2)]^N, N > 2$.

A more advanced real-time formalism was developed since 1975 by Takahashi, Umezawa and their collaborators[3]. It is called Thermo Field Dynamics (*TFD*) and its essential ingredients are:

(i) a doubling of the number of field operators and a corresponding enlargement of Hilbert space,

(ii) an expression of thermal averages as expectation values for a »temperature-dependent vacuum state«.

TFD allows a systematic perturbative calculation of Green functions in real time and, as shown more recently[4], can also be used to compute the free energy, be it in a more indirect way. It is in our opinion quite remarkable that point (i) above is the perturbative counterpart of a similar doubling discovered by Haag, Hugenholtz and Winnink[5] in the C* algebra approach to quantum statistical mechanics. This analogy was pointed out by Ojima[6] and would probably deserve to be analyzed further, because operator doubling is not an inescapable feature of finite temperature quantum systems but a consequence of the mathematical approach selected for their description [C* algebra in Ref. 5, representation of thermal averages as expectation values for a single temperature-dependent vacuum state in *TFD*]. After all, ordinary statistical mechanics can be and has been fully developed without doubling. It is not yet clear that this cannot be done for field theory.

It should also be noted that existing real-time and imaginary-time perturbative treatments of finite temperature field theory still lead to puzzling mathematical features. This was pointed out recently by Fujimoto[7] in the case of one-loop diagrams for scalar and fermion propagators (see note added in proof). Also renormalization questions and gauge invariance questions can be quite difficult in finite temperature calculations. Although there is no reason to expect any of these difficulties to be of fundamental nature, more work is needed before one can claim to have a good understanding of quantum field theories at positive temperature.

Another line of work which developed recently, lattice field theory, is also applicable at finite temperature and has given important results in quantum chromodynamics. As of now, it is essentially limited to the imaginary-time formalism.

In the present paper we concentrate on a somewhat special but quite interesting problem, the behaviour of supersymmetry (*SUSY*) at $T > 0$. Since *SUSY* relates bosons to fermions, its behaviour at finite temperature must obviously be affected by the difference between Bose-Einstein and Fermi-Dirac statistics. Hence, the problems of *SUSY* breaking must be significantly different at $T = 0$ and $T > 0$. Starting with the original papers of Das and Kaku[8] and Girardello et al.[9], much has been written on this subject[10,11]. While progress has been made by studying the manifestations of $SUSY$ at $T > 0$ in special models, the general situation is not fully clarified. As illustrated by recent papers of Fuchs[12] and Midorikawa[13], the nature of $SUSY$ breaking at $T > 0$ is still under debate and there is also a misconception on the use of the Klein operator $K = (-1)^F$, $F =$ fermion number. Our aim is to explore some of these matters in a model-independent way, using as a guide the comparison between $SUSY$ and the case of ordinary symmetries.

In Section 2 we discuss Ward-Takahashi identities and Goldstone modes at $T > 0$ for symmetry and $SUSY$. In Section 3, finite temperature $SUSY$ is discussed as a one-to-one relation between ensembles of bosonic and fermionic states. The question of $SUSY$ order parameter is also addressed and found to be quite different from the familiar situation for ordinary symmetries. The last section is devoted to concluding remarks.

2. *Thermal Goldstone modes*

In our view, the most interesting recent advance concerning finite temperature $SUSY$ has been in the study of supersymmetric Ward-Takahashi (WT) identities by Matsumoto et al.[14] and by Aoyama and Boyanovski[15]. These authors discovered that the WT identities of the Wess-Zumino model[16] contain singularities at four-momentum zero. The singularities only exist at $T > 0$ and are linked to transitions in the thermal distribution of real particles where a boson of given momentum is replaced by a fermion of the same momentum, or vice-versa. In other words, these transitions consist in creation or annihilation of particle-hole pairs of four-momentum $P_\mu = 0$ and fermion number $F = \pm 1$, which are called »thermal superpairs« by Matsumoto et al.[14] and are finite temperature analogues of goldstino modes. The resulting singularity at $P_\mu = 0$ occurs already for non-interacting fields, in which case it appears in the WT identity for the composite field ψA of the Wess-Zumino model. In presence of interaction, the singularity spreads to the WT identity for the Majorana field ψ, except in case of chiral symmetry where it remains in the WT identity for ψA (A is the scalar field).

