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PHYSICAL APPLICATIONS OF JULIA SETS

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We review some properties of iterated polynomials and rational fractions in the complex plane. We give several interpretations of the corresponding invariant sets, usually called Julia sets. These interpretations apply to two-dimensional electrostatic, one dimensional almost periodic Schrödinger equation, vibrational spectra of fractal structures, and location of the zeroes of the partition function for exactly renormalizable lattice spin systems.

1. Introduction

Among his last works, Vladimir Jurko Glaser, participated to the study of iterations of a special mapping related to a spin glass model[1]. His deep knowledge of the analytic function theory would have been most valuable for the future of the theory of iterations; and we dedicate to him this short review of some aspects of the complex iteration theory. We shall display a few examples of models in which a functional equation related to iteration theory plays a central role, and indicate to which kind of physical situations they are relevant.

Certainly the most important recent applications of iteration theory are in the domain of the onset of turbulence[2], in which the renormalization group approach has been very powerful, as it is in the analysis of iteration of real variables[3]. The

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iteration of polynomials and rational fractions is a long standing subject since the classical works by Julia and Fatou[4]. Motivated first by the question: to which root will converge the Newton iteration scheme for calculation of polynomials roots, the study of iterations in the complex plane has now been widely popularized, and produces fascinating fractal shapes[4]. The study of complex iteration has been extended to measures[6], and their relation to conformal dynamical systems extensively described[7]. A detailed analysis of the invariant sets under a polynomial (or rational fraction) transformation is now available including the dependence of the dynamics as a function of the parameter entering the transformation[8,9].

We shall give a short description of invariant sets and invariant measures under rational transformation and give first the electrostatic description of Julia sets for polynomials. Then we will review the orthogonality properties of iterated polynomials, which will display a quantum mechanical interpretation for real Julia sets. An analogous case will be mentioned, namely the vibration properties of some fractal structure. Finally a thermodynamical description will be reported for some rational fraction iterations. In all interpretations we will emphasize the special role played by renormalization group arguments, and show the corresponding functional equations, and shortly describe their singular behaviour.

2. *Definition of Julia sets for rational transformation*

Let $T(z)$ be a rational transformation, namely:

$$z' = T(z) = \frac{P(z)}{Q(z)}, \tag{1}$$

where P and Q are polynomials whithout common factor. The degree N of T is defined as the supremum of the degrees of P and Q and we assume $N \geqslant 2$. The Julia set $\mathcal{J}_T$ of T is defined as the closure of the set of the repulsive fixed points of any order of T. If we denote $T^{(0)}(z) = z$, the iterates of T will be defined recursively by $T^{(n)}(z) = T(T^{(n-1)}(z))$ for any n > 0. A fixed point is a solution of $T^{(k)}(z) = z$ for some $k > 0$, and it will be repulsive if the derivative $(T^{(k)}(z))'$ at this point has a modulus greater than one.

This definition of the Julia set $\mathcal{J}_T$ coincides with the more classical definition: $z \in \mathcal{J}_T$ if and only if the sequence $T^{(n)}(z)$ is not normal (in the sense of Montel, see Ref. 4 for details), in the vicinity of z. The Julia set is totally invariant under T, which means that if $z \in \mathcal{J}_T$, then $T(z) \in \mathcal{J}_T$, and also $T_i^{-1}(z) \in \mathcal{J}_T$, where the T_i^{-1} are the N inverses functions of T.

When T is a polynomial, then $\mathcal{J}_T$ is the boundary of the basin of attraction of infinity. It is convenient in this case to define the filled-in Julia set K_T, as the complement of the basin of attraction of infinity: in other words, $z \in K_T$ if and only if the sequence $T^{(n)}(z)$ is bounded. $\mathcal{J}_T$ is also the boundary of K_T. Let us comment slightly more the case where T is a polynomial of degree 2. By a linear change of coordinates, one can always reduce the problem to the one parameter family $f_c(z) = z^2 + c$[8,9]. Then there are two possibilities: either the critical point of f_c, which is $z_c = 0$, belongs to K_{f_c}, in which case K_{f_c} is connected, or $z_c = 0$ is attracted to infinity under f_c, in which case K_{f_c} is a completely disconnected set (Cantor set), sometimes called Fatou's dust. Now, in the complex plane of the variable c, the set of values of c such that K_{f_c} is connected is called the Mandelbrot $\mathcal{M}$-set. and it has amazing fractal properties. Nice pictures of Julia sets or $\mathcal{M}$-sets are available in Refs. 8, 9 and 4.

