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# IRREDUCIBILITY, ANALYTICITY AND UNITARITY OR ASYMPTOTIC COMPLETENESS IN MASSIVE QUANTUM FIELD THEORY: SOME GENERAL CONJECTURES AND RESULTS

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General conjectures and related results on the momentum-space analytic structure of multiparticle Green functions and collision amplitudes in massive quantum field theory are presented. The first conjecture is a set of formal expansions of Green functions, in each energy region, in terms of (regularized) Feynman-type integrals whose vertices are irreducible kernels. These expansions provide minimal decompositions of the (generally non holonomic) $S$-matrix and Green functions into elementary (regular, holonomic) contributions with specified analyticity and monodromy properties. Mathematical conjectures on discontinuities of these integrals are then presented and corresponding results are derived, in the sense of formal expansions, for the $S$-matrix and Green functions: unitarity relations, asymptotic completeness and $S$-matrix discontinuity formulae such as those needed in multiparticle dispersion relations. The analysis may be relevant, in axiomatic field theory and $S$-matrix theory (analyticity properties of multiparticle collision amplitudes), in constructive field theory (derivation of asymptotic completeness) and, from the mathematical viewpoint, in microlocal analysis.

## 1. *Introduction*

This work, which develops previous ideas of Ref. 1, presents general conjectures and related results on the momentum-space analytic structure of multiparticle collision amplitudes and Green functions in massive quantum field theory. As will appear below and as discussed again in Sect. 6, the analysis may be relevant in the following domains:

— analyticity properties, on the complex mass-shell, e. g. in axiomatic or constructive field theory and in $S$-matrix theory

— derivation of unitarity or asymptotic completeness relations in constructive field theory

— from the mathematical viewpoint, microlocal analysis[1] of solutions of non linear equations, and more precisely separation of non holonomic singularities of these solutions into well specified regular holonomic contributions. (The system of equations in the physical context is the infinite set of unitarity-type equations for the $S$-matrix or Green functions, a system which has its full mathematical interest).

As recalled below (Sect. 1.1), non perturbative results obtained so far in these domains either give only preliminary informations, or are limited to special cases ($2 \to 2$ and $3 \to 3$ processes in the low energy region), or depend more generally (e. g. in $S$-matrix theory) on heuristic considerations which are quite valuable, in particular as arguments of internal consistency, but are not satisfactory from a more fundamental viewpoint. On the other hand, relevant results of perturbation theory have provided important informations (analyticity properties of Feynman integrals and Landau singularities[6,4], perturbative derivation of unitarity[7], ...), but are themselves not complete, even from a formal viewpoint. The analysis of this paper relies on ideas derived from both perturbative and axiomatic field theory and $S$-matrix theory. Based on formal expansions in terms of irreducible kernels, it is intermediate between purely perturbative and non perturbative approaches and is also to a large extent heuristic: it mainly presents a number of conjectures not proved so far except in special cases, the formulation is not always clear and precise and modifications may be needed. However, it is hoped that the main ideas contain some truth, give some new insight on the general formalism and are amenable to a more precise mathematical study. For simplicity, only one type of (stable) scalar particle of mass $\mu > 0$ is considered. As is usual in axiomatic works in this domain, the question of existence of theories in space-time dimension 4 is not discussed: it is hoped that the analysis may be valid in dimension 2 and 3 and may be useful for possible adaptations to realistic theories in dimension 4.

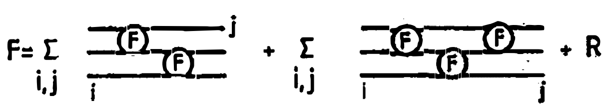
The general background is recalled in Sect. 1. 1 and the contents of this work are then presented in Sect. 1.2.

### 1.1. *General background*

We start below with axiomatic field theory and will then mention relevant results of constructive field theory and $S$-matrix theory.

[1] Microlocal is here understood in the mathematical sense proposed by M. Sato (which has no direct link with microlocality in field theory). Holonomicity and regular holonomicity are also understood in the sense of Sato[2,3] and are well specified criteria of simplicity, in relation in particular with monodromy properties. A detailed knowledge of these notions is not required in this paper. In the simplest case of a function $f$ that admits an infinite-sheeted structure around a given singularity, they mean that the vector space generated by the successive determinations of $f$ in its various sheets is finite-dimensional plus some regularity conditions ($f$ is then locally a *finite* sum of terms with singularities in $z^\alpha (\ln z)^n$ where $n$ is a $\geq 0$ integer). Feynman integrals have regular holonomic singularities[4]. The $S$-matrix and Green functions have in general non holonomic singularities[5].

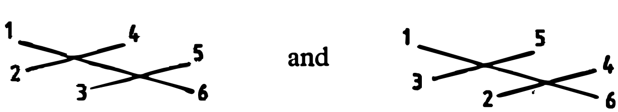
General off-shell results on the momentum-space algebraic and analytic structure of multiparticle Green functions (primitive domain of analyticity, ...) have been achieved in the »linear program« of axiomatic field theory, based on microcausality and the spectral condition: see Ref. 8 and references therein. However, results obtained in this framework are substantial, on the complex mass-shell, only for $2 \to 2$ processes and even in that case provide no detailed information on the singularity structure. They do not allow one e. g. to extricate the physical region Landau singularities[2]. The latter have been extricated so far for $2 \to 2$ and more recently $3 \to 3$ processes in the low energy region ($s < (3\mu)^2$ in the $2 \to 2$ case, $s < (4\mu)^2$ in the $3 \to 3$ case or $s < (5\mu)^2$ for an even theory, where $s$ is the squared center of mass energy of the channel) through the further use of axiom of asymptotic completeness and regularity assumptions. Two approaches, with similar results, have been developed: that of Refs. 10 and 11 (2-body case) and Ref. 11 ($3 \to 3$ case), which is more direct and uses the weakest regularity assumptions, and that of Ref. 12 (2-body case) and Ref. 13 ($3 \to 3$ case, even theory) which is more directly related to the analysis of this work and to which we come back below. In the 2-body case, it is shown that the $S$-matrix[10] or Green function[12,11] is analytic in the physical region between $s = (2\mu)^2$ and $s = (3\mu)^2$ (with a corresponding analytic continuation across the cut $s \geqslant 4\mu^2$ into an unphysical sheet) and has (in space-time dimension 4) a two sheeted, square-root type singularity at the 2-particle threshold $s = 4\mu^2$. In the $3 \to 3$ case, it is shown in Ref. 11 up to some technical limitations that the Green function F admits the decomposition:



(1)

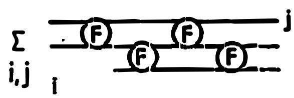
where $i = 1, 2, 3$ and $j = 4, 5, 6$ denote, respectively, incoming and outgoing particles, the terms in the sums $\sum$ are (possibly regularized[3]) Feynman type convo-

[2] The linear program ensures[9,8] that the $S$ matrix is, in the physical region, a sum of boundary values of analytic functions from complex on-shell domains depending on the real point considered. This sum reduces to one term in a limited part of the physical region. However, it can be expected on the basis of perturbation theory (and it is derived in $S$-matrix theory from macrocausality: see below) that there is only one boundary value almost everywhere, analytic outside $+ \alpha$-Landau singularities and obtained at almost all $+ \alpha$-Landau points from specified »plus $i\varepsilon$« directions (depending on the point considered). Points where this is no longer true do exist in the multiparticle case, such as those that lie on several Landau surfaces with plus $i\varepsilon$ rules that are conflicting on the complex mass-shell: e. g. the plus $i\varepsilon$ directions associated with the graphs

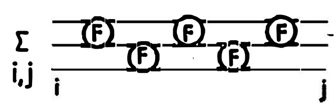


are opposite at all points $p$ such and that $p_1 = p_6, p_2 = p_4, p_3 = p_5$ in a $3 \to 3$ process with initial particles 1, 2, 3 and final particles 4, 5, 6. Such points lie in low-dimensional sets if the dimension $\nu$ of space-time is $> 2$.

lution integrals with 4-point Green functions at each vertex and R is analytic on shell, in the physical region, below the 4-particle threshold (i. e. $s < (4\mu)^2$), except at the 3-particle threshold ($s = (3\mu)^2$) and at the 2-particle subenergy thresholds $s_{ij} = (2\mu)^2$ ($i, j = 1, 2, 3$ or $i, j = 4, 5, 6$). The terms of the sums $\sum$ of (1) exhibit the one-particle and triangle $+$ $\alpha$-Landau singularities which are the only ones encountered on-shell, in the physical region, apart from the 3-particle and 2-particle thresholds already mentioned. The results of Ref. 13, obtained for an even theory and correspondingly at $s < (5\mu)^2$, give the same and also more detailed information: in particular, the terms




and




which can be, on shell, included in R but have their own singularities off-shell in the physical sheet, are exhibited in the right-hand side. The analysis of Ref. 13 also leads[5,1], as described in Sect. 2, to further formal expansions of direct interest in this work.