For the interacting model without chiral symmetry, Matsumoto et al. find that at one-loop level the singularity also remains in the WT identity for ψA, whereas Aoyama and Boyanovsky state that it disappears. The difference comes from the mass values used for the φ and A fields. The latter authors have $m_A \neq m_\psi$ whereas the former rearrange the Lagrangian to restore the $SUSY$ relation $m_A = m_\psi$. This is a point where there is room for clarification, but we presume that the ambiguity is a formal one; it should also be remembered that masses acquire imaginary parts at finite temperatures in presence of interaction. The existence of thermal excitation modes playing the role of goldstinos can be interpreted as breaking of $SUSY$ at $T > 0$, and the authors of Refs. 14 and 15 agree on this point to which we return later.

The results just reported are limited to Wess-Zumino models at one-loop level. We shall now try to obtain their essential contents in a model-independent way applicable both to $SUSY$ and to ordinary symmetries. In quantum field theory, WT identities have the following structure

$$\int d^4 z\, \partial^\mu_z \langle T\, S_\mu(z)\, \mathcal{O}_1(x, y, \ldots)\rangle = \langle \mathcal{O}_2(x, y, \ldots)\rangle. \tag{1}$$

$S_\mu(z)$ is the Noether (super)current of the (super)symmetry, $\mathcal{O}_1$ a polynomial in field operators at space-time points $x, y, \ldots$, and $\mathcal{O}_2(x, y, \ldots)$ a similar polynomial calculated from the effect of the (super)symmetry on $\mathcal{O}_1$. The symbol T denotes the usual time-ordered product. The average $\langle \ldots \rangle$ can be the thermal one, but can also be taken over other ensembles, because the *WT* identities are basically operator identities. In the interesting situations it is reasonable to assume that the space components $\langle T\vec{S}(z)\,\mathcal{O}_1(x, y, \ldots)\rangle$ vanish for $|\vec{z}| \to \infty$ at constant time $z_0 = t$ in the rest frame of the ensemble. By partial integration the terms $\mu = 1, 2, 3$ then vanish on the left-hand side of (1) and we are left with

$$\int \mathrm{d}t\, \partial_t \langle \mathrm{T}\, Q(t)\, \mathcal{O}_1(x, y, \ldots)\rangle = \langle \mathcal{O}_2(x, y, \ldots)\rangle \tag{2}$$

where we have introduced the (super) charge

$$Q(t) = \int \mathrm{d}\vec{z}\, S_0(z),\ z_0 = t. \tag{3}$$

Equation (2) is equivalent to

$$\lim_{t \to +\infty} \left(\langle \mathrm{T}\, Q(t)\, \mathcal{O}_1(x, y, \ldots)\rangle - \langle \mathrm{T}\, Q(-t)\, \mathcal{O}_1(x, y, \ldots)\rangle\right) = \langle \mathcal{O}_2(x, y, \ldots)\rangle. \tag{4}$$

By (super)symmetry of the theory, the operator $Q(t)$ is constant. But this only implies constancy of $\langle \mathrm{T}Q(t)\,\mathcal{O}_1(x, y, \ldots)\rangle$ when t varies without crossing the times $x_0, y_0, \ldots$ Whenever one of these times is crossed, the time ordering T of the operators is changed and the expectation value $\langle \mathrm{T}\, Q(t)\, \mathcal{O}_1(x, y, \ldots)\rangle$ changes by a finite amount. In other words, this expectation value is a step function of t with as many steps as there are times $x_0, y_0, \ldots$ in $\mathcal{O}$ riting Q for the constant operator $Q(t)$, we can reformulate (4) as

$$\langle [Q, \mathcal{O}_1(x, y, \ldots)]_\pm \rangle = \langle \mathcal{O}_2(x, y, \ldots)\rangle. \tag{5}$$

where the commutator $[,]_-$ applies in the case of symmetry, and the anticommutator $[,]_+$ for *SUSY*. In the former case, Q and $\mathcal{O}_1$ are bosonic operators (i. e, commute with F). In the latter they are fermionic (i. e., give $\Delta F =$ odd integer, usually ± 1), which causes the change of sign in the second T product of (4). $\mathcal{O}_2$ is always bosonic.

