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EXISTENCE THEOREMS FOR THE DIFFERENTIAL EQUATION OF TWO EFFECTIVE COUPLINGS

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For massless models of quantum field theory with two independent couplings in-variance under the renormalization group implies an ordinary differential equation which relates the effective couplings β and λ. In lowest order it is assumed that the β-function of g^2 has the form $b\, g^4$ with $b \neq 0$ while the β-function of λ is a general bilinear form in λ and g^2. For the exact system the solutions of the differential equation are studied near the singular point $\lambda = g = 0$. All solutions λ (g^2) are determined for which the ratio λ/g^2 stays bounded in the limit $g \to 0$. For small g an explicit form of the asymptotic behaviour is obtained in agreement with formal asymptotic expansions found previously.

1. Introduction

In this paper exact relations between effective couplings will be derived for massless renormalizable models of quantum field theory involving two dimensionless coupling constants. The effective couplings, denoted by g and λ are defined as functions of a scale variable k^2/K^2,

$$g = g\,(k^2/K^2),\ \lambda = \lambda\,(k^2/K^2)\ \ k^2 < 0,\ K^2 < 0,$$

where k is a momentum variable and K the normalization mass, both Euclidean. In addition, the effective couplings depend on the values of the coupling parameters which they assume at the normalization point $k^2 = K^2$. After elimination of the scale variable λ becomes a functions of g

$$\lambda = \lambda (g^2)$$

which satisfies the differential equation[1]

$$\beta_0 \frac{d\lambda}{dg^2} = \beta_1. \tag{1.1}$$

β_0 and β_1 denote the Callan-Symanzik functions associated with the couplings g^2 or λ, respectively. The same differential equation (1.1) is obtained for the coupling parameters themselves if the requirement is imposed that the coupling parameter λ be a function of g^2 independent of the normalization point[2]. The aim of the present paper is to determine all solutions of (1.1) with bounded ratio λ/g^2 for small g^2 and to classify them according to their asymptotic behaviour for $g^2 \to 0$. The β-functions are defined in a region

$$0 \leqq g^2 \leqq \zeta,\ 0 \leqq |\lambda| \leqq \zeta' \ (\zeta, \zeta' > 0). \tag{1.2}$$

For some models the admissible values of λ are restricted either to $\lambda \geqq 0$ or to $\lambda \leqq 0$. All partial derivatives of the β-functions are assumed to exist near the origin.* The formal expansions of the β-functions with respect to g^2 and λ can be set up with the partial derivatives at the origin as coefficients. For the form of the lowest order terms we assume as in Ref. 1 that

$$\begin{aligned} \beta_0 &= b g^4 + \ldots, && b \neq 0, \\ \beta_1 &= c_1\lambda^2 + c_2\lambda g^2 + c_3 g^4 + \ldots, && c_1 \neq 0. \end{aligned} \tag{1.3}$$

We restrict ourselves to the case $b \neq 0$ since different mathematical methods[3,4] are required for treating the differential equation (1.1) if $b = 0$. It will further be assumed that

$$\Delta = (c_2 - b_0)^2 - 4\, c_1 c_3 \geqq 0, \tag{1.4}$$

which is a necessary condition for asymptotic relations between small effective couplings[5].

In order to facilitate the proof of existence theorems in cases where the sign of λ is restricted we extend the differential equation to values λ of opposite sign. Suppose a model is only defined for $\lambda \geqq 0$ ($\lambda \leqq 0$), then we continue the β-func-

* This means that each partial derivative exists in (1.2) in a sufficiently small neighborhood of the origin $g^2 = \lambda = 0$ including points with $g^2 = 0$ and/or $\lambda = 0$. At $g^2 = 0$ derivatives with respect to g^2 are defined from the right. Likewise derivatives with respect to λ are defined one-sided if the sign of λ is restricted by the model. There need not be a common neighborhood of the origin in which all partial derivatives exist.

tions to $\lambda < 0$ ($\lambda > 0$) by a polynomial in g^2 and λ given by a finite number of terms of the power series expansion. The order of the polynomial is chosen such that sufficiently many partial derivatives defined for $\lambda \to +0$ or $\lambda \to -0$, respectively, match at $\lambda = 0$ as are needed for carrying through the existence proof. Solutions with inadmissible sign are then disregarded at the end.

In Ref. 1 the differential equation (1.1) was solved by power series expansions

$$\lambda = \sum_{n=1}^{\infty} \lambda_n g^{2n} \tag{1.5}$$

and expansions involving logarithms

$$\lambda = \sum_{n=1}^{m-1} \lambda_n g^{2n} + \lambda_m g^{2m} + d\, g^{2m} \ln g^2 + \ldots \tag{1.6}$$

or non-integral powers

$$\lambda = \sum_{n=1}^{\infty} \lambda_n g^{2n} + d\, g^{2|\xi|+2} + \ldots \tag{1.7}$$

The characteristic exponent ξ is defined by the lowest order terms of the β-functions

$$\xi = -\frac{c_1}{b} (\varrho_+ - \varrho_-), \tag{1.8}$$

where $\varrho_+ \geqq \varrho_-$ denote the roots of

$$c_1 x^2 + (c_2 - b)\, x + c_3 = 0. \tag{1.9}$$

$|\xi|$ is the smallest non-integral exponent occuring in the expansion of λ/g^2. An integral value of ξ indicates that the asymptotic expansion of λ may involve logarithms.

Expansions of the form (1.5—7) were derived by checking each order of the differential equation. The actual existence of corresponding exact solutions was not established in Ref. 1. Since the power series expansions of the β-functions are presumably only asymptotic, the expansions for $\lambda(g^2)$ cannot be expected to converge. Therefore solutions of (1.1) may not be constructed by convergent power series. On the other hand the existence of solutions at points $g^2 > 0$ is easily established by applying the Picard-Lindelöf theorem. The required hypotheses follow for sufficiently small values of g^2 and λ from the differentiability of the β-functions and the fact that $\beta_0 \neq 0$ due to (1.3) with $b \neq 0$. The existence of solutions is thus guaranteed, but no information on their asymptotic behaviour for $g^2 \to +0$ is provided by this method.

At $g^2 = \lambda = 0$ the differential equation (1.1) is singular since β_0 and β_1 vanish there. Therefore, the standard existence theorems cannot be applied at the origin. In this paper suitable transformations will be developed which completely remove the singularity at $g^2 = 0$. By this method it is possible to find all solutions with bounded ratio λ/g^2 along with their asymptotic behaviour for $g^2 \to +0$. The main result will be that in the neighborhood of $g^2 = 0$ all solutions of (1.1) with bounded ratio λ/g^2 are represented by the asymptotic expansions found in Ref. 1.