An other important general property deserves to be mentioned here. Indifferent fixed points are solutions of $T^{(k)}(z_0) = z_0$ such that $(T^{(k)}(z))' = e^{2i\pi\alpha}$ has modulus one. When the argument α is rational, or irrational satisfying a kind of Liouville condition [see Ref. 4, example (3.21)], then z belongs to $\mathcal{J}_T$. If one the other hand, α is irrational fulfilling a diophantine condition [see Ref. 4], it has been proven by Siegel that $z_0 \notin \mathcal{J}_T$ and that there is an open set around z such that the following equation, named Schröder equation:

$$\begin{cases} K(T^{(k)}(z)) = e^{2i\pi\alpha}\, T^{(k)}(z) \\ K(z_0) = 0,\ K'(z_0) = 1, \end{cases} \tag{2}$$

has a solution $K(z)$, which is analytic in the vicinity of z_0. In other words using $K(z)$ as a new coordinate function, the action of $T^{(k)}$ reduces to a rotation around z_0 with irrational angle, and there exist invariant curves corresponding to circles in coordinates $K(z)$. When T is a rational fraction more complicated situations can occur, and there may exist domains topologically equivalent to an annulus, in which we can find a coordinate system for which $f^{(k)}$ acts as an irrational rotation (Herman Rings). We refer to the classical papers and to review article[4] for a general discussion, including for instance examples where $\mathcal{J}_T$ is the full complex plane, or Riemann sphere.

On the Julia set, there exists a natural invariant measure[6], which can easily be described as follows: take an arbitrary point z_0 in the complex plane: then compute its N preimages, that is the N solutions of $T(z) = z_0$, call them $z_1^{(i)}$, $i = 1 \ldots N$. Then compute the preimages of each point $z_1^{(i)}$ and call them $z_2^{(i)}$, $i = 1 \ldots N^2$ and so one. At order k, there are N^k preimages of z_0, called $z_k^{(i)}$, $i = 1, \ldots N^k$. Now define for any $k \geq 0$ a discrete measure $\mu(k)$ having point masses as each $z_k^{(i)}$ will equal weights $\frac{1}{N^k}$ in such a way that the total mass be one. We therefore have a sequence of probability measures, and one can prove[6] that this sequence has a (weak) limit independent of z_0, which is the natural asymptotic distribution of predecessors. Note that there may exist exceptional starting points z_0 (at most two) for which this statement is not true. This limiting measure allows using small computers to perform plots of Julia sets: take an arbitrary starting point on $\mathcal{J}_T$ (for instance a repulsive fixed point of order one), compute its preimages, plot them, then choose one of them at random and consider this point as the new starting point and repeat the procedure. The measure represents also the asymptotic distribution of fixed points of large order[6]. We shall now describe more explicitly some properties of this measure: let T_i^{-1}, $i = 1, \ldots N$, be an assignment of branches for the inverse function. The only ambiguities are related to the existence of critical points, that is point z_c such that $T'(z_c) = 0$. In such case $T(z_c)$ is called a critical value. We can always perform the choice of the branches of the inverse function in such a way that for $i \neq j$, and for any set E in the complex plane, $T_i^{-1}(E) \cap \cap T_j^{-1}(E)$ contains at most a finite number of critical points, which will have measure zero. Under this circumstances, the above mentioned measure is invariant:

$$\mu(E) = \mu(T^{-1}E), \tag{3}$$

and balanced[6, 10]:

$$\mu(E) = N\,\mu(T_i^{-1}(E)), \tag{4}$$

for any inverse branch, and any Borel subset E of the complex plane. The measure has no point mass, and as we shall see in the next section, it is nothing else than the equilibrium harmonic measure on the Julia set in the case where T is a polynomial.