The difference between the two cases is immediately visible on Eq. (5). Take the ensemble to be invariant for the (super)symmetry, which means that

$$[Q, \varrho]_- = 0 \tag{6}$$

where ϱ is the density matrix of the ensemble. In the case of symmetry the left-hand side of (5) then vanishes by the cyclic property of the trace:

$$\langle [Q, \mathcal{O}_1]_- \rangle = \mathrm{Tr}(Q\,\mathcal{O}_1\,\varrho) - \mathrm{Tr}(\mathcal{O}_1\,Q\,\varrho) = \mathrm{Tr}(Q\,\mathcal{O}_1\,\varrho) - \mathrm{Tr}(\mathcal{O}_1\,\varrho\,Q) = 0. \tag{7}$$

Hence the manifestation of symmetry is

$$\langle \mathcal{O}_2 (x, y, \ldots) \rangle = 0 \tag{8}$$

both at finite and at zero temperature. The situation is different for $SUSY$ because (7) then contains the anticommutator, with the result

$$\langle \mathcal{O}_2 (x, y, \ldots) \rangle = 2 \,\mathrm{Tr}\, (Q\, \mathcal{O}_1\, \varrho). \tag{9}$$

In general, this does not vanish at finite temperature. It vanishes for unbroken $SUSY$ at $T = 0$, because then

$$\varrho = |0\rangle \langle 0|, \quad Q\,|0\rangle = 0 \tag{10}$$

where $|0\rangle$ is the ground state (vacuum state in field theory).

A non-vanishing value of (9) means that $\langle \mathrm{T} Q(t)\, \mathcal{O}_1 (x, y, \ldots) \rangle$, which we saw to be a step function in t, has different values for $t \to \pm\infty$. This implies a singularity structure for the same Green function expressed in energy-momentum variables, namely a zero energy pole which one naturally expects to occur when some momentum vanishes, as is the case in the examples treated in Refs. 14 and 15. The physical meaning of the singularity is an excitation mode with a dispersion law $E = E(\vec{p}) \to 0$ for momentum $\vec{p} \to 0$ [of course $E(\vec{p})$ cannot be expected to have the relativistic form since a system at $T > 0$ is not Lorentz invariant; in addition one may have $\mathrm{Im} E(\vec{p}) \neq 0$]. This *»thermal goldstino mode«* is fermionic because Q and $\mathcal{O}_1$ are fermionic operators. Its actual physical nature depends on the system and to some degree on the operator $\mathcal{O}_1$. For example, as mentioned above, for the Wess-Zumino model without interaction, it occurs for $\mathcal{O}_1 = \psi A$ whereas it occurs for $\mathcal{O}_1 = \psi$ in the model with non-chiral-invariant interaction.

It is interesting to compare the above analysis with the occurrence of a bosonic Goldstone mode when an ordinary symmetry is spontaneously broken. In this case, the symmetric ensemble is replaced by an appropriate subensemble which is not invariant under the symmetry. The operators Q and $\mathcal{O}_1$ are bosonic, the commutator must be taken in (5), but (6) does not hold for the subensemble, so that the expectation values (7) and (8) are in general $\neq 0$. The argument presented for $SUSY$ in the previous paragraph then applies again. The different limiting values of the step function $\langle \mathrm{T} Q(t)\, \mathcal{O}_1 \rangle$ for $t \to +\infty$ and $-\infty$ signal a Goldstone mode with $E(\vec{p}) \to 0$ for $\vec{p} \to 0$. The mode is of course bosonic in the present case which can occur both at $T = 0$ and $T > 0$. The fact that a $E = \vec{p} = 0$ mode occurs here only for a spontaneously broken symmetry whereas for $SUSY$ it occurs generally at $T > 0$ supports the view that $SUSY$ should be regarded as broken at finite temperature even when unbroken at $T = 0$. As shown in the next section, however, there are differences between the two cases which have apparently not been noticed before, and they make the question of automatic $SUSY$ breaking at $T > 0$ somewhat more subtle.