2. *Non-integral characteristic exponent*

In this section we will determine all solutions of (1.1) near $g^2 = 0$ which have a bounded ratio

$$\varrho = \frac{\lambda}{g^2} \tag{2.1}$$

in case that the characteristic exponent ξ defined by (1.8) is not an integer. The difference

$$\delta = \varrho - \varrho_+, \text{ if } \xi < 0, \tag{2.2}$$

$$\delta = \varrho - \varrho_-, \text{ if } \xi > 0,$$

satisfies the differential equation

$$\frac{\beta_0}{g^2} \frac{\mathrm{d}\,\delta}{\mathrm{d}g^2} = c_1 \delta^2 + b\,|\xi|\,\delta + g^2 R, \quad c_1 \neq 0, \tag{2.3}$$

where R exists at $g^2 = \lambda = 0$ along with the partial derivatives. Wintner derived a general theorem referring to differential equations of a similar type which states the existence of solutions vanishing for $g^2 \to +0$ and their asymptotic behaviour in this limit*. But the hypotheses of the theorem do not apply to the case considered here since a term quadratic in δ would be excluded. If a quadratic term is present there is in general also a solution δ with non-vanishing limit for $g^2 \to +0$ which would not be covered by a theorem of this kind. On the other hand more detailed information is available for the renormalization group equation in so far as the β--functions have finite partial derivatives of any order at the origin. Making use of this transformations can be constructed which completely remove the singularity at $g^2 = 0$. Some care must be taken that no solution of the original differential equation (1.1) is missed by treating the transformed equation.

If ξ is not an integer there are two formal power series expansions which solve (1.1) order by order[1]. For the ratio (2.1) they have the form

$$\varrho = \varrho_+ + \sum_{j=1}^{\infty} \varrho_{+j}\, g^{2j} \tag{2.4}$$

$$\varrho = \varrho_- + \sum_{j=1}^{\infty} \varrho_{-j}\, g^{2j}. \tag{2.5}$$

* See Ref. 6, page 164.

By subtracting a sufficient number of terms from ϱ a remainder σ_n is defined by

$$\varrho = \varrho_+ + \varrho_{+1} g^2 + \dots + \varrho_{+(n-1)} g^{2n-2} + g^{2|\xi|} \sigma_n \text{ for } \xi < 0 \tag{2.6}$$

and

$$\varrho = \varrho_- + \varrho_{-1} g^2 + \dots + \varrho_{-(n-1)} g^{2n-2} + g^{2\xi} \sigma_n \text{ for } \xi > 0. \tag{2.7}$$

Recursively σ_n can be shown to satisfy the differential equation

$$\frac{d\sigma_n}{dg^2} = \frac{N}{D}, \quad D = \frac{\beta_0}{g^4} = b + g^2 S \tag{2.8}$$

$$N = c_1 g^{2|\xi|-2} \sigma_n^2 + g^{2n-2|\xi|-2} U + \sigma_n V + g^{2|\xi|} \sigma_n^2 P$$

$$S = S_0 + \varrho S_1 + \varrho^2 S_2 + \varrho^3 S_3 \tag{2.9}$$

$$U = U_0 + \varrho U_1 + \dots + \varrho^{n+2} U_{n+2} \tag{2.10}$$

$$V = V_0 + \varrho V_1 + \varrho^2 V_2 + \varrho^3 V_3. \tag{2.11}$$

$S_0, S_1, S_2, U_0, \dots, U_{n+1}$ and V_0, V_1, V_2 are functions of g^2 with all derivatives existing near $g^2 = 0$. U_{n+2}, V_3 and S_3 depend on g^2 and

$$\lambda = \varrho g^2 \tag{2.12}$$

with all partial derivatives existing near $\lambda = g^2 = 0$. P is a polynomial and

$$g^{2|\xi|} \sigma_n = \varrho - \varrho_\pm - \dots - \varrho_{\pm(n-1)} g^{2r-2}. \tag{2.13}$$

By (2.7—2.11) and the continuity of U_j, V_j, S_j the functions U, V and S are uniformly bounded in

$$0 \leqq g^2 \leqq \zeta$$

for any function σ_n with bounded ϱ

$$|\varrho| \leqq r$$

provided $\zeta > 0$ is chosen sufficiently small.

We begin by constructing solutions of (2.6) for $|\xi| < 1$.

(i) *Case* $|\xi| < 1$

With the variable $g^{2|\xi|}$ the differential equation of σ_n takes the form

$$\frac{d\sigma_n}{d g^{2|\xi|}} = \frac{N}{D}$$

$$N = c_1 \sigma_n^2 + g^{2n-4|\xi|} U + g^{2-2|\xi|} \sigma_n V + g^2 \sigma_n^2 P \tag{2.14}$$

$$D = b|\xi| + |\xi| g^2 S.$$

If n is chosen

$$n > 2\,|\xi| \tag{2.15}$$

the ratio N/D is continuous at points σ_n, $g^2 = 0$ where σ_n is arbitrary and continuity with respect to g^2 is defined from the right ($g^2 \to +0$). Although (1.1) originally was derived for $g^2 > 0$ only it is now natural to include the points $g^2 = 0$.

We begin with the discussion of solutions σ_n which stay bounded for $g^2 \to +0$. Let $M > 0$ be arbitrary. We try to determine all solutions σ_n of (2.14) which are bounded by

$$|\sigma_n| \leqq M \text{ in } 0 \leqq g^{2|\xi|} \leqq \eta, \tag{2.16}$$

$$M > 0,\ \eta > 0.$$

The bound M is arbitrary, η will be chosen as a suitably small number depending on M. For any value of M it can be shown that

$$\frac{\partial N/D}{\partial \sigma_n}$$

is bounded in some rectangle

$$0 \leqq g^{2|\xi|} \leqq \eta',\ |\sigma_n| \leqq M, \tag{2.17}$$

with $\eta' > 0$ sufficiently small for given M. This implies a Lipschitz condition for N/D in the rectangle (2.17). Accordingly there is exactly one solution σ_n through every point of (2.16) with a suitable number $\eta \leqq \eta'$*. In particular, there is exactly one solution $\sigma_n\,(g^2)$ with

$$\sigma_n = \sigma_{n0} \text{ at } g^2 = 0 \quad \text{for} \quad |\sigma_{n0}| \leqq M \tag{2.18}$$

defined in the interval $0 \leqq g^{2|\xi|} \leqq \eta$. For the corresponding solution λ of (1.1) the ratio λ/g^2 has the form

$$\begin{aligned} \varrho &= \varrho_+ + \varrho_{+1}\, g^2 + \dots + \varrho_{+(n-1)}\, g^{2n-2} + g^{2|\xi|}\,\sigma_n, \text{ if } \xi < 0, \\ \varrho &= \varrho_- + \varrho_{-1}\, g^2 + \dots + \varrho_{-(n-1)}\, g^{2n-2} + g^{2|\xi|}\,\sigma_n, \text{ if } \xi > 0. \end{aligned} \tag{2.19}$$