3. *Julia sets for polynomials: Electrostatic description*

Let T be a polynomial of degree $N \geqslant 2$, and for simplicity assume T be monic, that is its highest degree coefficient is equal to one:

$$T(z) = z^N + t_1 z^{N-1} + \dots t_N. \tag{5}$$

When z goes to infinity, $T(z)$ behaves as z^N, so one may ask whether a suitable change of coordinates allows to express the action of T as the action of z^N. We therefore look for a function $B(z)$, which behaves as z when z is large, and such that $B(z)/z$ is analytic in the vicinity of infinity:

$$B(z) = z\left\{1 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots + \frac{b_n}{z^n} + \dots\right\}, \tag{6}$$

and fulfills the functional equation:

$$B(T(z)) = (B(z))^N. \tag{7}$$

$B(z)$, sometimes called the Böttcher function is the requested change of coordinate. It is rather easy to show that for $|z|$ large enough a solution of (7) is given by:

$$B(z) = \lim_{n\to\infty} \{T^{(n)}(z)\}^{\frac{1}{N^n}}, \tag{8}$$

where the $1/N^n$ root is choosen so to behave as z for z going to infinity. Furthermore (7) allows to extend $B(z)$ to the full basin of attraction of infinity denoted $A(\infty)$, but algebraic branch point will occur at any critical value of T and its iterates $T^{(n)}(z)$. If no critical value of T belongs to the basin of attraction of infinity, then $B(z)$ is analytic in $A(\infty)$, the filled-in Julia set is connected, and $B(z)$ maps conformally $A(\infty)$, the exterior of K_T, toward the exterior of the unit disk. If some critical values of T belongs to $A(\infty)$, then $B(z)$ is multivalued, but $|B(z)|$ is still well defined and $\ln |B(z)|$ is a harmonic function, vanishing on the Julia sets. Hence $\ln |B(z)|$ is nothing else than the Green's function[11] of the complement of K_T.

From classical results in harmonic analysis, one easily checks that the $\ln |B(z)|$ is nothing but the logarithmic potential generated by the equilibrium measure on the Julia set $\mathcal{J}_T$ and that this equilibrium measure is nothing else than the balanced invariant measure introduced in the previous section to describe the asymptotic distribution of predecessors. In fact we define the analytic function,

$$G(z) = \int \ln (z - y) \; \mathrm{d}\mu (y) \tag{9}$$

which has a logarithmic singularity around infinity, and we then have:

$$B(z) = \mathrm{e}^{G(z)}. \tag{10}$$

One can deduce (10) from (9) by using the invariance (3) and balance properties (4) of the measure. Indeed we have from (9), using (3) and (4), and modulo $2ik\pi$:

$$G(T(z)) = \int \ln (T(z) - y) \, \mathrm{d}\mu (y)$$

$$= \Sigma \int \ln [z - T_i^{-1}(y)] \, \mathrm{d}\mu (y)$$

$$= N \int \ln (z - y) \, \mathrm{d}\mu (y).$$

Therefore we get:

$$G(T(z)) = N\, G(z) + 2\mathrm{i}\, k\, \pi, \tag{11}$$

which gives the Böttcher equation (7). This allows us to identify the real part of $G(z)$ with the electrostatic two dimensional potential generated by a charge distribution associated to the measure μ:

$$\mathrm{Re}\, G(z) = \int \ln |z - y| \, \mathrm{d}\mu (y). \tag{12}$$

Taking the derivative of (11), one obtains the following equations for the generating function of the measure:

$$g(z) = \frac{\mathrm{d}}{\mathrm{d}z} G(z) = \int \frac{\mathrm{d}\mu(y)}{z - y} = \sum_n \frac{\mu_n}{z^{n+1}}, \tag{13}$$

$$N\, g(z) = T'(z)\, g(T(z)), \tag{14}$$

where the μ_n are the moments of μ:

$$\mu_n = \int z^n \, \mathrm{d}\mu (z). \tag{15}$$

So we see that the functional equations (7), (11) or (14) allow to determine the electrostatic properties for the Julia set. Furthermore (14) allows to compute recursively the moments of the measure μ.

In order to bring back the description to a more usual scheme, one has to remember how electrostatic in two dimension is related to the usual three dimensional picture: we consider the potential generated by a grounded three dimensional cylindrical conductor in the vertical direction, the horizontal section of which is the Julia set. The electric charge per unit vertical length is equal to $\mu(E)$ on any subset E of the horizontal section. A capacity made of two cylindrical vertical plates, the section of the internal one being the Julia set, and the section of the external one being a very large circle, will have capacity one per unit vertical length.