3. *Supersymmetry as a relation between bosons and fermionic ensembles*

As far as *SUSY* is concerned, the above considerations centre on the fact that in general Eq. (5) gives

$$\langle [Q, \mathcal{O}_1]_+ \rangle = \langle \mathcal{O}_2 \rangle \neq 0 \tag{11}$$

at finite temperature. On the other hand, following Ref. 10, if Q and ϱ commute one has at all temperatures

$$\langle K [Q, \mathcal{O}_1]_+ \rangle = \langle K \mathcal{O}_2 \rangle = 0 \tag{12}$$

where K is the Klein operator $K = (-1)^F$, which anticommutes with Q and $\mathcal{O}_1$. Due to the well-known superselection rule[17] between bosonic (even fermion number F) and fermionic (odd F) states, we can also assume K to commute with ϱ. The first equality of (12) follows from the *WT* identity as operator relation. To obtain the second, one remarks that in presence of K, Eq. (7) is replaced by

$$\langle K [Q, \mathcal{O}_1]_+ \rangle = \mathrm{Tr}\,(K Q \mathcal{O}_1 \varrho) + \mathrm{Tr}\,(K \mathcal{O}_1 Q \varrho) = \mathrm{Tr}\,((K Q) \mathcal{O}_1 \varrho) -$$
$$- \mathrm{Tr}\,(\mathcal{O}_1 \varrho (K Q)) = 0. \tag{13}$$

What is the physical content of this type of relation? We first stress that an expectation value $\langle K \mathcal{O} \rangle = \mathrm{Tr}\,(K \mathcal{O} \varrho)$ is *not* some new type of thermal average of the operator $\mathcal{O}$, because ϱK is in general *not* a density matrix; it has negative eigenvalues whenever the ensemble described by ϱ contains fermionic states. In physical terms, $\langle K \mathcal{O} \rangle$ is the weighted *difference* between the averages of $\mathcal{O}$ in two subensembles of the ensemble ϱ, one formed by the bosonic states with density matrix ϱ_b and the other formed by the fermionic states with ϱ_f, the relative weights being $\mathrm{Tr}\,\varrho'_{b,f}$ where

$$\left.\begin{aligned} &\varrho'_b = \frac{1}{2}(1 + K)\,\varrho, \ \varrho'_f = \frac{1}{2}(1 - K)\,\varrho = \varrho - \varrho'_b, \\ &\varrho_{b,f} = \varrho'_{b,f}/\mathrm{Tr}\,\varrho'_{b,f}, \quad \mathrm{Tr}\,\varrho'_b + \mathrm{Tr}\,\varrho'_f = \mathrm{Tr}\,\varrho = 1. \end{aligned}\right\} \tag{14}$$

ϱ_b and ϱ_f are easily checked to be density matrices because ϱ and K commute. With (14) we immediately obtain the announced result:

$$\left.\begin{aligned} &\langle K \mathcal{O} \rangle = (\mathrm{Tr}\,\varrho'_b)\,\langle \mathcal{O} \rangle_b - (\mathrm{Tr}\,\varrho'_f)\,\langle \mathcal{O} \rangle_f \\ &\langle \mathcal{O} \rangle_{b,f} = \mathrm{Tr}\,(\mathcal{O}\,\varrho_{b,f}). \end{aligned}\right\} \tag{15}$$

By applying (15) to (12) we obtain

$$\left.\begin{aligned} &\langle [Q, \mathcal{O}_1]_+ \rangle_{b,f} = \langle \mathcal{O}_2 \rangle_{b,f}, \\ &\langle \mathcal{O}_2 \rangle_b = R\,\langle \mathcal{O}_2 \rangle_f \end{aligned}\right\} \tag{16}$$

where the ratio R defined by

$$R = \mathrm{Tr}\, \varrho_f' / \mathrm{Tr}\, \varrho_b' = (\mathrm{Tr}\, \varrho_b')^{-1} - 1 \tag{17}$$

depends only on the ensemble ϱ and not on $\mathcal{O}_1$, $\mathcal{O}_2$. On the other hand, we of course have

$$\langle \mathcal{O}_2 \rangle = (\mathrm{Tr}\, \varrho_b') \langle \mathcal{O}_2 \rangle_b + (\mathrm{Tr}\, \varrho_f') \langle \mathcal{O}_2 \rangle_f \tag{18}$$

so that the combined result is

$$\langle \mathcal{O}_2 \rangle = \frac{2}{1+R} \langle \mathcal{O}_2 \rangle_b = \frac{2R}{1+R} \langle \mathcal{O}_2 \rangle_f. \tag{19}$$

The physical content of the identity (12) is now apparent. If $\mathcal{O}_2$ is the operator obtained on the right-hand side of a supersymmetric WT identity, its expectation values in a supersymmetric ensemble ϱ and in its bosonic and fermionic subensembles are related by (19). As we shall see, this result puts into doubt some of the physical interpretations proposed in Refs. 13—15 for the inequality (11).