The approach of Refs. 12 and 13, which develops previous ideas of Ref. 14, is based on the introduction of Bethe-Salpeter type irreducible kernels, linked to Green functions via (possibly regularized: see below) Bethe-Salpeter type equations. In the $2 \to 2$ case, this equation reads:

$$F = G + F \circ G \tag{2}$$

or in a diagrammatical notation:


(2′)

where G is a 2-particle irreducible kernel in the $2 \to 2$ channel considered and the last term $F \circ G$ is a Feynman-type convolution integral, including (if needed) regularization[3]. Integration is made over a contour $\Gamma(k)$, which depends on the energy-momentum $k$ of the channel and is obtained by distortion of the euclidean space with which it coincides at infinity. The kernel G, as the Feynman-type operation o, depends on the choice of the regularization factor. For the equations of the $3 \to 3$ case, see Ref. 13 and Sect. 2 of this work (where they are recovered from formal expansions of F). The above Bethe-Salpeter type kernels cannot be expected to have a simple interpretation from a perturbative viewpoint in the regularized case (see further discussion in Ref. 15). However, a quasi-equivalence, up to technical assumptions corresponding to problems arising from zeroes of denominators in solution of Fredholm-type equations, is established in Refs. 12 and 13 between:

i) *asymptotic completeness* in the region considered, which can be characterized by well specified discontinuity formulae of Green functions. In the $2 \to 2$

[3] I. e. an analytic cut-off factor, equal to one on-mass-shell and with sufficient decrease at infinity in euclidean directions to ensure convergence, is included on each internal line.

case at $s < (3\mu)^2$ (and for an even theory), it takes the form of the unitarity formula (with external energy-momenta being allowed to take off-shell values)

$$F_+ - F_- = F_+ * F_- \tag{3}$$

where $F_+$ and $F_-$ are the plus $i\varepsilon$ and minus $i\varepsilon$ boundary values at $s > 4\mu^2$ (from above and below the cut $s \geqslant 4\mu^2$) of the physical sheet Green function F, and * denotes on mass shell convolution over two internal energy-momenta. (The restrictions of $F_+$ and $F_-$ to the mass-shell are the physical connected $S$-matrix and its hermitian conjugate).

In more general cases, asymptotic completeness is characterized by discontinuity formulae analogous to (3); they involve, however, different boundary values and do not have the form of a unitarity equation: see Sect. 5.1.

ii) *irreducibility properties (in the axiomatic, analytic sense)* of the Bethe-Salpeter type kernels involved. E. g. G in the $2 \to 2$ case is not singular at the 2-particle threshold ($s = 4\mu^2$) and is in fact analytic up to the 3-particle threshold. More precisely (3) is shown to be (quasi)-equivalent to the discontinuity equation:

$$G_+ - G_- = 0 \tag{4}$$

at $s < (4\mu)^2$, as a consequence of the »intertwinning« formula:

$$(F_+ - F_- - F_+ * F_-) \circ_- (1_- - G_-) = (1_+ + F_+) \circ_+ (G_+ - G_-) \tag{5}$$

where $\circ_+$ and $\circ_-$ denote integration over limiting contours $\Gamma_+(k)$ and $\Gamma_-(k)$ and $1_+$, resp. $1_-$, denotes the identity for the operation $\circ_+$, resp. $\circ_-$. Eq. (5) is itself easily checked from (2) and from the relation:

$$\circ_+ - \circ_- = * \tag{6}$$

which is independent of the regularization factor.

In the axiomatic approach, one starts from asymptotic completeness, which, together with regularity assumptions, entails irreducibility properties. The latter, plugged into the integral equation, provide in turn information on the analytic structure of Green functions in the region considered. In the constructive approach, irreducibility properties are directly established, and the above quasi-equivalence (or related versions of it) have been used, together with specific estimates, to establish asymptotic completeness for a class of models[4], again so far for $2 \to 2$[16,17] and $3 \to 3$[18] processes in the low energy region.

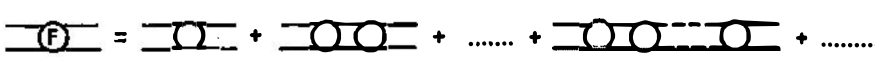
The works of Refs. 11 and 13 should be ultimately extended to more general cases and some steps in this direction have been considered in Ref. 13. However, one is still confronted in this program to a number of problems, not solved so far, from both algebraic and analytic viewpoints. As an introduction to our later analysis,

[4] No regularization is needed in the (2 or 3-dimensional) cases treated so far (super-renormalizable theories). For a further discussion in renormalizable theories, see Sect. 6.

we conclude this part on axiomatic field theory with the following alternative way[1] of understanding how the irreducibility of G in the 2 → 2 case yields the unitarity equation (3). We consider the Neumann series expansion of F in Eq. (2):

$$F = \sum_{n \geqslant 0} G^{0(n+1)} \tag{7}$$

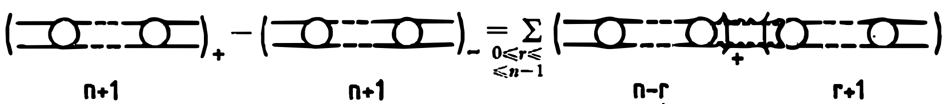
where $G^{o(n+1)} \doteq G o G \ldots o G$ ($n + 1$ factors), or diagrammatically:


(7′)

where  ≡ G. By successive applications of (6), one obtains, for each $n$:

$$(G^{o(n+1)})_+ - (G^{o(n+1)})_- = \sum_{0 \leqslant r \leqslant n-1} (G^{o(n-r)})_+ * (G^{o(r+1)})_- \tag{8}$$

or diagrammatically:




in view of the relations $(G^{o(n+1)})_+ \equiv G_+ \, o_+ \, G_+ \ldots o_+ \, G_+$, $(G^{o(n+1)})_- \equiv G_- \, o_- \ldots \ldots o_- \, G_-$ and of the assumed analyticity of G ($G_+ = G_-$).

By a simple formal reordering of terms, Eq. (8) provides in turn Eq. (3), at least in the sense of formal expansions in G: namely, the formal expansions obtained on each side of (3) when $F_+$ and $F_-$ are replaced by their formal expansions in G coincide for each number $n$ of kernels G involved. Q. E. D.

This is the type of approach that will be proposed here in more general cases, as outlined below in Sect. 1.2. Previously, we outline some further relevant elements of the general background, starting with $S$-matrix theory. Landau singularities are there directly linked in the physical region (= real mass-shell) to a property of macroscopic causality[19,20] and arise on the complex mass-shell from assumptions of maximal analyticity. General discontinuity formulae, either local around individual Landau surfaces, or global, such as those needed in multiparticle dispersion relations[5] in the 3 → 3 case, and related results have been on the other hand derived from the analysis of unitarity equations: see Refs. 6, 22 and 20 and references therein, and Sect. 5.1. We note, however, that apart again from special cases, such as the discontinuities around the one-particle[23] and triangle[24] Landau singularities encountered for the 3 → 3 $S$ matrix in the low energy region, and related results[24] fully analogous to (1), the $S$ matrix derivations make a crucial use of

[5] A derivation of part of these formulae, also called generalized optical theorems, has also been given[21] in field theory as a consequence of previous results (Ref. 8) and of asymptotic completeness, at least in an off-shell framework (see indications in Sect. 5.1).

ad hoc assumptions, such as (micro-)local separation of singularities in unitarity equations by classes of topological diagrams[6]. This property is not a direct consequence of edge-of-the-wedge theorems (or their general versions of essential support[27,28)] or hyperfunction[2)] theory), and one motivation of this work is to get a deeper understanding of its origin e. g. in field theory. Discontinuity formulae provide useful informations on the structure of the $S$-matrix and in some cases on the nature of its singularities[7]. However, they are not in general sufficient by themselves to give detailed information on the latter, apart from (holonomic) cases corresponding in space-time dimension 4 to singularities of (basic) graphs with sets of at most one or two lines between any pair of vertices.

This problem has been studied in Ref. 5, in both $S$-matrix and field theory approaches, in a simplified theory of the $m$-particle threshold in a $m \to m$ process with no subchannel interaction. In this model, the unitarity-type equation takes again (locally) the form of the discontinuity formula (3) where F is now the $m \to m$ Green function and $*$ denotes on-mass-shell convolution over $m$ internal energy-momenta. In the case $(m - 1)(\nu - 1)$ odd, where $\nu$ is the dimension of space-time, a (holonomic) two-sheeted, square root type structure is obtained, as in the case $m = 2$, $\nu = 4$, by an extension of the methods of Refs. 10 and 11 or of Ref. 12. However, the situation is different in the case $(m - 1)(\nu - 1)$ even, e. g. $m = 3$, $\nu = 4$, where the $S$-matrix or Green function is shown to have a non holonomic singularity. The introduction of irreducible kernels allows one in this case to characterize the structure and nature of the singularity through a series expansion of F into well specified elementary holonomic contributions, each of which having well specified monodromy properties[8] (they are more precisely in this model of the form $a_n(p)(\sigma^{\beta} \ln \sigma)^n$, $n = 0, 1, 2, \ldots, \sigma = s - (m\mu)^2$, $\beta = \frac{(m-1)\nu - m - 1}{2}$, where $a_n$ is locally analytic, plus possibly subdominant terms).

Two type of irreducible kernels have been considered in the simplified theory: kernels U that generalize the K matrix of the case $m = 2$, $\nu = 4$, are as the latter first defined on-shell[5)] but can be extended off-shell[30)], and Bethe-Salpeter type kernels G (The analysis of their links is given in Ref. 30). Although they have some advantages (the $n^{th}$ term is in this case of the form $a_n(p)(\sigma^{\beta} \ln \sigma)^n$ with no subdominant contribution and the expansion is convergent for $\beta > 0$), it has not been possible so far to define general kernels analogous to U in the non simplified theory and we shall below concentrate our attention to Bethe-Salpeter type kernels, whose introduction is natural in a field theory framework.

[6] Although this was not always recognized, this property (or related versions of it) has to be used as an assumption in all $S$-matrix works prior to Refs. 23 and 24, even in the simplest cases such as the derivation of the pole factorization theorem in the 3 → 3 case (one-particle singularity). Its proof requires the treatment of the »$u = 0$« problem (see Ref. 25; see also in a different approach[26)]) and of further difficulties, treated so far only in the cases mentioned above.