Functions σ_n and $\sigma_{n'}$ which coincide at $g^2 = 0$ lead to the same λ. Hence the construction of λ does not depend on the order n of the remainder. This fact can be used to show that the solution with $\sigma_n = 0$ at $g^2 = 0$ has a power series expansion in g^2. For (2.6) implies that in this case all derivatives of σ_n with respect to g^2 up to and including the order $n - 1$ vanish at $g^2 = 0$. Then

$$\varrho = \varrho_\pm + \varrho_{\pm 1}\, g^2 + \dots + \varrho_{\pm(n-1)}\, g^{2n-2} + g^{2|\xi|+2n-2}\,\frac{\sigma_n}{g^{2n-2}}$$

* For this the right-handed version of the Picard-Lindelöf theorem is used. See Ref. 7, page 8.

with

$$\lim_{g^2\to 0} \frac{\sigma_n}{g^{2n-2}} = \lim_{g^2\to 0} \frac{\mathrm{d}^{n-1}\sigma_n}{(\mathrm{d}g^2)^{n-1}} (\Theta g^2) = 0 \quad (0 \leqq \Theta \leqq 1).$$

Since n is arbitrary the expansion (2.5) follows for $\xi > 0$ and (2.4) for $\xi < 0$.

The condition $n > 2\,|\xi|$ was only needed for the existence proof. Since $|\xi| < 1$ the leading asymptotic behaviour of ϱ for $g \to 0$ is given by

$$\varrho = \varrho_+ + g^{2|\xi|} \sigma(g^2) \quad \text{for} \quad \xi < 0,$$
$$\varrho = \varrho_- + g^{2|\xi|} \sigma(g^2) \quad \text{for} \quad \xi > 0. \tag{2.20}$$

σ is defined by

$$\sigma = \sigma_1 = g^{-2|\xi|} (\varrho_\pm g^2 + \dots + g^{2|\xi|} \sigma_n).$$

σ and σ_n have the same limit for $g \to 0$. Hence there is exactly one solution ϱ for each value σ_0 of σ

$$\sigma = \sigma_0 \text{ at } g^2 = 0$$

and the solution ϱ with $\sigma_0 = 0$ permits the expansion (2.5) if $\xi > 0$ and (2.4) if $\xi < 0$.

So far we have restricted ourselves to bounded functions σ_n. Bounded σ_n always implies the weak coupling limits ϱ_- of ϱ if $\xi > 0$ and ϱ_+ if $\xi < 0$. But in Ref. 1 also power series solutions were found with the limit ϱ_+ for $\xi > 0$ and ϱ_- for $\xi < 0$. Such solutions cannot be obtained from bounded σ_n. We therefore extend our investigation to arbitrary solutions σ_n subject to the only condition that the functions ϱ be bounded,

$$|\varrho| = |\varrho_\pm + \varrho_{\pm 1} g^2 + \dots + \varrho_{\pm(n-1)} g^{2n-2} + g^{2|\xi|} \sigma_n| \leqq r \tag{2.21}$$

where $r > 0$ may be arbitrary.

We begin by proving that for any unbounded solution σ_n the limit

$$\lim_{g^2\to 0} |\sigma_n| = \infty \tag{2.22}$$

follows. To this end it is sufficient to show that an unbounded solution σ_n cannot have a finite cumulation point for $g \to 0$. Suppose ω were a finite cumulation point and $\gamma > 0$, then we choose the bound M in (2.16) such that $M > \omega + \gamma$ and $\omega - \gamma > -M$. Let $\sigma_n^\pm$ be the solutions determined by the initial condition

$$\sigma_n^\pm = \omega \pm \gamma \quad \text{at } g^2 = 0.$$

Choosing $\mu > 0$ sufficiently small the pieces of the curves $\sigma_n^\pm$ over the interval $0 \leqq g^{2|\xi|} \leqq \mu$ lie entirely within the rectangle (2.16). As a consequence the region

$$\sigma_n^- \leqq \sigma_n \leqq \sigma_n^+, \; 0 \leqq g^{2|\xi|} \leqq \mu \tag{2.23}$$

of the $g^{2|\xi|}$, σ_n plane is a subset of (2.16). All solutions passing through any point in the neighborhood (2.23) of ω have finite limits bounded by $\omega \pm \gamma$. This contradicts the hypothesis that ω is a cumulation point for an unbounded solution σ_n.

In order to study the existence of unbounded σ_n we introduce the inverse

$$\varphi = \sigma_n^{-1}. \tag{2.24}$$

Its limit corresponding to (2.22) is

$$\lim_{g^2 \to 0} \varphi = 0. \tag{2.25}$$

Therefore, we need only consider functions φ which are bounded for small g^2

$$0 \leqq |\varphi| \leqq d \text{ for } 0 \leqq g^{2\xi} \leqq \eta,$$

$$d > 0,\ \eta > 0.$$

The differential equation of φ is

$$-\xi (b + g^2 S) \frac{\mathrm{d}\varphi}{\mathrm{d}g^{2|\xi|}} = c_1 + g^{2n-4|\xi|} U \varphi^2 + g^{2-2|\xi|} V \varphi + g^2 P. \tag{2.26}$$

For bounded φ and ϱ this implies

$$\lim_{g^2 \to 0} \frac{\mathrm{d}\varphi}{\mathrm{d}g^{2|\xi|}} = \frac{\xi / |\xi|}{\varrho_+ - \varrho_-}. \tag{2.27}$$

Hence $\varphi \geqq 0$ (or $\varphi \leqq 0$) for sufficiently small g^2 if $\xi > 0$ ($\xi < 0$). With this the limits of unbounded σ_n with bounded ϱ become

$$\lim_{g^2 \to 0} \sigma_n = \infty \text{ if } \xi > 0,$$

$$\lim_{g^2 \to 0} \sigma_n = -\infty \text{ if } \xi < 0. \tag{2.28}$$

Accordingly the solutions φ to be considered may be restricted by $0 \leqq \varphi \leqq d$ if $\xi > 0$ and $-d \leqq \varphi \leqq 0$ if $\xi < 0$. Only the case $\xi > 0$ will be treated further since the corresponding analysis of the case $\xi < 0$ is similar. It is convenient to replace φ by the function

$$\chi = \varphi - \frac{1}{2} \frac{g^{2|\xi|}}{\varrho_+ - \varrho_-} \tag{2.29}$$

for which

$$\chi = \varphi \text{ and } \lim_{g^2 \to 0} \frac{\mathrm{d}\chi}{\mathrm{d}g^{2\xi}} = \frac{1}{2(\varrho_+ - \varrho_-)} \text{ if } \xi > 0. \tag{2.30}$$