We first discuss the ratio R and show that it can be taken to be $R = 1$ in the most interesting cases. Consider the eigenstates $|\varphi_n\rangle$ of the over-all density matrix ϱ

$$\varrho\, |\varphi_n\rangle = p_n\, |\varphi_n\rangle. \tag{20}$$

Since p commutes with K, each $|\varphi_n\rangle$ can be taken to be bosonic ($K = 1$) or fermionic ($K = -1$) and is then also an eigenstate of the subensemble density matrices $\varrho_{b,f}$. But ϱ also commutes with Q, meaning that the over-all ensemble is supersymmetric. Hence, for each $|\varphi_n\rangle$, either $Q \mid \varphi_n\rangle = 0$, or $Q \mid \varphi_n\rangle$ is an eigenstate with the same eigenvalue of ϱ and opposite eigenvalue of K. We call p the sum of the p_n of all eigenstates $|\varphi_n\rangle$ with $Q \mid \varphi_n\rangle \neq 0$, and p_b° (resp. p_f°) the sum of the p_n of the bosonic (resp. fermionic) eigenstates $|\varphi_n\rangle$ with $Q \mid \varphi_n\rangle = 0$. We have

$$p + p_b^\circ + p_f^\circ = 1, \quad \mathrm{Tr}\, \varrho_{b,f} = p_{b,f}^\circ + \frac{1}{2} p \tag{21}$$

which gives for the ratio (17)

$$R = (1 + p_f^\circ - p_b^\circ) / (1 + p_b^\circ - p_f^\circ). \tag{22}$$

In $SUSY$, a state obeying $Q \mid \varphi\rangle = 0$ is usually a zero energy ground state. If we take for the ensemble ϱ only excited states as is, e. g., the case for a microcanonical ensemble (ensemble of excited states of given total energy), we can assume that all these states obey $Q|\varphi\rangle \neq 0$. We have then $p_f^\circ = p_b^\circ = 0$ and $R = 1$ so that (19) reduces to

$$\langle \mathcal{O}_2 \rangle = \langle \mathcal{O}_2 \rangle_b = \langle \mathcal{O}_2 \rangle_f. \tag{23}$$

In unbroken $SUSY$, there is usually only one zero energy state and it is bosonic (vacuum state for a field theory). Then $p_f^\circ = 0$, $p_b^\circ > 0$ and $R < 1$ for the thermal ensemble; in addition $R \to 1$ for $T \to \infty$, $R \to 0$ for $T \to 0$, and at $T = 0$, Eq. (19) of course reduces to $\langle \mathcal{O}_2 \rangle = \langle \mathcal{O}_2 \rangle_b = 0$.

These considerations throw a new light on the thermal goldstino mode discussed in Section 2. We limit ourselves to an ensemble ϱ of states obeying $Q \mid \varphi \rangle \neq 0$, e. g., a microcanonical ensemble. Q then defines a one-to-one mapping of each of the subensembles ϱ_b, ϱ_f on the other. The thermal goldstino mode of vanishing energy momentum can be physically understood as what makes the difference in particle contents between a bosonic state of ϱ_b and the corresponding fermionic state of ϱ_f. In the non-interacting Wess-Zumino model, Q is essentially $\int d^3 \vec{z} A \gamma^\circ \psi$; this of course confirms the identification of the thermal goldstino with the thermal superpair of Matsumoto et al.[14].

We now return to the comparison of the $SUSY$ situation just discussed with what happens for a spontaneously broken symmetry, as already considered briefly at the end of Section 2. In the latter case, the overall ensemble is invariant for the symmetry but the subensembles corresponding to the various possible values of the order parameter(s) are not. We call ϱ_Φ the density matrices of these subensembles with Φ denoting a complete set of order parameter(s). The over-all density matrix is

$$\varrho = \int d m_\Phi \, \varrho_\Phi \tag{24}$$

with dm_Φ the invariant volume element. Whereas the charge operator Q (now bosonic) commutes with ϱ, it does not with the ϱ_Φ. Hence for appropriate choice of $\mathcal{O}_1$ the subensemble average

$$\langle \mathcal{O}_2 \rangle_\Phi = \mathrm{Tr} \, (\mathcal{O}_2 \, \varrho_\Phi) \tag{25}$$

does not vanish, but its overall average does

$$\langle \mathcal{O}_2 \rangle = \int dm_\Phi \, \langle \mathcal{O}_2 \rangle_\Phi = 0 \tag{26}$$

so that $\langle \mathcal{O}_2 \rangle_\Phi$ changes with Φ. This expectation value therefore exhibits the symmetry breaking and can be taken as an order parameter. The zero-energy momentum Goldstone modes can be understood in this context as making the difference between the states of a subensemble ϱ_Φ and those of $\varrho_{\Phi+\delta\Phi}$ with $\delta\Phi$ the result of an infinitesimal symmetry transformation generated by Q.