The local shape of the Julia set is in general not smooth, and the potential will have singularities close to the set. In order to analyse the nature of the singularity near a point $z \in \mathcal{J}_T$, we define[12] the Mellin transform:

$$M_z(s) = \int_{\mathcal{J}_T} (z - y)^{-s} \, \mathrm{d}\mu(y). \tag{16}$$

When z is a repulsive fixed point, one finds[12] that $M_z(s)$ is meromorphic, and holomorphic for Re s large enough, and from these properties, one deduces that $G(z)$ has a power law behaviour near any periodic repulsive fixed point. A more explicit example will be displayed in the next section.

4. *Real Julia sets for polynomials: The quantum mechanical description*

a) *Orthogonality properties of iterated polynomials*

Let us begin by an example. Consider the polynomial:

$$T(z) = z^2 - 2. \tag{17}$$

Setting $z = 2\cos\Theta$, and $z_1 = 2\cos\Theta_1 = T(z)$, we get modulo 2π, $\Theta_1 = 2\Theta$. For the n-th iterate $z_n = T^{(n)}(z) = 2\cos\Theta_n$, we get $\Theta_n = 2^n\Theta$. This remark allows to relate $T^{(n)}(z)$ to the usual Chebyschev polynomials:

$$T_p(\cos\Theta) = \cos p\,\Theta, \tag{18}$$

and we get:

$$T^{(n)}(z) = 2\,T_{2^n}(z/2). \tag{19}$$

So the iterated polynomials $T^{(n)}(z)$ belong to a sequence of orthogonal polynomials. This result is in fact much more general[10, 13–16]. In fact we have the following orthogonality properties with respect to the balanced invariant measure on the Julia set $\mathcal{J}_T$ associated with the monic polynomial T of degree N, under the additional hypothesis that the term of degree $(N-1)$ has a vanishing coefficient:

$$T^{(n)}(z) = P_{N^n}(z) = \pi_{N^n}(z), \tag{20}$$

where the polynomials P_k and π_k are monic polynomials of degree k defined by the orthogonality properties:

$$\int_{\mathcal{J}_T} P_k(z)\, P_l(z)\, \mathrm{d}\mu(z) = \delta_{kl} h_l, \tag{21}$$

$$\int_{\mathcal{J}_T} \bar{\pi}_k(z)\, \pi_l(z)\, \mathrm{d}\mu(z) = \delta_{kl} \tilde{h}_l. \tag{22}$$

The scalar product (22) is hermitian, but (21) is not, however the two scalar products, and the set of orthogonal polynomials coincide when the set is real. The additional condition (vanishing of the z^{N-1} term) can always be fulfilled through a translation, but (20) is not true if this condition is not achieved. However, we always have:

$$P_{pN^k}(z) = P_p(T^{(k)}(z)). \tag{23}$$

$$\pi_{pN^k}(z) = \pi_p(T^{(k)}(z)). \tag{24}$$

We recover the Chebyschev case for $T(z) = z^2 - 2$, in which case the Julia set is the closed interval $[-2, +2]$, and the invariant measure $\frac{\mathrm{d}x}{\pi\sqrt{4 - x^2}}$ which up to a trivial change of variable is the defining measure for Chebyschev polynomials. Notice that the normalization parameter h_l and $\tilde{h}_l$ in (21) and (22) arise from the fact that P and π are monic polynomials. They satisfy $h_{lN} = h_l$, $\tilde{h}_{lN} = \tilde{h}_l$.

We observe that the sequence of iterated polynomials $1, T(z) \ldots T^{(k)}(z) \ldots$, has degrees $0, N, \ldots, N^k, \ldots$ and the above results show that its is possible to define intertwining polynomials for the lacking degrees in such a way that the sequence is a complete sequence of orthogonal polynomials. The balanced invariant measure gives a solution to this question when the coefficient of z^{N-1} in T vanish. If we only look to the so called formal orthogonal polynomials[17], it is possible to find a solution in every case, and in general the solution is not unique, depending on $(N - 1)$ arbitrary parameters[18]. But all possible solutions satisfy (23) and (24).