Again we have bounded $\chi \geqq 0$ for unbounded σ_n and bounded ϱ provided $\xi > 0$ and g^2 sufficiently small. Accordingly we need only consider solutions with

$$0 \leqq \chi \leqq d \quad \text{for} \quad 0 \leqq g^{2\xi} \leqq \eta \tag{2.31}$$

where $d > 0$ is arbitrary and $\eta > 0$ sufficiently small. By this (2.21) is automatically fulfilled with

$$r = |\varrho_-| + |\varrho_{-1}|\,\eta^{1/\xi} + \ldots + |\varrho_{-(n-1)}|\,\eta^{(n-1)/\xi} + 2(\varrho_+ - \varrho_-).$$

χ satisfies the differential equation

$$\frac{\mathrm{d}\chi}{\mathrm{d}g^{2\xi}} = \frac{N'}{D'} \tag{2.32}$$

with

$$N' = \frac{c_1}{2} + \frac{g^{2n}}{4(\varrho_+ - \varrho_-)^2}\,U + g^{2n-4\xi}\,U\chi^2 + \frac{g^{2(n-\xi)}}{\varrho_+ - \varrho_-}\,U\chi + \frac{1}{2}\,\frac{g^2}{\varrho_+ - \varrho_-}\,V +$$

$$+ g^{2(1-\xi)}\,V\chi + g^2 P + \frac{1}{2}\,\frac{\xi g^2}{\varrho_+ - \varrho_-}\,S$$

$$D' = -b\,\xi - g^2\,\xi\,S.$$

Since we only need solutions with $\chi \geqq 0$ we may modify the right-hand side of (2.32) for $\chi < 0$. U, V, S and P depend on χ through the variable ϱ*. We redefine ϱ for negative χ by setting

$$\varrho = \varrho_- + \varrho_{-1}\,g^2 + \ldots + \varrho_{-(n-1)}\,g^{2n-2} + \Theta(\chi)\,\frac{g^{2\xi}}{\chi + \frac{1}{2}\,g^{2\xi}\,(\varrho_+ - \varrho_-)^{-1}} \tag{2.33}$$

$$\Theta(\chi) = 1 \ \text{for}\ \chi \geqq 0, \ \Theta(\chi) = 0 \ \text{for}\ \chi < 0.$$

By this definition the discontinuity of ϱ at $\chi = g^2 = 0$ is not removed, but ϱ becomes a bounded function for all χ. This will be sufficient for our purpose. For the derivative

$$\frac{\partial}{\partial\chi}\,\frac{N'}{D'}$$

can be shown to be continuous in

$$0 \leqq g^{2\xi} \leqq \eta', \quad |\chi| \leqq D, \tag{2.34}$$

* For the dependence of P on ϱ see (2.13).

provided η' is made small enough for given $D > 0$. Hence for each given value χ_0 with $|\chi_0| < \eta$ where $\eta > 0$, $\eta \leqq \eta'$ there is exactly one solution χ with

$$\chi(0) = \chi_0. \tag{2.35}$$

Solutions for negative χ_0 are of no interest since the form of the differential equation was changed in that case. Solutions with positive χ_0 correspond to bounded functions σ_n and have already been dealt with before. The only new solution is the one determined by

$$\chi(0) = 0.$$

The corresponding function σ_n is the only unbounded solution of (2.14) with bounded ϱ. Because of (2.5) and (2.29) the limit of ϱ becomes

$$\lim_{g^2 \to 0} \varrho = \varrho_- + \lim_{g^2 \to 0} \frac{g^{2\xi}}{\chi + \frac{1}{2} g^{2\xi} (\varrho_+ - \varrho_-)^{-1}} = \varrho_+. \tag{2.36}$$

It can be verified that all derivatives of this solution ϱ with respect to g^2 exist at $g^2 = 0$*. Hence the solution λ of (1.1) with unbounded σ_n leads to an asymptotic expansion

$$\varrho = \varrho_+ + \sum_{n=1}^{\infty} \varrho_{+n} g^{2n}. \tag{2.37}$$

Similarly for $\xi < 0$ there is just one solution of unbounded σ_n. It allows for a power series expansion of ϱ with limit ϱ_- for $g^2 \to 0$.

Using (2.20) we may now restate the result concerning unbounded σ_n ($n > 2|\xi|$) in terms of σ. There is exactly one solution λ of (1.1) with unbounded σ. It allows for the expansion (2.4) if $\xi > 0$ and (2.5) if $\xi < 0$. The weak coupling limits of σ are

$$\lim_{g^2 \to 0} \sigma = \infty \qquad \text{if } \xi > 0,$$

$$\lim_{g^2 \to 0} \sigma = -\infty \qquad \text{if } \xi < 0.$$

The results of this section are summarized by the following theorem:

Theorem 1. If the characteristic exponent ξ lies within the interval $0 < |\xi| < 1$ all solutions λ of (1.1) with bounded ratio λ/g^2 are of the form

$$\lambda(g^2) = \varrho_- g^2 + g^{2\xi+2}\sigma, \text{ if } \xi > 0,$$
$$\lim_{g^2 \to 0} \sigma = \sigma_0, \ \sigma_0 \text{ finite or } \sigma_0 = \infty, \tag{2.38}$$

* To this end the iterated solution of (2.32) is used whose convergence is established in the proof of the Picard-Lindelöf theorem. See Ref. 7, page 8.

and

$$\lambda(g^2) = \varrho_+ g^2 + g^{2|\xi|+2}\sigma, \text{ if } \xi < 0,$$

$$\lim_{g^2 \to 0} \sigma = \sigma_0, \quad \sigma_0 \text{ finite or } \sigma_0 = -\infty. \tag{2.39}$$

For each value σ_0 there is exactly one solution defined on some interval

$$0 \leqq g^2 \leqq \zeta, \quad \zeta > 0.$$

For vanishing or diverging σ_0 the solution possesses a power series expansion

$$\lambda(g^2) = \varrho_+ g^2 + \sum_{n=1}^{\infty} \varrho_{+n} g^{2n+2},$$

$$\text{if } \xi > 0, \sigma_0 = \infty \text{ or } \xi < 0, \sigma_0 = 0, \tag{2.40}$$

and

$$\lambda(g^2) = \varrho_- g^2 + \sum_{n=1}^{\infty} \varrho_{-n} g^{2n+2},$$

$$\text{if } \xi > 0, \sigma_0 = 0 \text{ or } \xi < 0, \sigma_0 = -\infty. \tag{2.41}$$

(ii) *Case* $|\xi| > 1$

In this case the existence of solutions can be established by using the differential equation of σ_n in its original form (2.8). Taking $n > |\xi| + 1$ a Lipschitz condition is verified for the right-hand side of (2.8) so that there is exactly one solution σ_n through every point of a rectangle (2.16).