[7] The most simple cases are the well known square-root type singularities at 2-particle thresholds, the poles corresponding to graphs with one internal line, and the triangle singularities which are logarithmic in dimension 4. Further more general and more refined results in the (limited) class mentioned below have been given in Ref. 29.

[8] Analogous expansions can also be written, as in Eq. (7), in the case $(m - 1)(\nu - 1)$ odd, but they are less interesting from the viewpoint of analyticity properties, since each term has in this case the same type of singularity (square-root).

The various studies that have been mentioned indicate that, although a number of results can be achieved without recourse to irreducible kernels, their introduction is useful and is probably needed for several purposes, such as the detailed investigation of the multisheeted analytic structure of the $S$ matrix and Green functions, and the derivation of asymptotic completeness in constructive theory. They indicate, on the other hand, that at least for the first of these two purposes, the use of regularized (non renormalized) irreducible kernels should be sufficient in theories that require renormalization (and involve corresponding convergence problems in convolution integrals), without recourse to »renormalized« irreducible kernels[9]. Although the latter have their own interest, the analysis given in this work will correspondingly be made in terms of possibly regularized, non renormalized kernels (see further discussion in Sect. 6).

### 1.2. *Description of contents*

The present work has two related parts. The first one proposes a general formalism of formal series expansions of Green functions in terms of (possibly regularized) Feynman-type convolution integrals whose vertices are suitable irreducible kernels, and which should provide, in each energy region, »minimal« decompositions of the (generally non holonomic) $S$ matrix and Green functions into well specified elementary holonomic contributions. The second one then analyzes (from the viewpoint of formal series expansions) the way irreducibility properties (in the analytic sense) of the above kernels should yield unitarity or asymptotic completeness relations, as also various $S$-matrix discontinuity formulae such as those mentioned in Sect. 1.1. Concerning unitarity, this analysis can be considered as an analogue, in the case of formal expansions in terms of (regularized) irreducible kernels (and in a somewhat different approach), of the previous perturbative results[10] of Ref. 7. It is hoped that, besides its own interest, it will be helpful in the further development of the axiomatic and constructive programs in field theory, and it also sheds a further light on the property of separation of singulatities assumed in the $S$-matrix approach (see Sect. 1.1), which will appear as a potential consequence of global properties of graph by graph separation of singularities (that will generalize the discontinuity equation (8)).

In Sect. 2, we first consider, as an illustration of the ideas, the $3 \to 3$ case in the low energy region and for an even theory. Some important features, not present in the $2 \to 2$ case, already occur, alghouth there are still a number of simplifications in comparison with the general case outlined below. The formal expansions can be derived in this case from the results and integral equations of Ref. 13. They will be introduced here as a particular case of the general conjecture descri-

[9] The latter are from the perturbative viewpoint sums of the renormalized Feynman amplitudes of graphs with the corresponding irreducibility properties. For their recent introduction from an axiomatic viewpoint, at least for the simplest class of renormalizable theories and in the case of 2-particle irreducibility, and a corresponding analysis (in this case) of the links between irreducibility and asymptotic completeness, see Refs. 15 and 31. (The links between 2-partice irreducible regularized and renormalized kernels are examined in Ref. 15). An application to the massive Gross-Neveu model in space-time dimension 2 (whose rigorous construction has been recently achieved[32]) is given in Ref. 33.

[10] The latter have the advantage of providing actual proofs, whereas our analysis relies so far on conjectures.

bed below and are used at the end in the converse direction to recover these integral equations: it is hoped that this approach my be helpful to understand the type of equations to be considered in more general cases.

The general conjecture on formal expansions is described in Sect. 3. For each Green function F, each physical $m \to n$ process and each energy region $s < [(r + 1)\,\mu]^2$, it takes the form:

$$F = \Sigma_G^{(r)} F_G^{(r)} \tag{9}$$

where the sum runs over a suitable class of graphs G (depending on $r$) and $F_G^{(r)}$ is a (finite) sum of Feynman-type[11] (regularized) integrals $F_{G,\alpha}^{(r)}$ in which each vertex of G is replaced by a bubble b that represents a kernel with specified indices of »total« irreducibility (see Sect. 3.1) in each channel. These indices are also indices of reducibility in some cases. A more general form of the conjecture, needed in the further analysis of unitarity relations and related results, is a set of similar expansions applying to kernels with given indices of (total) irreducibility (or reducibility). In a perturbative framework without renormalization, and assuming correspondingly that no regularization is needed, each such kernel is formally the sum of Feynman amplitudes of all graphs that satisfy these irreducibility or reducibility properties, and Eq. (9) is from that viewpoint a partial resummation of the perturbative series: it corresponds more precisely, in a well specified sense, to the best possible regrouping of Feynman integrals that have in the region $s < [(r + 1)\,\mu]^2$, a common analytic and monodromic structure, in the neighborhood of the physical region (and a further domain of the complex mass-shell), corresponding to a common »skeleton graph« G. In particular they have, for each G, the same Landau singularities, namely those associated with G and graphs obtained from G by contraction.

In the axiomatic framework, it is assumed, as part of the conjecture, that independently of the perturbative background (and of questions of renormalizability), and being given if needed for convergence an analytic regularization factor, there exists a set of kernels satisfying analyticity properties associated with their degrees of irreducibility, such that formal expansions of the same algebraic form (9) as above still be satisfied. As a purely mathematical conjecture, we also assume that, in view of the analyticity properties of irreducible kernels, each term $F_{G,\alpha}$ or $F_G$ has again the same analytic and monodromic structure, in the region considered, as if these kernels were constants, and is in particular holonomic with regular singularities in the sense of Ref. 2 as the (regularized or renormalized) Feynman integral[3)] of the graph G. Eq. (9) will thus exhibit, as announced, a decomposition of F into well specified (regular) holonomic contributions, which is moreover »minimal« in the region considered, in a well specified sense.

Although the above conjecture on formal expansions has no longer a perturbative basis if regularization is needed, our belief in its validity is based on the one hand on the results already mentioned of the $2 \to 2$ and $3 \to 3$ cases, and on the other hand on the fact that the expansions (9) will indeed appear to be natural so-

[11] Feynman-type integrals are defined in general in a way analogous to Ref. 13, with Feynman propagators or two-point functions (that have a pole at the physical mass $\mu$) attached to each internal line. All internal lines of the graphs considered join two different bubbles.

lutions of unitarity (or asymptotic completeness) equations in the region considered. Concerning unitarity, this will follow from the mathematical conjecture (graph by graph unitarity) described in Sect. 4.1, which expresses the discontinuity $(F_{G,\alpha})_+ - (F_{G,\alpha})_-$ of each term $F_{G,\alpha}$ in the region considered in a way that generalizes Eq. (8). This conjecture relies again on the idea that, in view of their analyticity properties, the internal structure of the irreducible kernels involved at each vertex should not modify the basic structure of the discontinuity equations. The algebraic analysis of the way graph by graph unitarity yields unitarity equations for the $S$ matrix or Green functions (in the sense of formal expansions) is then given in Sect. 4.2.

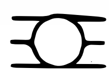
An analogous analysis, based on further discontinuity formulae[12] conjectured in Sect. 5.2. and relating various boundary values of $F_G$, allows one in turn (Sect. 5.3) to show other types of discontinuity formulae of the $S$ matrix and Green functions, e. g. those corresponding in field theory to asymptotic completeness or those of $S$-matrix theory occurring e. g. in multiparticle dispersion relations in the $3 \to 3$ case. Preliminary indications on these various formulae are given in Sect. 5.1.

Complementary comments, in particular on the relevance of the analysis in domains mentioned in Sect. 1.1, are presented in Sect. 6.

## *2. 3→3 case in the low energy region (even theory)*

In this section, we consider as in Ref. 13 a $3 \to 3$ process in the low energy region, in an even theory (The latter restriction introduces non negligible simplifications). For simplicity, the expansion is written below for the function $F_1$ which is 1-particle irreducible in the $3 \to 3$ channel considered. The general analysis of Sect. 3, leads in this case to introduce only purely irreducible kernels, where irreducibility is meant with respect to the channel corresponding to initial and final lines on the left and right respectively, namely:

— the $2 \to 2$ 2-*p*. i. kernel 

— the $3 \to 3$ totally 3-*p*. i. kernel L =  in the sense defined in Sect. 3.1 (L is the kernel $F^{(3)}$ defined there in Remark 4).

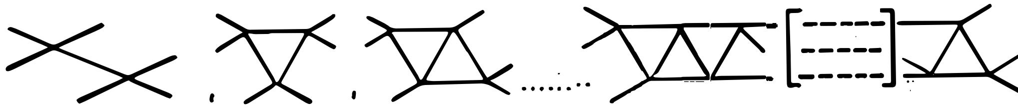
The expansion of $F_1$ then takes the form:

$$F_1 = \Sigma_G F_G \tag{10}$$

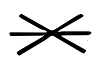
where the sum runs over the following graphs G:

— all graphs G including only $2 \to 2$ vertices and associated with $n$-loop truss-bridge diagrams, $n > 0$:

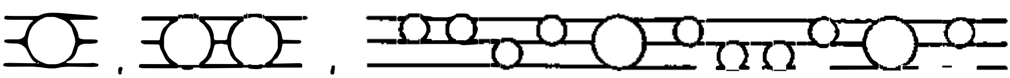
[12] Part of the formulae of Sects. 4 and 5 are known for (regularized or renormalized) Feynman integrals, while others are also implicitly conjectured for the latter.