b) *Real Julia sets*

Assume now that T is a monic polynomial, with real coefficients, and that there exists an interval S in the real line such that when $y \in S$, all solution x_l of the equation $T(x) = y$ are real and belong to S. Under these hypothesis the Julia set of T is real and included in S. The invariant balanced measure has a real support, included in S, an we associate to it the set of orthogonal monic polynomials:

$$\int_S P_k(x)\, P_l(x)\, \mathrm{d}\mu(x) = h_l\, \delta_{kl}, \tag{25}$$

which satisfy:

$$P_k(T(x)) = P_{Nk}(x). \tag{26}$$

One way of computing the polynomials P_n is first to get the moments of the measure from equation (13) and (14), and then use the classical formulas[19] relating the orthogonal polynomials to the moments. One gets in particular:

$$P_0 = 1, \; P_1(x) = x + \frac{a}{N}, \tag{27}$$

where a is the coefficient of z^{N-1} in T. Then (26) tells us that:

$$P_1(T^{(l)}(x)) = T^{(l)}(x) + \frac{a}{N} = P_{N^l}(x), \tag{28}$$

which is the general form to be compared to (20) when a vanishes.

The polynomials P_n fulfill the usual three terms recursion relation[19]: for $n \geqslant 0$, using the convention $P_{-1} = 0$, we have:

$$P_{n+1}(x) = (x - B_n) P_n(x) - R_n P_{n-1}(x), \tag{29}$$

with

$$B_n = \frac{1}{h_n} \int_S x P_n^2(x) \, d\mu(x), \qquad R_n = h_n / h_{n-1}. \tag{30}$$

The coefficients B_n and R_n occur in the continued fraction expansion of the generating function $g(z)$ given in (13) and (14):

$$g(z) = 1/(z - B_0 - R_1/(z - B_1 - \ldots/(z - B_n - R_{n+1}/(z - B_{n+1} - \ldots))\ldots)). \tag{31}$$

The equation (14) can be iterated, and we get

$$g(z) = N^{-k} T^{(k)\prime}(z) g(T^{(k)}(z)). \tag{32}$$

We define as usual the Padé approximant $[p - 1/p]_g(z)$ to the function g around infinity, as the rational fraction, the numerator and the denominator of which have degree $(p - 1)$ and p, respectively, such that for z large, we have:

$$g(z) - [p - 1/p]_g(h) \sim 0(z^{-(2p+1)}). \tag{33}$$

Comparing (32) and the expansion (13) shows that:

$$[pN^k - 1/pN^k]_g(z) = \frac{T^{(k)\prime}(z)}{N} [p - 1/p]_g(T^{(k)}(z)). \tag{34}$$

The [0/1] Padé approximant is easily computed from the first moments, and we have (cf. 27):

$$[0/1]_g(z) = \left(z + \frac{a}{N}\right)^{-1}. \tag{35}$$

We therefore get:

$$[N^k - 1/N^k]_g (z) = \frac{T^{(k)'} (z)}{N^k (T_k (z) + a/N)}. \qquad (36)$$

So the rational fraction in (36) has the same continued fraction expansion as $g(z)$ in equation (32), but truncated, which means that R_{N^k+1} is replaced by 0 in the right hand side.

c) The quantum mechanical interpretation

We shall now give an operator form[20] for the three terms recursion relation (29). We define the infinite Jacobi matrix $H_{i,j}$, $i \geqslant 0, j \geqslant 0$, all elements of which vanish except for $i = 0, 1, 2....$:

$$H_{ii} = B_i, H_{i,i+1} = 1, H_{i+1,i} = R_{i+1}. \qquad (37)$$

We define the column vector $\varphi(x)$ by its components:

$$\varphi_n (x) = P_n (x), \quad n \geqslant 0, \qquad (38)$$

and finally we define the »decimation« operator D by its action on any column vector:

$$(D \varphi)_n = \varphi_{Nn}. \qquad (39)$$

Then equation (29) can be rewritten as:

$$H \varphi (x) = x \varphi (x), \qquad (40)$$

and equation (26) is equivalent to the operator equation:

$$H D = D T(H). \qquad (41)$$

If we use normalized polynomials $\hat{P}_n$ (x), satisfying:

$$\int_S \hat{P}_k (x) \hat{P}_l (x) \mathrm{d}\mu (x) = \delta_{kl} \qquad (42)$$

and the corresponding φ, we see that now we have:

$$\hat{H} \hat{\varphi} (x) = x \hat{\varphi} (x), \qquad (43)$$

where H is now a hermitian semi-infinite matrix, defined by:

$$\hat{H}_{ii} = B_i, \hat{H}_{i,i-1} = \hat{H}_{i-1,i} = \sqrt{R_i}. \qquad (44)$$