How does this compare with the $SUSY$ situation? To the subensembles ϱ_Φ correspond the subensembles ϱ_b and ϱ_f of $SUSY$. That the latter form a discrete set instead of a continuum is only a superficial difference; we could for $SUSY$ consider for example the continuum of mixtures

$$\varrho_\alpha = \alpha \, \varrho_b + (1 - \alpha) \, \varrho_f, \; 0 < \alpha < 1. \tag{27}$$

To the order parameter $\langle \mathcal{O}_2 \rangle_\Phi$ of Eq. (25) correspond in $SUSY$ the expectation values $\langle \mathcal{O}_2 \rangle_b$, $\langle \mathcal{O}_2 \rangle_f$ of (15), or the continuous interpolation for the ensemble (27)

$$\langle \mathcal{O}_2 \rangle_\alpha = \mathrm{Tr} \, (\mathcal{O}_2 \, \varrho_\alpha) = \alpha \, \langle \mathcal{O}_2 \rangle_b + (1 - \alpha) \, \langle \mathcal{O}_2 \rangle_f. \tag{28}$$

Here, however, we find a substantial difference between $SUSY$ and ordinary symmetry. In the latter case we saw that the overall average $\langle \mathcal{O}_2 \rangle$ vanishes, but the subensemble averages $\langle \mathcal{O}_2 \rangle_\Phi$ are in general $\neq 0$ and change with Φ, this being the property which exhibits the symmetry breaking and makes $\langle \mathcal{O}_2 \rangle_\Phi$ an order parameter. For $SUSY$ on the contrary, the overall average $\langle \mathcal{O}_2 \rangle$ does not vanish and for the most interesting case of positive energy ensembles (e. g., microcanonical ensembles) the subensemble averages $\langle \mathcal{O}_2 \rangle_b$, $\langle \mathcal{O}_2 \rangle_f$, $\langle \mathcal{O}_2 \rangle$ are all equal by Eq. (23), although the corresponding subsensembles are not invariant under the supersymmetry. *This disqualifies the subensemble averages of $\mathcal{O}_2$ as order parameters.* It also reduces the meaning of the statement that $\langle \mathcal{O}_2 \rangle \neq 0$ is a sign of $SUSY$ breaking. In fact the equality $\langle \mathcal{O}_2 \rangle_b = \langle \mathcal{O}_2 \rangle_f = \langle \mathcal{O}_2 \rangle$ is non-trivial evidence that $SUSY$ holds good for excited states. Its physical interpretation is also in full accord with the very nature of $SUSY$, which is to express that bosonic and fermionic states have similar dynamical properties despite their differences in spin and statistics. It is true that the latter differences imply $\langle \mathcal{O}_2 \rangle \neq 0$ and therefore the existence of a thermal goldstino mode. One can say with the authors of Refs. 13—15 that this indicates $SUSY$ breaking in analogy with the relation between symmetry breaking and Goldstone modes. But then one could say that even in a single $SUSY$ multiplet with a single mass value, $SUSY$ is broken, because one goes from a boson to a fermion by creation of a »goldstino« composed of a boson hole and a fermion. The argument then reduces to a matter of terminology. We return to it in more physical terms in the concluding section.

In the above discussion, we concentrated on the case where the ensemble ϱ is such that $R = 1$, i. e., where the »generalized Witten index«[18] $p_b^\circ - p_f^\circ$ vanishes. To interpret what happens in general, the simplest procedure is to separate ϱ into a first subensemble spanned by all eigenstates $|\varphi_n\rangle$ which obey $Q \mid \varphi_n \rangle = 0$, and the orthogonal subensemble (this entails a similar separation of $\varrho_{b,f}$). Our previous considerations apply to the latter. The first subensemble is rather trivial since it contains only states which are unrelated to each other by $SUSY$ transformations; these states are usually the zero-energy ground state(s) or vacuum state(s). When expressed in terms of quantum states, the contents of $SUSY$ is all to be found in the second subensemble, to which our earlier treatment fully applies.