Next we discuss unbounded solutions σ_n with bounded function ϱ. As in the case $|\xi| < 1$ the limit (2.22) follows for any such σ_n. Therefore we need only consider the inverse $\varphi = \sigma_n^{-1}$ with vanishing limit (2.25). For $|\xi| > 1$ it is more involved to establish (2.27). First we prove by induction that

$$\lim_{g^2 \to 0} \frac{\varphi}{g^{2m}} = 0 \text{ if } \lim_{g^2 \to 0} \varphi = 0 \tag{2.42}$$

for any integer m with $0 < m < \xi$. φ satisfies the differential equation

$$-(b + g^2 S)\frac{d\varphi}{dg^2} = c_1 g^{2|\xi|-2} + g^{2n-2|\xi|-2} U\varphi^2 + V\varphi + g^{2|\xi|} P. \tag{2.43}$$

Taking the limit $g^2 \to 0$ we obtain

$$\lim_{g^2 \to 0} \frac{d\varphi}{dg^2} = 0 \text{ if } \lim_{g^2 \to 0} \varphi = 0.$$

Hence (2.42) follows for $m = 1$. We now consider integers $m' < m' + 1 < \xi$ assuming that (2.42) has been proved for $m = m'$. From (2.43):

$$- (m' + 1)(b + g^2 S) \frac{\mathrm{d}\varphi}{\mathrm{d}\, g^{2m'+2}} =$$

$$= c_1 g^{2|\xi| - 2m' - 2} + g^{2n - 2m' - 2|\xi| - 2} U \varphi^2 + V g^{-2m'} \varphi + g^{2|\xi| - m'} P.$$

Then all terms vanish in the limit $g^2 \to 0$ on the right-hand side so that (2.42) follows for $m = m' + 1$. In order to prove (2.27) we introduce $g^{2|\xi|}$ as variable of (2.43):

$$-\xi (b + g^2 S) \frac{\mathrm{d}\varphi}{\mathrm{d}\, g^{2|\xi|}} = c_1 + g^{2n - 4|\xi|} U \varphi^2 + g^{2 - 2|\xi|} V \varphi + g^2 P. \qquad (2.44)$$

Let m be the largest integer below $|\xi|$. Then

$$m < |\xi| < m + 1$$

o that

$$\lim_{g^2 \to 0} g^{2(1 - |\xi|)} \varphi = \lim_{g^2 \to 0} g^{2(n+1 - |\xi|)} \cdot \lim_{g^2 \to 0} \frac{\varphi}{g^{2m}} = 0.$$

Hence applying the limit $g^2 \to 0$ to (2.44) we get the desired relation (2.27). Thus $\varphi \geqq 0$ (or $\varphi \leqq 0$) for sufficiently small g^2 if $\xi > 0$ ($\xi < 0$). The limits of unbounded σ_n with bounded ϱ become (2.28).

According to (2.27) the solutions φ may be restricted by $0 \leqq \varphi \leqq d$ if $\xi > 0$ and $-d \leqq \varphi \leqq 0$ if $\xi < 0$. In the following discussion we only treat the case $\xi > 0$. Instead of φ we introduce the function χ defined by (2.29) and the variable $g^{2\varepsilon}$ with $0 < \varepsilon < 1$. In terms of the n w variables the differential equation has the form

$$\frac{\mathrm{d}\chi}{\mathrm{d}\, g^{2\varepsilon}} = \frac{N}{D}$$

$$N = \frac{1}{2} c_1 g^{2(\xi - \varepsilon)} + \frac{g^{2(n + \xi - \varepsilon)}}{4(\varrho_+ - \varrho_-)^2} U + g^{2(n - \xi - \varepsilon)} \chi^2 U + \frac{g^{2(n - \varepsilon)}}{\varrho_+ - \varrho_-} \chi U +$$

$$+ \frac{1}{2} \frac{g^{2(\xi + 1 - \varepsilon)}}{\varrho_+ - \varrho_-} V + g^{2(1 - \varepsilon)} V \chi + g^{2(\xi + 1 - \varepsilon)} P + \frac{\xi g^{2(\xi + 1 - \varepsilon)}}{2(\varrho_+ - \varrho_-)} S \qquad (2.45)$$

$$D = -b\varepsilon - g^2 \varepsilon S.$$

The right-hand side is modified again for $\chi < 0$ by inserting (2.33) for the variable ϱ. Then a Lipschitz condition holds and there is exactly one solution through every point of a rectangle

$$0 \leqq g^{2\varepsilon} \leqq \zeta, \ |\chi| < C, \ C > 0. \qquad (2.46)$$

The only solution corresponding to an unbounded σ_n with bounded ϱ is determined by

$$\chi = 0 \text{ at } g^2 = 0.$$

For this solution the weak coupling limit of ϱ becomes

$$\lim_{g^2 \to 0} \varrho = \varrho_+ .$$

All derivatives of ϱ exist at $g^2 = 0$ so that an asymptotic expansion (2.37) holds.

Similarly for $\xi < 0$ there is just one solution of unbounded σ_n. It allows for a power series expansion of ϱ in g^2 with limit ϱ_- for $g^2 \to 0$.

For the existence proof we took integers $n > |\xi| + 1$. But for the formulation of the results it is more natural to use the largest integer n below $|\xi| + 1$

$$n < |\xi| + 1 < n + 1. \tag{2.47}$$

Only then the remainder $g^{2|\xi|} \sigma_n$ is of higher order than the other terms of (2.4—5). The relation between σ_n, with n being the largest integer below $|\xi| + 1$, and any $\sigma_{n'}$ with $n' > |\xi| + 1$ is

$$\sigma_n = \varrho_{\pm n} g^{2(n-|\xi|)} + \dots + \varrho_{\pm(n'-1)} g^{2(n'-1-|\xi|)} + \sigma_{n'} .$$

Hence σ_n and $\sigma_{n'}$ have the same weak coupling limits and all statements concerning solutions of (1.1) apply to the remainder with (2.47) as well. The final results are summarized by:

Theorem 2. If the characteristic exponent ξ is not an integer and $|\xi| > 1$ all solutions λ of (1.1) with bounded ratio λ/g^2 are of the form

$$\lambda(g^2) = \varrho_- g^2 + \dots + \varrho_{-n} g^{2n} + g^{2\xi+2} \sigma_n \text{ if } \xi > 0 \tag{2.48}$$

$$\lim_{g^2 \to 0} \sigma_n = \sigma_0, \qquad \sigma_0 \text{ finite or } \sigma_0 = \infty,$$

and

$$\lambda(g^2) = \varrho_+ g^2 + \dots + \varrho_{+n} g^{2n} + g^{2|\xi|+2} \sigma_n \text{ if } \xi < 0 \tag{2.49}$$

$$\lim_{g^2 \to 0} \sigma_n = \sigma_0, \quad \sigma_0 \text{ finite or } \sigma_0 = -\infty.$$

n is the largest integer less than $|\xi| + 1$. For each value σ_0 there is exactly one solution defined on some interval

$$0 \leqq g^2 \leqq \zeta .$$

For vanishing or diverging σ_0 the solution possesses a power series expansion (2.40) or (2.41), respectively.