It appears[16,18,21] that $\hat{H}$ fulfills the same equation as H:

$$\hat{H} D = D T(\hat{H}). \tag{45}$$

Now we give a physical interpretation of equation (45) : $\hat{H}$ being a hermitian tridiagonal matrix, it can be interpreted as the vibration mode equation for a linear inhomogeneous chain of masses connected with springs[21]. Such a matrix appears also in a simplified tight-binding approximation for electron in a one dimensional disordered crystal. The interesting feature of our particular case is that the spectrum of H is the Julia set itself, and as we shall see in the following discussion, we get an explicit example of a Schrödinger-like discrete operator with a singular continuous spectrum.

From now one, we shall restrict ourselves to the case $T(z) = z^2 - \lambda$, with λ real > 2, for which the Julia set is real and of Lebesgue measure zero. In this case the recursion relation for the coefficients R_n occuring in H or $\hat{H}$, which can be deduced for instance from (41), have the simple following form[14-16]:

$$\begin{cases} R_{2n} + R_{2n+1} = \lambda \\ R_{2n} R_{2n-1} = R_n. \end{cases} \tag{46}$$

In addition we have $B_n = 0$ for any $n > 0$.

We initialize the recursion using the convention $R_0 = 0$, and we get $R_1 = \lambda$, $R_2 = 1$, $R_3 = \lambda - 1$, $R_4 = (\lambda - 1)^{-1}$, and so one. A remarkable property of these coefficients is their almost periodicity property[14,20,23], defined by:

$$\lim_{k \to \infty} R_{p2^k+s} = R_s, \tag{47}$$

the limit being reached uniformly with respect to k and s. Therefore our model gives an explicit solvable example of almost periodic operator, a subject extensively studied in the recent years for solid state physics problems as Peierls instability for one dimensional conductors[24], or electronic properties of crystals under magnetic field[25]. Quasiperiodic operators have singular continuous spectrum[26] under some special circumstances but our model is the only one in which the explicit behaviour of wave functions, can be described[20]. We summarize the properties of the operator $\hat{H}$, called quadratic map hamiltonian, assuming $\lambda > 2$:

i) the spectrum is purely singular continuous with support on the Julia set,

ii) the wave functions are the orthogonal polynomials $P_n(x)$, where x is the energy variable,

iii) the resolvant in the state $|\psi_0 >$ such that $\psi_{0,n} = \delta_{n,0}$, is nothing else than the functions g defined in (13):

$$\langle \psi_0 | (z - H)^{-1} | \psi_0 \rangle = \int_{J} (z - x)^{-1} \, d\mu(x) = g(z), \tag{48}$$

iv) $\mu(x)$ is the integrated density of state,

v) the Green's function defined in equation (12) is nothing else than the Lyapunov exponent $\gamma(x)$ which governs the high n behaviour of the wave function at energy x:

$$\frac{1}{n} \ln |\psi_n(x)| \sim \gamma(x), \tag{49}$$

in particular $\gamma(x)$ vanishes on the spectrum,

vi) the state $\psi_n(x)$ displays some kind of chaotic behaviour as n varies (see Ref. 20 for details).

We also use here the ζ function associated to the operator H. Let x_{end} be the upper end of the spectrum, x_{end} is also the largest fixed point of T:

$$x_{end} = \frac{1}{2} + \sqrt{\lambda + 1/4}. \tag{50}$$

A similar discussion holds for the lower end of the spectrum $(-x_{end})$. We define the ζ function as:

$$\zeta(s) = \int_{\mathcal{J}} (x_{end} - s)^s \, d\mu(x), \tag{51}$$

which is the Mellin transform of the measure mentioned in Section 2, and which has been shown to be a meromorphic function[12]. From the analysis of the poles of the function $\zeta(s)$, one deduces that the density of states does not behaves as a simple power law, as deduced from simple scaling arguments[25]: for $x \lessdot x_{end}$ we should not simply write:

$$\frac{d\mu(E)}{dE} = \sigma(E) \sim C(x_{end} - x)^{\delta}, \tag{52}$$

but we must include oscillating corrections:

$$\sigma(E) \sim (x_{end} - x)^{\delta} \{C_0 + \sum_{k=1}^{\infty} C_k \cos [k \tau \ln (x_{end} - x) + \varphi_k]\}, \tag{53}$$

where:

$$\delta = \frac{\ln 2}{\ln (2 x_{end})}, \quad \tau = \frac{2\pi}{\ln (2 x_{end})}. \tag{54}$$

These logarithmic oscillating corrections arise from complex poles of the Mellin transform which occur to sit on a semi-infinite periodic lattice. We refer to Ref. 12 for details.