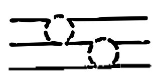
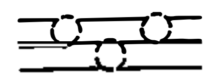


— all graphs G in which one or more of the 2 → 2 vertices are replaced by arbitrary sequences  .

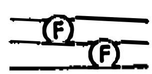
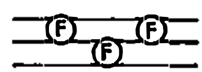
— all graphs composed of sequences (from left to right) of the graphs above and of 3 → 3 vertices  .

There is only one (regularized) Feynman-type convolution integral $F_G$ for each G. Typical terms $F_G$ are e. g.:



Apart from the terms of the form  and 

where  denotes either  or any sequence 

whose sums are equal to  and  , respectively

in view of (7), the singularities of all other terms are restricted on shell, in the physical sheet, to $s = (3\mu)^2$. One thus recovers (formally) the result (1). One reobtains similarly the results of Ref. 13 already mentioned off-shell. In non physical sheets, singularities of the other terms (that can be conjectured to be Landau, or more precisely modified[13] Landau singularities as introduced in Ref. 4) should become effective away from $s = (3\mu)^2$, probably in more and more remote sheets according to the complication of the graph G.

We now show on the example of this section how irreducibility yields unitarity in the sense of formal expansions. (For asymptotic completeness, see Sect. 5). As a particular case of the conjecture of Sect. 4.1, each term $F_G$ should satisfy, as a consequence of the irreducibility properties of the bubbles, the discontinuity equation:

$$(F_G)_+ - (F_G)_- = \sum_{(G_1,G_2)} (F_{G_1})_+ * (F_{G_2})_- \tag{11}$$

at $s < (5\mu)^2$, where the sum $\Sigma$ runs over all ways of dividing G into two successive (connected or not connected) subgraphs with 3 intermediate lines. E. g. if $F_G =$

$=$  ,

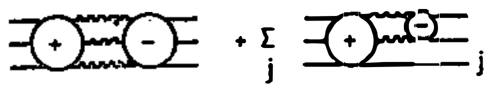
[13] The usual Landau definition leads to »surfaces« whose non $+\ \alpha$-branches may in some cases cover full open sets in the region considered, whereas the actual singular set is expected to be of codimension one.

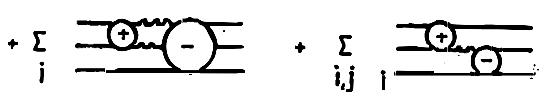
$$(F_G)_+ - (F_G)_- = \qquad (12)$$

where  indicates on mass-shell convolution. Formulae (11) and (12) should hold in particular in an (off-shell) neighborhood of the physical region.

The expansion (10) and the discontinuity formula (11) for each $F_G$ entail in turn, as a particular case of the analysis of Sect. 4.2, the usual unitarity equation for $F_1$ in the region considered,

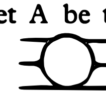
$$(F_1)_+ - (F_1)_- =$$

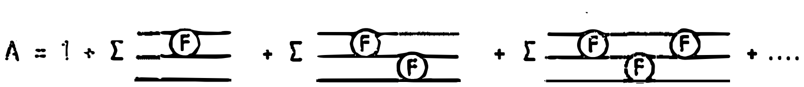




(13)

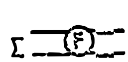
where external energy-momenta may vary in a neighborhood of the mass-shell, the bubbles $+$, resp. $-$, refer either to $(F_1)_+$ or to the $2 \to 2$ function $(F_{2,2})_+$, resp. to $(F_1)_-$ or $(F_{2,2})_-$, and the sums $\Sigma$ run over the various ways of associating initial, resp. final particles to the external lines. The result is obtained in the sense of formal expansions in terms of irreducible kernels; the formal expansions obtained on each side of (13) when $(F_1)_+$, $(F_1)_-$, $(F_{2,2})_+$, $(F_{2,2})_-$ are replaced by the expansions provided by Eqs. (10) and (7), and by applying (11) in the left hand side coincide for each topological graph G.

The integral equations of Ref. 13, considered there on the basis of axiomatic arguments, can be recovered from the expansion (10) as follows. Let A be the (formal) sum of all terms of the expansion that include no bubble , to which the non connected terms  are added. I. e.:



(14)

Let F′ be the non connected $1$-$p$. i. function in the $3 \to 3$ channel considered:

$$F' = F_1 + 1 + \Sigma$$  (15)

The expansion (10) gives:

$$F' = A + ALA + ALALA + \dots$$

$$= A\,[\mathbf{1} + \sum_{n \geqslant 0} (LA)^{o(n+1)}] \tag{16}$$

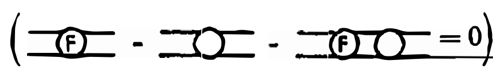
where products are to be understood in the sense of convolutions with respect to the operation o (e. g. ALA means AoLoA).

By a (formal) resummation of the expression in the right hand side of (16) and o-multiplication on both sides with $\mathbf{1}$ — LA, Eq. (16) provides in turn the integral equation:

$$F' = F'\,LA + A \tag{17}$$

between F′ and L, or:

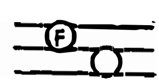
$$F'\,\Lambda = F'\,L + \mathbf{1} \tag{18}$$

where $\Lambda = A^{o(-1)}$ ($Ao\,\Lambda = \mathbf{1}$). From (2) , one checks (formally) that:

$$\Lambda = \mathbf{1} - \Sigma$$  (19)

$$F'\,\Lambda = \mathbf{1} + F_1\,\Lambda - T_{out} \tag{20}$$

where, following the notation of Ref. 13,

$$T_{out} = \Sigma$$  (21)

Finally, by taking the connected parts of both sides in Eq. (18) (and writing $F'\,L = F'\,\Lambda\Lambda^{-1}\,L$), one obtains:

$$F_1\,\Lambda - T_{out} = \Lambda^{-1}\,L + [F_1\,\Lambda - T_{out}]\,\Lambda^{-1}\,L \tag{22}$$

which is the integral Fredholm-type equation between $F_1\,\Lambda - T_{out}$ and $\Lambda^{-1}\,L$ considered in Ref. 13.

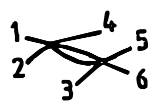
## 3. *Formal expansions*

### 3.1. *Preliminary definitions*

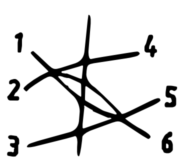
Being given a set of external lines, a channel g is a partition of this set into two subsets, each having at least one element. The kernels $F_{\underline{\nu}}$ introduced below have indices $\nu_g$ of »total irreducibility« in each channel, the index $\nu_g$ being also in some cases (specified by $\underline{\nu}$) an index of »total reducibility«. In a perturbative framework (without renormalization or regularization) $F_{\underline{\nu}}$ is the sum of all Feynman amplitudes of graphs G that have the same set of external lines and are compatible with $\underline{\nu}$, namely:

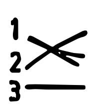
i) G is »totally ($\nu_g$-1) particle-irreducible« in the channel g, $\forall$g. By definition, G is *r-p.* i., resp. totally *r-p.* i., in the channel g if it cannot be divided with respect to g[14] into two successive connected, resp. connected *or* not connected, multiple scattering subgraphs $G_1$, $G_2$ (with at least one orientation of all lines of G) by »cutting« $r' \leqslant r$ intermediate lines. The latter include internal lines of G, but may also include some incoming or outgoing lines of G, as will be explained on the examples below.

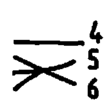
ii) Whenever $\nu_g$ is also an index of reducibility, resp. total reducibility, there is at least one way of dividing G as above with respect to g (into two connected, resp. two connected or not connected subgraphs) with $\nu_g$ intermediate lines.

*Example*: the graph 


is *r-p.* i. in the channel (123; 456), $\forall r$, (i. e. is not reducible in this channel). It is totally 1, 2 and 3-*p.* i., but is *not* totally 4-*p.* i., 5-*p.* i., ... in the channel (123; 456) in view of the possible cutting:




with the two subgraphs $G_1 =$ 

, $G_2 =$ 

;

its index of total reducibility in this channel is 4. In the channel (124; 356), the index of reducibility or of total reducibility is 2.

*Remarks*:

1) Total *r*-particle irreducibility in a channel g entails irreducibility properties in the usual sense in various channels. For instance, total 3-particle irreducibility in the channel (123; 456) entails:

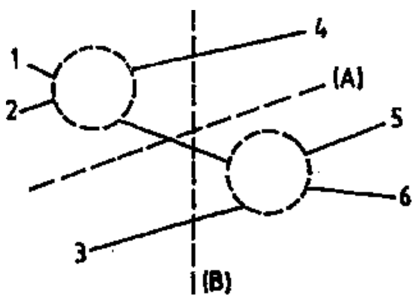
— 3-particle irreducibility in the channel (123; 456)

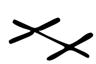
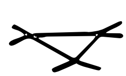
[14] I. e. $G_1$ and $G_2$ must contain respectively the *m* incoming and *n* outgoing external lines of G.

— 2-particle irreducibility in any channel of the form ($ij$; k 456) where $\{i, j, k\} = \{1, 2, 3\}$, or ($123\,k'$; $i'j'$) where $\{i'j'k'\} = \{4, 5, 6\}$.

— 1-particle irreducibility in any crossed channel such as (124; 356).