3. *Integral characteristic exponent*

(i) *Case* $\xi = 0$

In order to remove the singularity of the differential equation (1.1) in the case $\xi = 0$ we introduce a function τ by

$$\varrho = \varrho_0 - \frac{b}{c_1} \frac{1}{\tau + \ln g^2} \tag{3.1}$$

where

$$\varrho_0 = - \quad c_2 - b \tag{3.2}$$

denotes the coinciding roots $\varrho_\pm$ of (1.9). τ satisfies the differential equation

$$\left(1 + \frac{g^2}{b} S\right) \frac{\mathrm{d}\tau}{\mathrm{d}g^2} = \frac{c_1}{b^2} (\tau + \ln g^2)^2 \, U + \frac{1}{b} (\tau + \ln g^2) \, S - \frac{S}{b} \tag{3.3}$$

with functions S, U as in (2.9) and (2.10) ($n = 1$). Explicitly U is given by

$$U = T - \varrho_0 \, S \tag{3.4}$$

$$\beta_1 = c_1 \lambda^2 + c_2 \lambda g^2 + c_3 g^4 + g^6 \, T. \tag{3.5}$$

Replacing g^2 in (3.3) by the variable $g^{2\varepsilon}$ with $0 < \varepsilon < 1$ the differential equation of τ takes the form

$$\frac{\mathrm{d}\tau}{\mathrm{d}\, g^{2\varepsilon}} = \frac{N}{D}$$

$$N = \frac{c_1}{b^2} g^{2(1-\varepsilon)} (\tau + \ln g^2)^2 U + \frac{1}{b} g^{2(1-\varepsilon)} (\tau + \ln g^2) \, S - \frac{1}{b} g^{2(1-\varepsilon)} \, S \tag{3.6}$$

$$D = \varepsilon \left(1 + \frac{g^2}{b} S\right)$$

for which the right-hand side satisfies a Lipschitz condition in a rectangle

$$0 \leqq g^{2\varepsilon} \leqq \eta', \quad |\tau| \leqq M. \tag{3.7}$$

$M > 0$ is arbitrary and $\eta' > 0$ sufficiently small for given M. This implies the existence and uniqueness of solutions in a domain (3.7) with η' replaced by a number $\eta > 0$ with $\eta \leqq \eta'$.

It remains to discuss solutions λ of (1.1) for which τ is unbounded while ϱ stays bounded. Accordingly we consider solutions of (3.3) for sufficiently small g^2 which are subject to the condition

$$|\varrho| = |\varrho_0 - \frac{b}{c}\frac{1}{\tau + \ln g^2}| \leqq A \tag{3.8}$$

with arbitrary $A > 0$. Similarly as for non-integral ξ it can be shown that

$$\lim_{g^2 \to 0} |\tau| = \infty \tag{3.9}$$

for any unbounded solution τ with bounded ϱ. Instead of τ we introduce the inverse

$$\psi = \tau^{-1} \tag{3.10}$$

which satisfies the differential equation

$$\frac{\mathrm{d}\psi}{\mathrm{d}g^{2\varepsilon}} = \frac{N'}{D'} \tag{3.11}$$

$$N' = \frac{c_1}{b^2} g^{2(1-\varepsilon)} (1 + \psi \ln g^2)^2 \, U + \frac{1}{b} g^{2(1-\varepsilon)} \psi - \frac{1}{b} g^{2(1-\varepsilon)} \psi^2 \, S$$

$$D' = -\varepsilon \left(1 + \frac{g^2}{b} S\right).$$

The solution ψ corresponding to unbounded τ and bounded ϱ has the limit

$$\lim_{g^2 \to 0} \psi = 0 \tag{3.12}$$

(see (3.9)). Accordingly we need only study bounded solutions ψ of (3.11) with bounded ϱ. A regular version of (3.11) is obtained if in the expressions U and S the variable ϱ (also occuring through $\lambda = g^2\varrho$) is replaced by

$$\hat{\varrho} = \varrho_0 - \frac{b}{c_1} \cdot \frac{1}{\psi^{-1} + \ln g^2} = \varrho \tag{3.13}$$

if

$$\psi < \frac{1}{|\ln g^2| + r} \quad \text{or} \quad \psi > \frac{1}{|\ln g^2| - r} \tag{3.14}$$

and

$$\hat{\varrho} = \varrho_0 \tag{3.15}$$

if

$$\frac{1}{|\ln g^2| + r} \leqq \psi \leqq \frac{1}{|\ln g^2| - r}. \tag{3.16}$$

$r > 0$ is arbitrary with the bound η in (3.13) chosen such that

$$\ln g^2 < -r$$

implying

$$|\ln g^2| + r > 0.$$

Then $\hat{\varrho}$ is a bounded function of g^2 in (3.13) since

$$\left| \frac{1}{\psi^{-1} + \ln g^2} \right| < \frac{1}{r}$$

if ψ satisfies one of the inequalities (3.14). With $\hat{\varrho}$ substituted for ϱ we obtain a differential equation

$$\frac{\mathrm{d}\psi}{\mathrm{d}g^{2\varepsilon}} = \frac{N''}{D''} \tag{3.17}$$

for which the right-hand side satisfies a Lipschitz condition in a rectangle

$$0 \leqq g^{2\varepsilon} \leqq \eta', \quad |\psi| \leqq C, \ \eta', C > 0. \tag{3.18}$$

Hence there is exactly one solution ψ'' with

$$\lim_{g^2 \to 0} \psi'' = 0. \tag{3.19}$$

We want to show that there is also only one solution ψ' of (3.15) with

$$\lim_{g^2 \to 0} \psi' = 0 \quad \text{and } \varrho \text{ bounded.} \tag{3.20}$$

The differential equations (3.11) and (3.1) imply

$$\frac{\mathrm{d}\psi}{\mathrm{d}g^{2\varepsilon}} = 0 \text{ at } g^2 = 0 \text{ (for bounded } \psi' \text{ and } \varrho), \tag{3.21}$$

$$\frac{\mathrm{d}\psi''}{\mathrm{d}g^{2\varepsilon}} = 0 \text{ at } g^2 = 0 \text{ (for bounded } \psi''). \tag{3.22}$$

Combining this with (3.19) and (3.20) we get

$$\lim_{g^2 \to 0} \psi' \ln g^2 = 0, \tag{3.23}$$

or

$$\lim_{g^2 \to 0} \psi'' \ln g^2 = 0, \tag{3.24}$$

respectively. With this information it becomes possible to prove that a solution ψ' of (3.11) with (3.20) also solves (3.17) with (3.19) and vice versa. Let ψ denote one of these solutions. Negative values of ψ imply (3.16) so that $\varrho = \hat{\varrho}$. For $\psi \geqq 0$ we use (3.23) or (3.24) which also imply (3.16) for sufficiently small η in (3.13). Hence in any case the right-hand sides of (3.11) and (3.17) are identical and so are the solutions ψ' and ψ'' with the properties (3.20) or (3.19). This completes the proof that there is precisely one solution ψ of (3.11) with the property (3.20). As a consequence there is one and only one solution of (1.1) for which τ is unbounded with bounded ϱ.