We shall end this section with another example where an exact renormalization group will play the same role as equation (45) in our previous example. We consider a fractal structure which have self similar properties, namely the celebrated Sierpinski gasket[5, 28, 29]. We assume this structure made with masses and springs and using decimation arguments one gets for the density of states a functional equation similar to (14). More precisely the density of state is governed by an equation of the following form[29]:

$$\Phi(x) = \frac{1}{3}\Phi[x(x+5)] + \frac{1}{18}\ln\left(\frac{\mu+2}{(\mu+5)(\mu+6)^3}\right). \tag{55}$$

Here the transformation $T(x)$ is still a quadratic polynomial: $T(x) = x(x+5)$, but the equation (55) has an inhomogeneous term. As a consequence, the spectrum of the vibrating Sierpinski Gasket is a pure point spectrum, with accumulation points on the Julia set of $T(x)$[28, 30]. Nevertheless, the low energy spectrum still displays oscillating corrections to the power law behaviour[31].

5. *Thermodynamical interpretation*

We also find functional equations of a form analogous to (7), (13) or (55) in thermodynamic models on self similar lattices. We will here only briefly report on the so called diamond hierarchical lattices[32, 33]. We first generate the lattice by iterating the following transformation: each bond in the lattice connects to points that we call A and B, so that we denote the link AB. For each bond AB, add two new points C and D to the lattice, remove the bond AB, and replace it by four bonds AC, CB, AD, DB. Start the process with an elementary »lattice«: L_1 made of two points linked by one bound. After n steps, we get a lattice L_n with 4^{n-1} bonds, and in L_n there are $2 \cdot 4^{n-2}$ new sites which were not in L_{n-1} (for $n > 1$). So the total number of sites in L_n is $N_n = 2 + 2\{\sum_{i=0}^{n-2} 4^n\} = \frac{2}{3}(4^{n-1}+2)$. Number the sites from 1 to N_n, and attach to each site a spin variable σ_i which can take q different values from 1 to q. To a configuration, that is any assignment of values to the variables σ_i, we define the usual energy function of the Potts model:

$$\mathscr{H}\{\sigma_i\} = -\sum_{<i,j>} \mathcal{J}\,\delta_{\sigma_i,\sigma_j}, \tag{56}$$

where the sum extends on each pair of neighbouring sites (or equivalently on each bound i, j in L_n).

Then the partition function of the model, is as usual defined by summing over all possible configurations $\{\sigma_i, i \in L_n\}$:

$$Z_n(y) = \sum_{(\sigma_i)} \exp(-\beta\mathscr{H}\{\sigma_i\}), \tag{57}$$

where we set:

$$y = \exp\beta\mathcal{J}. \tag{58}$$

However, one can find a recursion relation. Indeed, if one performs first the sum over the spins values which belong to L_n and not to L_{n-1}, one finds that it is possible to deduce the function Z_n for the Potts model over L_n from the expression of Z_{n-1} for the Potts model over L_{n-1}. We get[33]:

$$Z_n(y) = Z_{n-1}(y')\,[2y + q - 2]^{2\cdot 4^{n-2}}, \tag{59}$$

where:

$$y' = T(y) = \left(\frac{y^2 + q - 1}{2y + q - 2}\right)^2. \tag{60}$$

We initialize the recursion relation by computing directly Z_1:

$$Z_1(y) = q(y + q - 1). \tag{61}$$

The usual spin 1/2 Ising model is obtained for the particular value $q = 2$. As observed in Ref. 33, the partition function $Z_n(y)$ is a polynomial of degree 4^{n-1}, the zeros of which are just obtained from the zeroes of $Z_{n-1}(y)$ by taking then the four inverse images under the rational fraction T defined in (60). Therefore the set of zeroes of Z_n are just predecessors of the unique zero of Z_1, and they accumulate on the Julia set. The limiting distribution of zeros is just the balanced invariant measures on the Julia set. Nice pictures of this set for several values of q are displayed in Ref. 33.