4. Concluding remarks

Our discussion showed that the quantity $\langle \mathcal{O}_2 \rangle$ of Eq. (11), although nonvanishing, cannot be taken as order parameter to distinguish between the bosonic and fermionic subensembles $\varrho_{b,f}$, which (for ensembles of states $|\varphi\rangle$ with $Q \mid \varphi \rangle \neq 0$) are the subensembles transformed into each other by $SUSY$ transformations. In fact, a natural »order parameter« for this distinction is the parity of the fermion number F, i. e., $K = (-1)^F$. Unfortunately, it is not represented by a local operator as is the case for the order parameters of ordinary symmetries, and it is therefore of little use for large systems as considered in thermodynamics. We do not know of a local or quasi-local order parameter adapted to supersymmetric systems.

As we argued at the end of the previous section, the statement of Refs. 13—15 that $SUSY$ is always broken at finite temperature reflects rather poorly the actual situation which (for $R = 1$) is one of perfect symmetry between the bosonic and fermionic subensembles. Guided once more by the analogy with ordinary symmetries, we believe that the physical issue is whether it is possible, by external action on the system, to bring it into one of the subensembles $\varrho_{b,f}$ (in the way that an external magnetic field can be used to orient the spins of a ferromagnet). This is certainly possible for a finite system since one can control its value of K. As far as we can see, it seems impossible in the thermodynamic limit of an infinite system, since the fermion number contained in any finite volume will constantly fluctuate and we do not see how it could be controlled by local or quasi-local external actions. If this is true, the appropriate terminology would be to call $SUSY$ unbroken at $T > 0$ for infinite systems.

The latter conclusion is also reached when using another definition of (super) symmetry breaking, as advocated, e. g., by Fuchs[12]: a (super)symmetry is broken if and only if there exists some measurable quantity represented by an operator R whose change δR under a (super)symmetry transformation has non-vanishing expectation value. As explained in Refs. 10 and 12, this form of $SUSY$ breaking requires

$$\langle K [Q, R]_+ \rangle \neq 0$$

or with our notations of the previous section

$$(\mathrm{Tr}\, \varrho_b') \langle [Q, R]_+ \rangle_b \neq (\mathrm{Tr}\, \varrho_f') \langle [Q, R]_+ \rangle_f,$$

whereas by Eq. (13) we know that the equality sign holds. This argument has a weakness, however. In the case of $SUSY$, the operator R is fermionic and therefore, by the fermionic superselection rule[17], does not represent a non-trivial measurable quantity; it has zero expectation value in both subensembles ϱ_b and ϱ_f.

Our last comment is on the use of »supertraces« of the type $\langle K Q \rangle \doteq \mathrm{Tr}\,(K \mathcal{O}\, \varrho)$ which has been repeatedly criticized as leading to unphysical correlation functions[15], or having nothing to do with usual thermal averages[14] or with reality[13]. None of these criticisms applies to Ref. 10 or the present paper. On the contrary, we have shown that the operator K can be used to obtain-non-trivial dynamical relations between the expectation values $\langle \mathcal{O} \rangle_b$ and $\langle \mathcal{O} \rangle_f$ which are of course physical. The fact that a thermal supertrace $\langle K \mathcal{O} \rangle$ can be calculated by using periodic instead of antiperiodic boundary conditions for fermionic operators in imaginary time is a mathematical feature of no physical relevance; imaginary time itself is an unphysical tool for functional integration. What matters is that supertraces are to be understood as differences between expectation values in two physical ensembles (one bosonic and the other fermionic) which $SUSY$ relates to each other.

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*) *Note added in proof:* The preprint in Ref. 7 has been withdrawn by the author, some of the puzzles having been resolved.

KVANTNE TEORIJE POLJA PRI KONAČNIM TEMPERATURAMA

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Nakon općenitih primjedbi o suvremenim razvojima kvantne teorije polja pri konačnim temperaturama, diskutiramo razne aspekte supersimetričnih teorija u usporedbi sa slučajem običnih simetrija: Ward-Takahashijevim identitetima, Goldstoneovim modovima, parametrima uređenja i lomljenja supersimetrije.