We finally show that the solution ϱ with unbounded τ allows for a power series expansion. To this end we introduce the remainder σ_n by

$$\varrho = \varrho_0 + \varrho_1 g^2 + \dots + \varrho_{n-1} g^{2n-2} + \sigma_n \tag{3.25}$$

and set

$$\sigma_n = -\frac{b}{c_1}\frac{1}{\tau_n + \ln g^2}. \tag{3.26}$$

The functions τ and τ_n are related by

$$-\frac{b}{c_1}\frac{1}{\tau + \ln g^2} = \varrho_1 g^2 + \dots + \varrho_{n-1} g^{2n-2} - \frac{b}{c_1}\frac{1}{\tau_n + \ln g^2}. \tag{3.27}$$

We are interested in the solution of (1.1) with

$$\lim_{g^2 \to 0} |\tau_n| = \infty \tag{3.28}$$

which implies

$$\lim_{g^2 \to 0} \psi_n = 0 \quad \text{for} \quad \psi_n = \tau_n^{-1}. \tag{3.29}$$

ψ_n satisfies the differential equation

$$-\varepsilon\left(1 + \frac{g^2}{b} S\right)\frac{d\psi_n}{dg^{2\varepsilon}} = \frac{c_1}{b^2} g^{2(n-\varepsilon)} (1 + \psi_n \ln g^2)^2 U - \frac{1}{b} g^{2(1-\varepsilon)} \psi_n (1 +$$

$$+ \psi_n \ln g^2) S - \frac{1}{b} g^{2(1-\varepsilon)} \psi_n^2 S + \frac{1}{c_1} g^{2(1-\varepsilon)} \psi_n^2 P \tag{3.30}$$

in the notation of equation (2.8). From this and (3.29)

$$\lim_{g^2 \to 0} \frac{\psi_n}{g^{2(1-\varepsilon)}} = 0 \tag{3.31}$$

follows. This information may be used to start an inductive procedure for determining the weak coupling behaviour of ψ_n. In terms of the variable g^2 the differential equation (3.30) takes the form

$$-\left(1+\frac{g^2}{b}S\right)\frac{d\psi_n}{dg^2}=-\frac{S}{b}\psi_n+\frac{c_1}{b^2}g^{2n-2}(1+\psi_n\ln g^2)^2\,U-$$

$$-\frac{1}{b}\psi_n(1+\psi_n\ln g^2)\,V+\frac{1}{c_1}\psi_n^2\,P. \tag{3.32}$$

Inserting (3.31) on the right-hand side we find

$$\lim_{g^2\to 0}\frac{\psi_n}{g^2}=0 \text{ for } n>1,$$

if (3.29) holds. Using this result again on the right-hand side of (3.32) an estimate for ψ_n/g^4 is found. By induction we finally arrive at the converging limit

$$\lim_{g^2\to 0}\frac{\psi_n}{g^{2n}}=-\frac{c_1}{n\,b^2}\lim_{g^2\to 0}U$$

implying

$$\psi_n=o(g^{2n}). \tag{3.33}$$

Since n can be chosen arbitrarily large in the form

$$\varrho=\varrho_0+\varrho_1 g^2+\ldots+\varrho_{n-1}g^{2n-2}-\frac{b}{c_1}\frac{\psi_n}{1+\psi_n\ln g^2} \tag{3.34}$$

of (3.25) we find an asymptotic series

$$\varrho=\sum_{j=0}^{\infty}\varrho_j g^{2j} \tag{3.35}$$

for the solution of (1.1) with τ diverging in the weak coupling limit.

We summarize the results by the following theorem:

Theorem 3. If the characteristic exponent ξ vanishes all solutions of (1.1) with bounded ratio λ/g^2 are of the form

$$\lambda(g^2)=\varrho_0 g^2-\frac{b}{c_1}\frac{g^2}{\tau+\ln g^2}$$

$$\lim_{g^2\to 0}\tau=\tau_0,\ \tau_0 \text{ finite or } \lim_{g^2\to 0}|\tau|=\infty. \tag{3.36}$$

For every finite value of τ_0 and for diverging τ there is exactly one solution defined on some interval

$$0 \leqq g^2 \leqq \zeta, \ \zeta > 0. \tag{3.37}$$

For diverging τ the solution possesses a power series expansion

$$\lambda(g^2) = \varrho_0 g^2 + \sum_{n=1}^{\infty} \varrho_n g^{2n+2}. \tag{3.38}$$

(ii) *Case* $\xi \neq 0$

In case of integral non-vanishing ξ a formal expansion of $\varrho = \lambda/g^2$ was derived in Ref. 1 which also involves logarithms:

$$\begin{aligned} \varrho &= \varrho_+ + \sum_{j=1}^{|\xi|-1} \varrho_{+j} g^{2J} + d_+ g^{2|\xi|} \ln g^2 + \dots \text{ if } \xi < 0, \\ \varrho &= \varrho_- + \sum_{j=1}^{|\xi|-1} \varrho_{-j} g^{2J} + d_- g^{2\xi} \ln g^2 + \dots \text{ if } \xi > 0. \end{aligned} \tag{3.39}$$

The remainder τ defined by

$$\varrho = \varrho_+ + \sum_{j=1}^{|\xi|-1} \varrho_{+j} g^{2J} + d_+ g^{2|\xi|} \ln g^2 + g^{2|\xi|} \tau \text{ if } \xi < 0, \tag{3.40}$$

$$\varrho = \varrho_- + \sum_{j=1}^{\xi-1} \varrho_{-j} g^{2J} + d_- g^{2\xi} \ln g^2 + g^{2\xi} \tau \text{ if } \xi > 0$$

satisfies a differential equation of the form

$$\frac{d\tau}{dg^{2\xi}} = \frac{N}{D}, \ 0 < \varepsilon < 1,$$

$$\begin{aligned} N = c_1 g^{2(|\xi|+\varepsilon)} \sigma^2 + g^{2(1-\varepsilon)} \Pi_0 + g^{2(|\xi|+\varepsilon)} \sigma \Pi_1 + g^{2(1-\varepsilon)} R + g^{2(1-\varepsilon)} \sigma V \\ + g^{2(|\xi|+1-\varepsilon)} P \sigma^2 - c g^{2(1-\varepsilon)} S, \end{aligned} \tag{3.41}$$

$$\sigma = d_\pm \ln g^2 + \tau,$$

$$D = b\varepsilon + \varepsilon g^2 S$$

with functions S, V as in (2.9—10). R is of the form

$$R = R_0 + \varrho R_1 + \dots + \varrho^{|\xi|+3} R_{|\xi|+3}$$

where $R_0, \ldots, R_{|\xi|+2}$ are functions of g^2 with all derivatives existing near $g^2 = 0$. $R_{|\xi|+3}$ depends on g^2 and $\lambda = \varrho g^2$ with all partial derivatives existing near $\lambda = g^2 = 0$. Π_0 is a polynomial in g^2. P and Π_1 are polynomials in g^2 and the variable (2.13).