The interesting feature of this distributions of zeroes is that it meets the real axis in the variable y in two different points for q small enough ($q < 3$). For the Ising model for instance the corresponding ferromagnetic phase transition occur at $y_A \cong 3.38$ and the antiferromagnetic transition at $y_A \cong 0.2956$. We also check that these critical points are repulsive fixed points for T, and the critical exponents are related to the derivative of the transformation at the critical point[34]. For instance the specific heat critical index at the ferromagnetic transition is given by $(2 - \alpha) = \ln 4/\ln T'(y_F)$. The detailed analysis of the singular behaviour also displays critical oscillations of the same nature than those already mentioned in the Section 3, and can also be studied by Mellin transform techniques[34,12].

It is interesting to recall that these oscillations are related to the fact that the renormalization group which governs the critical behaviour is discrete and not continuous[36]. In fact any functional equation including a real transformation $T(y)$:

$$F(y) = \frac{1}{k(y)} F(T(y)) + h(y,) \tag{62}$$

where k and h are analytic around the repulsive fixed point of T, will generate a singular behaviour of the following form around the fixed point y_c:

$$F(y) \cong (y - y_c)^\varrho \,[C_0 + \sum_k C_k \cos\{k\,\tau \ln(y - y_c)\}], \tag{63}$$

with:

$$\varrho = \frac{\ln(k(y_c))}{\ln T'(y_c)}, \quad \tau = \frac{2\pi}{\ln T'(y_c)}. \tag{64}$$

Determining the coefficients C_k is a very hard problem[12], but there occurrence can be observed easily by computation, which is done by using recursively the functional equation (62) and analyzing numerically its behaviour near y_c.

6. Conclusion

This review of the physical interpretation of the theory of iteration of rational fractional displays the richness of the subject. Of course its mathematical aspects shows also a great variety, extending from the abstract theory of dynamical system, and geometry, up to number theory[35]. In the physical applications we have mentioned, the common aspect is the applications of the renormalization group approach. In fact the models exhibited are just made in order to have an exact renormalization group, expressible in terms of a polynomial or a rational fraction transformation. On the other hand these models allow a better intuition based on physical grounds on the mathematical aspects of the theory of iteration of rational fractions. The various notations suitable for each description and the resulting correspondences are summarized in the Table 1.

TABLE 1

Electrostatic description	Quantum mechanical description	Thermodynamical descriptions
$\mu(E)$: electric charge supported by the set E	$\mu(E)$: integrated density of states	$\mu(E)$: integrated density of zeros of the partition function
$G_{ES}(z) = \int_{\mathcal{J}} \ln \lvert z - x \rvert \, d\mu =$ = Electrostatic Green's function (Real part)	$L(z) = \int_{\mathcal{J}} \ln \lvert z - x \rvert \, d\mu$ = »Lyapunov« function (Real part)	$F(z) = \int_{\mathcal{J}} \ln (z - x) \, d\mu$ = Free energy per site
Böttcher function = exponential of the complex Green's function	Fredholm determinant = exponential of the complex »Lyapunov« function	Partition function per site = exponential of the free energy
$g(z) = \int_{\mathcal{J}} \frac{d\mu}{1 - zx}$ Generating function of the moments	$R(z) = \int_{\mathcal{J}} \frac{d\mu}{1 - zx}$ Resolvant, or quantum mechanical Green's function	$U(z) = \int_{\mathcal{J}} \frac{d\mu}{1 - zx}$ Internal energy
$M_z(s) = \int_{\mathcal{J}} (z - x)^s \, d\mu$ Mellin transform	$\zeta_z(s) = \int_{\mathcal{J}} (z - x)^s \, d\mu$ zeta function	$M_z(s) = \int (z - x)^s \, d\mu$ Mellin transform

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FIZIKALNA PRIMJENA JULIA SKUPOVA

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Prikazana su neka svojstva iteriranih polinoma i racionalnih razlomaka u kompleksnoj ravnini. Dano je nekoliko interpretacija odgovarajućih invariantnih skupova, koji se obično zovu Julia skupovi. Te interpretacije mogu se primijeniti u dvodimenzionalnoj elektrostatici, jednodimenzionalnoj, gotovo periodičnoj Schrödingerovoj jednadžbi, vibracionim spektrima fraktalnih struktura, te lokalizaciji nultočka particione funkcije za egzaktno renormalizabilan sistem rešetke spina.