In fact, if e. g. in the last case one could divide G into two connected parts containing 1, 2, 4 and 3, 5, 6, respectively, by cutting one line (cutting A), there would also exist a cutting with 3 lines (cutting B) that would divide G into two (non connected) subgraphs in the channel (123; 456):




From the viewpoint of analyticity properties, it follows from the above definitions and from analyticity properties associated with irreducibility (see Sect. 3. 2.) that $F_{\underline{\nu}}$ is in particular analytic, for each channel g, in a neighborhood of the physical region (and of a further domain of the complex mass shell) in the region $s_g < [\nu_g\, \mu]^2$. We emphasize that this property is *not* satisfied for usual indices of irreducibility, rather than total irreducibility: the 3-particle irreducible function in the channel (123; 456) has e. g. pole and triangle singularities associated with graphs of the form  and  in the region $s < (4\mu)^2$.

2) Indices in each channel are not independent. For instance, if we consider again a 6-point function and the channels $g_1 = (1234; 56)$, $g_2 = (123; 456)$, and if $\nu_{g_1}$ is an index of total reducibility, possible indices $\nu_{g_2}$ of total irreducibility must satisfy, as easily checked:

$$\nu_{g_2} \leqslant \nu_{g_1} + 1 \tag{23}$$

If indices are not consistent, e. g. if (23) is not satisfied in the example above, then by convention:

$$F_{\underline{\nu}} \equiv 0. \tag{24}$$

If, on the other hand, $\nu_{g_1}$ is a pure index of total irreducibility, then (23) is not necessarily satisfied, but

$$F_{\underline{\nu}} \equiv F_{\underline{\nu}'} \tag{25}$$

where $\underline{\nu} \equiv \underline{\nu}'$ except that $\nu'_{g_1}$ is equal to $\nu'_{g_2} - 1$ and is an index of total irreducibility.

3) The kernels $F_{\underline{\nu}}$ will always be assumed to satisfy the following rule, which is trivially satisfied in the perturbative framework:

$$F_{(\nu_g)_{\text{irred.}}} = F_{(\nu_g)_{\text{red.}}} + F_{(\nu_g+1)_{\text{irred.}}} \tag{26}$$

where all indices $\nu_{g'}$, $g' \neq g$ are common in the three terms and are left implicit, and where the notation, irred., resp. red., means that $\nu_g$ (or $\nu_g + 1$) is a pure index of total irreducibility in g, resp. is also an index of (total) reducibility.

Similarly:

$$F_{(\nu_g)_{\text{irred.}}} = F_{(\nu_g)_{\text{red.}}} + F_{(\nu_g+1)_{\text{red.}}} + \dots + F_{(\nu_g+k)_{\text{red.}}} + F_{(\nu_g+k+1)_{\text{irred.}}}, \ \forall k > 0 \tag{27}$$

4) Being specified a channel g, $F^{(r)}$ will denote below the particular case of $F_{\underline{\nu}}$ corresponding to a pure index of total irreducibility $\nu = r + 1$ and pure indices of (total) irreducibility $\nu_{g'} = 1$ for $g' \neq g$.

### 3.2. *Formal expansions of Green functions*

It is assumed in the remainder of this section that there exists a class of regularization factors such that the conjectures below be valid. This factor may be equal to one if there is no problem of convergence. The kernels $F_{\underline{\nu}}$ and the terms $F_G$ or $F_{G,\alpha}$ depend on the choice of this factor. Unless otherwise stated explicitly, all indices are indices of *total* irreducibility, or reducibility.

*Conjecture* 1 (algebraic expansion)

There exists a set of kernels $F_{\underline{\nu}}$ such that each (connected) Green function F admits, for each channel g and each positive integer $r$, the formal expansion:

$$F = \sum_G^{(r)} F_G^{(r)} \tag{28}$$

$$F_G^{(r)} = \sum_\alpha F_{G,\alpha}^{(r)} \tag{29}$$

where each term $F_{G,\alpha}^{(r)}$ is a (regularized) Feynman-type[11] integral associated with the graph G in with each vertex is replaced by a bubble b that represents a kernel $F_{\underline{\nu}_b}$ (where $\underline{\nu}_b$ may depend on G and $\alpha$). More precisely:

a) The sum in (28) runs over all connected graphs G such that, being given any internal line $l$ of G, there is at least one way of dividing G with respect to the channel g into two connected or not connected successive subgraphs with a set of at most $r$ intermediate lines (or less) that contains $l$.

b) For each G, the sum $\sum$ in (29) is a finite sum over all possible ways $\alpha$ of attributing indices $\underline{\nu}_b$ to each bubble b such that the following properties be satisfied:

(P) any way of dividing G with respect to g that goes now »across« one or more bubbles, i. e. crosses lines of one or more subgraphs $G_b$, if each bubble b is replaced by a subgraph $G_b$ compatible with $\underline{\nu}_b$, includes at least $r + 1$ intermediate lines

(P′) for each b and each channel $g_b$, the index $\nu_{b,g_b}$ is either a pure index of (total) irreducibility equal to $(\nu_{b,g_b})_o$ or is also an index of reducibility and is then $< (\nu_{b,g_b})_o$, where $(\nu_{b,g_b})_o$, which depends on G but not on $a$, is defined as follows. Being given b, $g_b$ let $\nu_{b,g_b}$ be any possible index of reducibility (with here no constraint) and let $C_{\nu_{b,g_b}}$ be the class of all possible ways of attributing indices $\nu_{(b,g_b)'}$ to all remaining $(b, g_b)' \neq (b, g_b)$ such that property (P) be satisfied. Then $(\nu_{b,g_b})_o$ is the minimal value such that

$$C_{\nu_{b,g_b}} = C_{(\nu_{b,g_b})_0}, \ \forall \ \nu_{b,g_b} > (\nu_{b,g_b})_0.$$

The verification of this conjecture in the perturbative framework is outlined at the end of this subsection. We first present remarks and then further complements on its content.

*Remarks*:

1) Examples of terms $F_G$ have been exhibited in Sect. 2 for a $3 \to 3$ process, in the low energy region ($r = 3$) and in an even theory. If the last condition is dropped, new terms have already to be considered at $r = 3$ ($s < (4\mu)^2$), e. g.

where the indices $\underline{\nu}_b$ have been omitted. Examples for which several terms $F_{G,\alpha}$ are needed for a given graph G occur for instance in a $2 \to 2$ process at $r = 4$ ($s <$ $< (5\mu)^2$) in the channel g = (12; 34), e. g. terms of the form shown in Fig. 1.

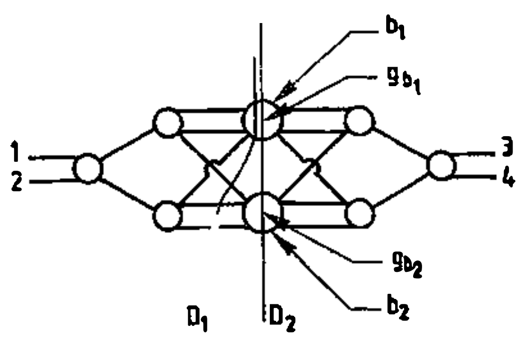



Fig. 1.

Each internal line $l$ of G is such that G can, as easily checked, be divided with respect to g into two subgraphs with at most 4 lines that contain $l$ as required by condition a) of Conj. 1. On the other hand, $\nu_{b_1, g b_1}$ must be $\geqslant 2$: otherwise the division $D_1$, which crosses $b_1$ would have less than 5 intermediate lines. Similarly, $\nu_{b_2, g b_2} \geqslant 2$. But one cannot have both $\nu_{b_1, g b_1} = \nu_{b_2, g b_2} = 2$: otherwise the division $D_2$ (which crosses $b_1$ and $b_2$) would have less than 5 intermediate lines. Among the terms $F_{G,\alpha}$, one thus finds terms with indices of (total) pure irreducibility 3 for $(b_{1, g b_1})$ and $(b_{2, g b_2})$, as also terms with an index $\nu_{b_1, g b_1}$ of (total) reducibility equal to 2 and an index $\nu_{b_2, g b_2}$ of (total) pure irreducibility equal to 3, or conversely.

2) If, being given two sets $\{\underline{\nu}_b\}$, $\{\underline{\nu}'_b\}$, $F\underline{\nu}_b = F\underline{\nu}'_b$, $\forall b$, there is only one term $F_{G,\alpha}$ in the expansion.

3) There is one term $F_G$ associated with the trivial graph G with only one vertex: it is equal to the kernel $F^{(r)}$ which is totally $r$-$p$. i. in the channel g (see Remark 4) in Sect. 3.1).

4) it can be checked that the index $(\nu_{b, g_b})_0$ is equivalently defined as follows. Let us consider all possible ways $\alpha'$ of attributing indices (either of pure irreducibility or also of reducibility) to all $(b, g_b)' \neq (b, g_b)$ in a way such that Property (P) be satisfied for any division that does not cross b in the channel $g_b$. For each $\alpha'$, let $\nu_{b, g_b}(\alpha')$ be the minimal value such that (P) still be satisfied for divisions that cross b in the channel $g_b$. Then, $(\nu_{b, g_b})_0 = \mathrm{Min}_{\alpha'}\, \nu_{b, g_b}(\alpha')$.

Conjecture 1 is complemented by the following analyticity assumption:

*Conjecture* 2 (analyticity properties of irreducible kernels)

$F_{\underline{\nu}}$ is analytic in the primitive domain. Moreover, if $r_g$ is the usual (not total) irreducibility in the channel g implied by $\underline{\nu}$, then $F_{\underline{\nu}}$ satisfies the analyticity properties associated with $r_g$ (i. e. analyticity in cut-energy planes, with cuts starting at $[(r_g + 1)\mu]^2$. More precisely, if the total energy-momentum $k_g$ of the channel g is of the form $((k_g)_0, \vec{0})$, $F_{\underline{\nu}}$ is analytic in the cut $(k_g)_0$ plane, with the cut $((k_g)_0) = (r + 1)\mu + \varrho$, $\varrho \geqslant 0$, $\forall g$. (See details in Ref. 13).