The right-hand side of (3.41) satisfies a Lipschitz condition in a rectangle (3.7) so that the existence and uniqueness of solutions is implied in a rectangle (3.7) with η' replaced by $\eta > 0$ with $\eta \leqq \eta'$. This covers all solutions which stay bounded for $g^2 \to 0$.

As in the previous cases unbounded solutions τ have the property

$$\lim_{g^2 \to 0} |\tau| = \infty \tag{3.42}$$

so that

$$\varphi = \tau^{-1} \tag{3.43}$$

has the limit

$$\lim_{g^2 \to 0} \varphi = 0. \tag{3.44}$$

As in the case of non-integral $|\xi| > 1$ it can be shown that the limit (2.27) holds. Then

$$\lim_{g^2 \to 0} \tau = \infty \text{ if } \xi > 0,$$

$$\lim_{g^2 \to 0} \tau = -\infty \text{ if } \xi < 0. \tag{3.45}$$

Instead of φ the function χ defined by (2.29) is introduced for which (2.30) holds. If $\xi > 0$, we need only consider solutions χ in a domain (2.31). χ satisfies the differential equation

$$\frac{d\chi}{dg^{2\varepsilon}} = -\varphi^2 \frac{N}{D} - \frac{\xi}{\varepsilon} \frac{g^{2(\xi-\varepsilon)}}{\varrho_+ - \varrho_-} \quad (\xi > 0). \tag{3.46}$$

Since only non-negative solutions χ are of interest, we modify the right-hand side of (3.45) for $\chi < 0$ by substituting

$$\varrho = \varrho_- + \sum_{j=1}^{\xi-1} \varrho_{-j} g^{2j} + d_- g^{2\xi} \ln g^2 + \Theta(\chi) \frac{g^{2\xi}}{\chi + \frac{1}{2} g^{2\xi} (\varrho_+ - \varrho_-)^{-1}}$$

$$\Theta(\chi) = 1 \text{ for } \chi \geqq 0, \ \Theta(\chi) = 0 \text{ for } \chi < 0 \tag{3.47}$$

for the variable ϱ. Then the right-hand side of (3.46) satisfies a Lipschitz condition. There is exactly one solution with

$$\chi(0) = 0.$$

This solution determines the only unbounded solution τ of (3.41) with bounded ϱ. By (3.47) the limit of ϱ is

$$\lim_{g^2 \to 0} \varrho = \varrho_+ \tag{3.48}$$

for this unbounded solution. As for non-integral ξ it can be shown that an asymptotic expansion

$$\varrho = \varrho_+ + \sum_{n=1}^{\infty} \varrho_{+n} g^{2n} \tag{3.49}$$

holds.

Likewise an asymptotic expansion

$$\varrho = \varrho_{-} + \sum_{n=1}^{\infty} \varrho_{-n} g^{2n} \tag{3.50}$$

follows if the coefficient $d_- = 0$. Then the coefficient

$$\varrho_{-\xi} = \tau_0 \tag{3.51}$$

can be chosen arbitrarily by taking the solution τ of (3.41) with initial value τ_0 at $g^2 = 0$.

With a similar discussion of the case $\xi < 0$ the final results can be summarized by the following theorem:

Theorem 4. If the characteristic exponent ξ is a non-vanishing integer all solutions λ of (1.1) with bounded ratio λ/g^2 are of the form

$$\lambda(g^2) = \varrho_- g^2 + \sum_{j=1}^{\xi-1} \varrho_{-j} g^{2j+2} + d_- g^{2\xi+2} \ln g^2 + g^{2\xi+2} \tau$$

$$\lim_{g^2 \to 0} \tau = \tau_0, \; \tau_0 \text{ finite or } \tau_0 = \infty, \text{ if } \xi \text{ is a positive integer,} \tag{3.52}$$

and

$$\lambda(g^2) = \varrho_+ g^2 + \sum_{j=1}^{|\xi|-1} \varrho_{+j} g^{2j+2} + d_+ g^{2|\xi|+2} \ln g^2 + g^{2|\xi|+2} \tau$$

$$\lim_{g^2 \to 0} \tau = \tau_0, \quad \tau_0 \text{ finite or } \tau_0 = -\infty, \text{ if } \xi \text{ is a negative integer.} \tag{3.53}$$

For each value of τ_0 there is exactly one solution defined in some interval

$$0 \leqq g^2 \leqq \zeta, \; \zeta > 0.$$

For vanishing $d_{\pm}$ and finite τ_0 the solution possesses a power series expansion

$$\lambda(g^2) = \varrho_- g^2 + \sum_{j=1}^{\infty} \varrho_{-j} g^{2j+2} \text{ if } \xi > 0, d_- = 0, \tag{3.54}$$

$$\lambda(g^2) = \varrho_+ g^2 + \sum_{j=1}^{\infty} \varrho_{+j} g^{2j+2} \text{ if } \xi < 0, \ d_+ = 0, \tag{3.55}$$

with $\varrho_{-\xi} = a_0$ or $\varrho_{+|\xi|} = a_0$, respectively. Thus there is a one parametric family of solutions with power series expansions if ξ is a positive integer and $d_- = 0$ or ξ is a negative integer and $d_+ = 0$.

For diverging τ_0 the solution possesses a power series expansion

$$\lambda(g^2) = \varrho_+ g^2 + \sum_{n=1}^{\infty} \varrho_{+n} g^{2n+2} \text{ if } \xi > 0, \ \tau_0 = \infty, \tag{3.56}$$

and

$$\lambda(g^2) = \varrho_- g^2 + \sum_{n=1}^{\infty} \varrho_{-n} g^{2n+2} \text{ if } \xi < 0, \ \tau_0 = -\infty. \tag{3.57}$$

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TEOREMI EGZISTENCIJE ZA DIFERENCIJALNU JEDNADŽBU S DVA EFEKTIVNA VEZANJA

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Za modele kvantne teorije polja bez mase s dva nezavisna vezanja, invarijantnost na renormalizacionu grupu znači diferencijalnu jednadžbu koja povezuje efektivna vezanja g i λ. U najnižem redu pretpostavljeno je da β-funkcija od g^2 ima oblik bg^4 sa $b \neq 0$, dok je β-funkcija od λ općeg bilinearnog oblika u λ i g^2. Za egzaktni sistem, rješenja diferencijalne jednadžbe proučavana su u blizini singularne točke $\lambda \approx g \approx 0$. Određena su sva rješenja $\lambda(g^2)$ za koja omjer λ/g^2 ostaje vezan u granici $g \to 0$. Za male vrijednosti od g dobiven je eksplicitni oblik asimptotskog ponašanja u skladu s formalnim asimptotskim razvojima nađenima ranije.