As a consequence of Property (P) in Conjecture 1 and of Conjecture 2, we also conjecture, as already mentioned in Sect. 1, the following property:

*Conjecture* 3:

If Conj. 2 holds for the irreducibile kernels involved at each vertex of a term $F^{(r)}_{G,\alpha}$ of (29), this term has, at $s < [(r + 1)\mu]^2$, the same analytic and monodromic structure as the (regularized) Feynman integral of the graph G, in a neighborhood of the physical region (and of a further domain of the complex mass-shell). It is in particular holonomic with regular singularities in that region.

Condition a) of Conjecture 1, together with the conditions on the sum $\Sigma_\alpha$, entails on the other hand that the decomposition (28), (29) of F is the *minimal* expansion in holonomic terms of this type, i. e. with the minimal number of terms, at least with possible degrees of pure (total) irreducibility depending only on the graph G. As a matter of fact, Conj. 1 can also be stated, in view of (27), in other equivalent forms by introducing different indices of pure (total) irreducibility. If indices $(\nu_{b,g_b})_0'$ larger than $(\nu_{b,g_b})_0$ are considered, the sum $\Sigma_\alpha$ runs over a larger number of terms with possible indices of (total) reducibility larger than $(\nu_{b,g_b})_0$ (and less than $(\nu_{b,g_b})_0'$.) On the other hand, one may alternatively consider in some cases further regroupings of terms $F_{G,\alpha}$, leading to decompositions of $F_G$ with a smaller number of terms $F_{G,\alpha'}$ in which indices of pure irreducibility may depend on the term considered and be smaller than $(\nu_{b,g_b})_0$. However, there is apparently no simple and general rule of this type, and this does not seem useful for our later purposes (namely using irreducibility properties to establish, in a given region, unitarity-type relations), since irreducibility indices equal to $(\nu_{b,g_b})_0$ will still be encountered in some of the terms.

We conclude this subsection by outlining how Conj. 1 can be checked in a perturbative framework (without renormalization or regularization). First, consider all terms $F_{G,\alpha}$, and let each $F_{\underline{\nu}_b}$ be replaced formally by the corresponding sum of Feynman integrals. One has to show that every Feynman integral in the perturbative series for F is obtained, and is obtained only once. Let G′ be any given Feynman graph. We divide the set of its internal lines into two subsets that will be called »skeleton lines« and »bubble lines«, respectively. A line $l$ of G′ is a skeleton line if there exists at least one way of dividing G′ into two (connected or non connected) subgraphs with respect to the channel g with a set of at most $r$ intermediate lines that contains $l$. Bubble lines are the remaining ones. It can be checked that this procedure determines in a unique way the »skeleton graph« G (obtained by contracting all bubble lines of G′). This procedure also determines connected subgraphs $G_b$ associated to each vertex of G. For each $G_b$, one determines in each channel $g_b$ an index $\nu_{b,g_b}$ of (total) reducibility as the minimal number of intermediate lines needed to divide $G_b$ into two (connected or not connected) subgraphs with respect to $g_b$. The indices $\underline{\nu}_b$ of the term $F_{G,\alpha}$ to which the Feynman integral of G′ belongs are then uniquely determined as follows: an index of (total) reducibility equal to $\nu_{b,g_b}$ if $\nu_{b,g_b} < (\nu_{b,g_b})_0$, a pure index of (total) irreducibility equal to $(\nu_{b,g_b})_0$ otherwise. Conversely, the term $F_{G,\alpha}$ thus determined clearly produces the Feynman integral of G′ once (and only once). Q.E.D.

### 3.3. *Formal expansions of irreducible kernels*

The following conjecture is a generalization of Conjecture 1 in two directions. On the one hand it applies to general kernels $F_{\underline{\nu}}$, and on the other hand the expansion is made with respect to several channels: Conjecture 1 is the particular case obtained when $F_{\underline{\nu}} = F$, $\nu_g' = r + 1$ in a given channel g, $\nu'_{g'} = \nu_{g'} = 1$ in other channels. As explained at the end, the general form of Conj. 4, as also Eqs. (26), (27), can be derived from a particular case of this conjecture ($\nu_g' - \nu_g$ at most equal to one for each g).

*Conjecture* 4 (formal $\nu'$-expansions of $F_{\underline{\nu}}$)

Let $\nu' = \{\nu'_g\}$ be a set of indices $\nu'_g$ such that $\nu'_g \geq \nu_g$, $\forall$g. Then:

$$F_{\underline{\nu}} = \sum_G F^{(\nu')}_{G,\underline{\nu}}, \qquad (30)$$

$$F^{(\nu')}_{G,\underline{\nu}} = \sum_\alpha F^{(\nu)'}_{G,\underline{\nu},\alpha}. \qquad (31)$$

a) The sum $\Sigma$ in (30) runs over connected graphs G such that:

i) G is totally $(\nu_g - 1)$-particle irreducible in each channel g.

ii) G has an index $\nu_g$ of (total) reducibility if $\nu_g$ is an index of (total) ireducibility in $\underline{\nu}$ and if $\nu_g < \nu'_g$.

iii) being given any internal line $l$ of G, there is at least one way of dividing G into two (connected or not connected, successive) subgraphs, with respect to at least one channel g such that $\nu_g < \nu'_g$, with a set of strictly less than $\nu'_g$ intermediate lines that contains $l$.

b) The sum $\Sigma$ in (31) is, for each G, a finite sum over all ways $\alpha$ of attributing indices $\underline{\nu}_b$ to each bubble b such that:

(P1) Any way of dividing G with respect to g that now crosses one or more bubbles (in the same sense as in Sect. 3.2) includes at least $\nu'_g$ intermediate lines.

(P2) If $\nu'_g = \nu_g$ and if $\nu_g$ is an index of (total) reducibility, there exists at least one way of dividing G with respect to g, possibly across one or more bubbles, with $\nu_g$ intermediate lines.

(P') $\nu_{b,g_b}$ is either a pure index of (total) irreducibility equal to $(\nu_{b,g_b})_o$ or is also an index of (total) reducibility $< (\nu_{b,g_b})_o$; $(\nu_{b,g_b})_o$ is defined as in Conjecture 1, (P) being replaced by (P1) and (P2).

The perturbative derivation of Conjecture 4 is analogous to that given at the end of Sect. 3.2 and is thus omitted.

*Remarks*:

1) The expansion (30), (31) is trivial ($F_{\underline{\nu}} \equiv F_{\underline{\nu}}$) if $\nu'_g = \nu_g$, $\forall$g. The same result holds if $\underline{\nu}$ can be replaced by an equivalent set $\underline{\nu}^{(1)}$ such that $\nu'_g = \nu^{(1)}_g$, $\forall$g.

2) The analyticity properties of Conj. 2 are the basic ones associated with the notion of irreducibility. The expansions (30) and (31), when assumed to hold in a given region, provide more detailed information on the analytic structure of $F_{\underline{\nu}}$.

3) The set $C(\nu_{b,g_b})_o$ may be empty in some cases, e. g. if (P2) can be satisfied only by crossing b in the channel $g_b$: the number of lines to be cut cannot be arbitrarily high in this case and $(\nu_{b,g_b})_o$ is the minimal index such that (P2) can no longer be satisfied.

More generally, it can be checked that $C(\nu_{b,g_b})_o$ is the set $C$ of configurations such that (P1) is satisfied for divisions that do not cross b in the channel $g_b$ and such that (P2) is already fully satisfied by these divisions. The index $(\nu_{b,g_b})_o$ is the minimal index such that (P1) is still satisfied »across $(b, g_b)$«, $\forall$ C $\in C$, and such that there is no configuration C′ $\notin C$ which, completed by $(\nu_{b,g_b})_o$, would satisfy (P1) and (P2).

We now conclude with the following lemma:

*Lemma* 1 If conjecture 4 is assumed in cases such that $\nu'_g$ is equal either to $\nu_g$ or to $\nu_g + 1$, $\forall$g, then it holds in general, as also Eqs. (26) and (27).

*Proof*

Eq. (26) is shown by considering the $\nu'$-expansion of $F_{(\nu_g)\mathrm{irred}}$, with $\nu'_g = \nu_g + 1$ and $\nu'_{g'} = \nu_{g'}$, $\forall g' \neq g$. Condition a) iii) of Conj. 4 (which applies by assumption) entails that, apart from the trivial graph with no internal line, all graphs G in the expansion have an index of reducibility $\nu_g$ in g. Their sum is thus equal to $F_{(\nu_g)\mathrm{red.}}$. The trivial graph gives $F_{(\nu_g+1)_{\mathrm{irred.}}}$ in view of Condition (P1). Eq. (27) follows from (26).

We next prove the general form of Conj. 4 by induction. Namely, we show below that, if $F_{\underline{\nu}}$ satisfies a $\nu'$-expansion, then it also satisfies a $\nu''$-expansion with $\nu''_{g_0} = \nu'_{g_0} + 1$ for a given $g_0$, $\nu''_g = \nu'_g$, $\forall g \neq g_0$. To that purpose, it is convenient to consider the corresponding decompositions $E'$, $E''$ obtained from (31) and (27) in which indices of pure irreducibility are all equal to $\nu_{\max} = \mathrm{Sup}_g\,(\nu'_g + 1)$ (and indices of reducibility are less than $\nu_{\max}$). For each term $F_{G,\underline{\nu},\alpha'}$ of $E'$ and each bubble b, let us consider all channels $g_b^{(1)}, \ldots, g_b^{(\varrho)}$ such that there is at least one division with respect to $g_0$ that goes across $(b, g_b)$ and has exactly $\nu'_{g_0}$ lines. (We recall that any division with respect to $g_0$ that goes across one or more bubbles has $> \nu'_{g_0}$ lines. The number of lines to be counted when a bubble is crossed in a given channel is the corresponding index of reducibility: divisions that cross a bubble in a channel with the index of irreducibility $\nu_{\max}$ can be disregarded since $\nu_{\max} > \nu'_{g_0}$). For each bubble b, we make a $\nu'_b$-expansion of the kernel $F_{\underline{\nu}_b}$, where $\nu'_{b,g_b} = \nu_{b,g_b} + 1$ if $g_b$ is one of the channels $g_b^{(1)}, \ldots, g_b^{(\varrho)}$, and $\nu'_{b,g_b} = \nu_{b,g_b}$ otherwise. We now check that (i) each term obtained belongs to $E''$ and that (ii) each term of $E''$ is obtained once (and only once).

i) We check here more specifically Property a) iii) of Conj. 4 (other properties are checked similarly). If the internal line $l$ is an original internal line of the term $F_{G,\underline{\nu},\alpha'}$, the result is straightforward. If $l$ is an internal line arising from the $\nu'_b$-expansion of a kernel $F_{\underline{\nu}_b}$, there is (by Property a) iii) of Conj. 4 applied to this expansion) at least one way of crossing b in one of the channels $g_b^{(1)}, \ldots, g_b^{(\varrho)}$ with a set of $\nu_{b,g_b}$ lines that contains $l$. By definition of these channels, there is in turn at least one division with respect to $g_0$ that goes across $(b, g_b)$, has $\nu'_{g_0}$ $(< \nu''_{g_0})$ lines, and crosses lines of graphs $G_{b'}$ of the $\nu'_{b'}$-expansions whenever it crosses other bubbles b′. (Property a) ii) applied to these expansions). The result is thus established.

ii) Conversely, consider one term $F_{G,\underline{\nu},\alpha''}$ of $E''$. The set of internal lines of G can be divided into classes in a way analogous to that described at the end of

Sect. 3.2.: »skeleton lines« $l$ such that G can be divided with respect to one at least of the channels g with a set of strictly less than $\nu'_g$ lines that contains $l$, and remaining ones for which one can only find a division with respect to $g_0$ with $\nu'_{g0}$ ($< \nu''_{g0}$) lines. This allows one to define in a unique way a new graph G whose internal lines are the skeleton lines just defined and new bubbles B. For each bubble B, indices of (total) irreducibility or also of reducibility are defined in a unique way as follows. For each channel $g_B$, one considers the minimal number of lines obtained in divisions with respect to $g_B$ (that may cross bubbles b inside B in channels with reducibility indices). If this number is $< \nu_{max}$, it defines an index of reducibility in the channel $g_B$. In all other cases, the index $\nu_{max}$ of irreducibility is attributed to $g_B$. The term in the $\nu'$-expansion of $F_{\underline{\nu}}$ corresponding to the graph G and to the set of indices $\nu_{B,g_B}$ just defined is the only one that can give rise to $F_{G,\underline{\nu},\alpha''}$ through $\nu'_B$-expansions of the kernels $F_{\underline{\nu}B} \cdot F_{G,\nu,\alpha''}$ is thus obtained once and only once. Q. E. D.

## 4. *Irreducibility and unitarity*

### 4.1. *Graph by graph unitarity*

We consider a (regularized) Feynman-type integral denoted below $F_G$, e. g. one of the terms $F^{(r)}_{G,\alpha}$ involved in the $r$-expansion (28), (29) of F. Conj. 3 of Sect. 2 means in particular that in a neighborhood of the physical region and (of a further domain of the complex mass shell) singularities encountered at $s < ([(r+1)\,\mu]^2$ are Landau singularities; more precisely $+\,\alpha$-Landau singularities in the physical region and other branches of (modified: see footnote 13) Landau singularities in other sheets. Conjecture 3 also entails that there is, at sufficiently small values of $s$ below all thresholds, e. g. at $s < (2\mu)^2$ for a term of the form

or $s < (3\mu)^2$ for a term of the form

 ,

a common analytic function which, by plus i$\varepsilon$, resp. minus i$\varepsilon$, analytic continuations around all normal thresholds and other singularities encountered gives the boundary values denoted $(F_G)_+$ and $(F_G)_-$, respectively. The latter can be defined in a field theory context as integrals over well defined contours $\Gamma_+(k)$ and $\Gamma_-(k)$, with plus i$\varepsilon$, resp. minus i$\varepsilon$, propagators or 2-point functions attached to each internal line of G. (Other analytic continuations will be considered in Sect. 5).

*Conjecture* 5 (graph by graph unitarity)

If the analyticity properties of Conjecture 2 are assumed for the irreducible kernels involved at each vertex of a (regularized) Feynman type integral $F_G$ and if Property (P) as stated in Conjecture 1 applies, then the following discontinuity formula holds at $s < [(r+1)\mu]^2$, in a real neighborhood of the physical region (and a further domain of the complex mass shell):

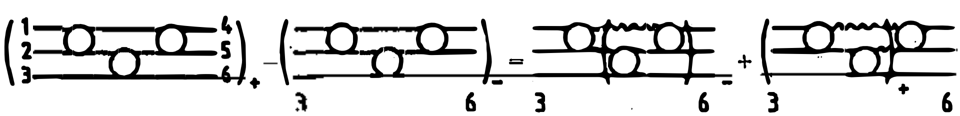
$$\Delta F_G \equiv (F_G)_+ - (F_G)_- = \sum_{(G_1,G_2)} (-1)^{N_c(G_2)+1} (F_{G_1})_+ * (F_{G_2})_- \qquad (32)$$

where the sum $\Sigma$ runs over all ways of dividing G with respect to g, with at most $r$ intermediate lines, into two (connected or non connected) subgraphs $G_1$, $G_2$ each of which having at least one non-trivial[15] connected component; $N_c(G_2)$ is the number of non trivial connected components in $G_2$. If $G_1$, resp. $G_2$ is not connected, $(F_{G_1})_+$, resp. $(F_{G_2})_-$, is the product of the (non-trivial) connected components (each one depending on its own energy-momenta variables). The operation $*$ denotes a convolution integral over on-mass-shell values of the energy-momenta of intermediate lines joining non trivial components.

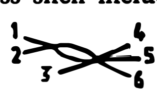
*Examples* (indices of irreducibility or reducibility are left implicit)

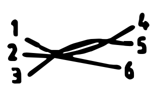
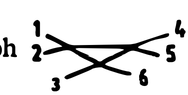
In all examples below, the channel g corresponds to incoming and outgoing lines on the left and right sides, respectively,

a)



(33)

for $r \geqslant 3$. We note that singularities encountered on the mass shell include the normal thresholds associated with the two contracted graphs 

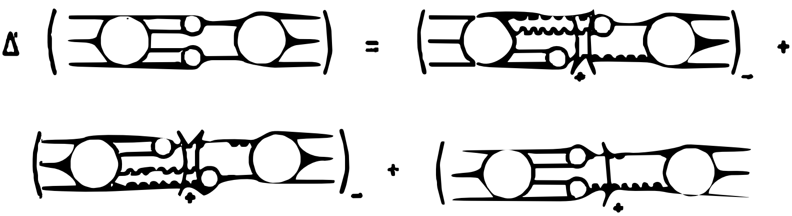
and  , and the triangle singularity of the graph  .

The discontinuity, in the form above, contains only the two terms associated with the two normal thresholds.

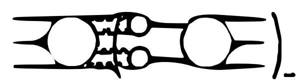
Another example occurring for one of the terms $F_G$ of Sect. 2 has been given in Eq. (12).

[15] Trivial components are those with one incoming and one outgoing line.

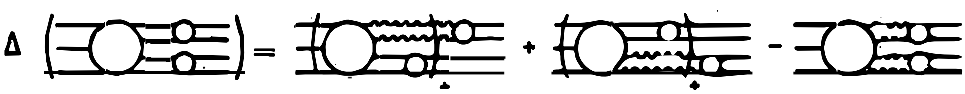
b) If the even character of the theory is removed, another example in a $3 \to 3$ process at $s < (4\mu)^2$ $(r = 3)$ is



(34)

with the supplementary term  if $r > 3$.

c)



for $r \geqslant 4$. (35)

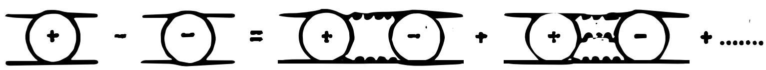
Conjecture 5 can be checked for the terms $F_G$ occuring in a $2 \to 2$ process at $r = 2$, as already indicated in Sect. 1 (Eq. (8)). It can be justified (see below) on the basis of perturbation theory arguments, but its proof, as that of Conj. 3, should depend only on the analyticity properties of irreducible kernels involved at each vertex, in view in particular of Condition (P) of Conjecture 1. It can possibly be derived by induction from a discontinuity formula on Feynman-type convolution integrals $F_1 \circ F_2$ proposed in Ref. 13 in a field theory context. An alternative proof might also be obtained[34] from Picard-Lefschetz theory and the analysis of intersection indices. It would directly explain, independently of the field theory context, why, e. g. in the example a) above, there is no contribution associated with the triangle singularity in (33).

In perturbation theory, unitarity at each order has been proved in Ref 7. An adaptation of the method (T-product formalism) should also allow one[35] to show »graph by graph unitarity«, in a sense similar to that of Conj. 5, for individual possibly renormalized Feynman integrals. In a theory with no renormalization, Conj. 5 can then be established as follows. Being given a term $F_G$, let us consider its (formal) expansion in terms of Feynman integrals (obtained from perturbative expansions of the kernels involved at each vertex), and let graph by graph unitarity be applied to each Feynman integral. The cuttings to be considered do not include internal lines arising from the bubbles but only original internal lines of G: there would be otherwise more than $r$ intermediate lines. A formal resummation then leads to the right hand side of (32).

### 4.2. *Formal derivation of unitarity relations*

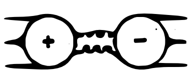
For simplicity, we only discuss below the physical unitarity relations for the $S$ matrix, or the analogous relations obtained for Green functions when the external energy-momenta are allowed to vary in a real off-shell neighborhood of the physical region. A similar analysis would yield more general unitarity-type relations, e. g. extended unitarity relations of the $S$-matrix for a $m \to n$ process (on the complex mass-shell) at $s$ real, $s < (m\mu)^2, (n\mu)^2$.

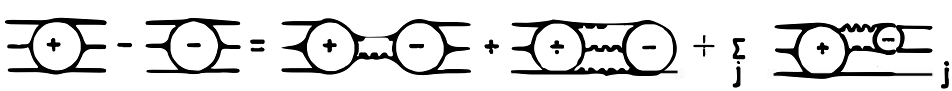
The $S$-matrix unitarity relations are the infinite set of equations between connected momentum-space $S$-matrix kernels and their hermitian conjugates derivated from $SS^{-1} = SS^{\dagger} = 1$ and the decomposition of $S$ (and $S^{\dagger}$) into connected components. A simple example, besides the standard $2 \to 2$ case where unitarity reads:



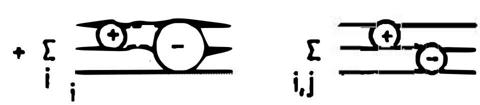
(36)

with sets of $\leqslant r$ intermediate lines in the region $s < [(r+1)\mu]^2$, is the $3 \to 3$ case at $s < (4\mu)^2$ which is analogous to Eq. (13) but includes the further term

 if the restriction to an even theory is removed. The left-hand side may be replaced by $F_+ - F_-$ in the region considered. The plus, resp. minus, bubbles stand for the connected kernels of $S$, resp. for the connected kernels of $S^{-1}$ multiplied by $-1$, or their off-shell extensions:



(37)



More generally, being given a $m \to n$ process, one obtains at $s < [(r+1)\mu]^2$ the relation:

$$\sum_{p \leqslant r} (F^{(n.c.)}_{m,p})_+ * (F^{(n.c.)}_{p,n})_- = 1_{m,n} \tag{38}$$

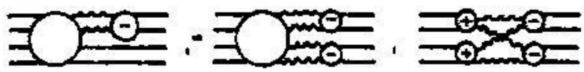
where $F^{(n.c.)}$ denotes the non connected Green function. The connected part of Eq. (38) can be written:

$$F_+ - F_- = \sum_{B} (-1)^{N_i(B_2)+1} (F_{B_1})_+ * (F_{B_2})_- \tag{39}$$

where the sum $\sum$ runs over connected graphs B with $m$ initial and $n$ final lines that can be divided (in a unique way) with respect to the $m \to n$ channel, with $p \leqslant r$ intermediate lines, into subgraphs $B_1$, $B_2$ composed of one or more connected components each of which has a single plus or minus bubble, respectively, or is a trivial line. More precisely, connected terms in which $B_1$, resp. $B_2$, has only trivial connected components (one incoming and one outgoing particle) are the terms

$F_-$, resp. $F_+$ which appear on the left-hand side of (39) and are thus excluded from the sum in the right-hand side. $F_{B_1}$ and $F_{B_2}$ are products of their connected components (each one depending on its own energy-momenta variables) and $N_C(B_2)$ is the total number of minus bubbles.

Examples of terms $(F_{B_1})_+ * (F_{B_2})_-$ in a $4 \to 4$ process at $s < (5\mu)^2$ are



We now state the following result whose proof will be given after some remarks and a preliminary lemma:

*Theorem* 1

If there exists a system of kernels $F_r$ satisfying the formal algebraic expansions stated in Conjectures 1 and 4 and if the analyticity properties of Conjecture 2 hold for the irreducible kernels involved in the $r$-expansion (28) and (29) of a given function F, for a given channel g and a given value of $r$, then Conjecture 5 (graph by graph unitarity) yields the unitarity relations (39) in the region $s < [(r+1)\mu]^2$ (in the sense of formal expansions of all Green functions in terms of irreducible kernels).

*Remarks*:

1) The non connected version (38) of the unitarity relations can also be derived from the same assumptions, in view in particular of Lemma 2 below.

2) The only analyticity properties that have to be assumed in order to establish unitarity in the region $s < [(r+1)\mu]^2$ are those of the irreducible kernels $F_r$ involved in the $r$-expansion of F.

3) It follows from Conjecture 3 on individual terms $F_{G,\alpha}$ (see also Sect. 4.1) that $F_+$ and $F_-$, which are equal to $\Sigma\,(F_{G,\alpha})_+$ and $\Sigma\,(F_{G,\alpha})_-$ (at least in a formal sense), are boundary values of analytic continuations of a common function, analytic on the complex mass-shell at sufficiently small values of $s$. These analytic continuations are obtained on the complex mass-shell by »plus $i\varepsilon$«, resp. »minus $i\varepsilon$«, distortions around the Landau singularities encountered, leaving aside those points where $i\varepsilon$ rules for different terms $F_G$ are conflicting (see Footnote 2).

The following Lemma, which is an extension to non connected functions of Conjecture 1 and follows from the algebraic statements of Conj. 1 and 4, will be useful in the proof of Theorem 1:

*Lemma* 2. Each non connected Green function admits, in a neighborhood of the physical region, the same algebraic expansion as that stated in Conjecture 1, except that the sum $\Sigma$ runs over all connected *or* non connected graphs G satisfying Condition a). (Condition b) is unchanged but applies to each connected or non connected graph G).

*Proof of Lemma* 2

The arguments are analogous to those used in the proof of Lemma 1 in Sect. 3.3 and we thus only outline the main steps. Being given the channel g, we consider any partition of the set of initial and final lines into subsets I and will show that the corresponding contribution to the non connected Green function admits the expansion (28) and (29) with a sum $\sum$ over non connected graphs whose connected components link together the initial and final lines of the subsets I. More precisely, we aim to show the equivalent expansion $E$ (obtained from (27)) in which indices of pure (total) irreducibility are all equal to $r$.

For each I, g determines a channel $g_I$ and we consider the $r$-expansion $E_I$ of $F_I$ with respect to $g_I$ with again indices of pure (total) irreducibility equal to $r$. This determines an expansion $E'$ which differs from $E$: e. g., for each internal line $l$ of a graph G in $E$, there exists a division of $G_I$ with respect to $g_I$, for a given set I, with a set of at most $r$ lines that contains $l$, but not necessarily a division of G with respect to g with such a set. In order to show that $E$ is equivalent to $E'$, we make for each term in $E$, a $\nu'_B$-expansion of each bubble B, where $\nu'_{B,g_B}$ is the minimal number such that any division with respect to $g_I$ that crosses B in the channel $g_B$ has more than $r$ intermediate lines (I refers to the relevant connected part that contains B). It is then checked that each term obtained by this procedure belongs to $E'$ and that each term of $E'$ is obtained once and only once. Being given a term in $E'$, the term in $E$ it comes from is well determined by considering skeleton lines $l$ which are here the internal lines $l$ of the graph G such that one can divide G with respect to g with a set of at most $r$ intermediate lines that contains $l$. (Indices of pure irreducibility equal to $r$ or indices of reducibility, smaller than $r$, are determined for the new bubbles in the same way as in the proof of Lemma 1).

We now conclude this section with the proof of Theorem 1, where we again only outline the main steps.

*Proof of Theorem* 1

On the one hand, a formal expansion of the left-hand side of (39) is obtained by using the $r$-expansion of $F_+$ and $F_-$, and by replacing each discontinuity $(F_{G,\alpha})_+ - (F_{G,\alpha})_-$ by a sum of $*$-convolutions according to Conjecture 5. We consider below the equivalent expansion $E$ in which indices of pure (total) irreducibility are equal to $r$. Similarly, a formal expansion of the right-hand side of (39) is obtained, e. g. by applying Lemma 2 to the non connected functions $F_+^{(n.c.)}$ and $F_-^{(n.c.)}$ in (38) and taking connected parts. The equivalent expansion with indices of pure irreducibility equal to $r$ will be denoted $E'$.

Each term, either in $E$ or in $E'$, is of the form $(F_{G_1,\beta})_+ * (F_{G_2,\gamma})_-$ and can be associated to a connected graph G, a cutting L of G with respect to g, with at most $r$ intermediate lines, into (connected or not connected) subgraphs $G_1$, $G_2$, and specifications of irreducible kernels at each vertex of G. In $E'$, the graphs G are such that, being given any internal line $l$ that belongs to $G_1$, resp. $G_2$, there always exists a division with respect to g with a set of at most $r$ intermediate lines that include $l$ and all belong to $G_1$, resp. $G_2$. In $E$, this division may in some cases have to include lines of both $G_1$ and $G_2$. Secondly, any division (with respect to g) that crosses a bubble must always have more than $r$ intermediate lines in $E$, whereas this is true only for divisions inside $G_1$, or inside $G_2$, in $E'$. However, the two expansions can be identified as follows.

Being given a term in $E'$, with its set of indices $\nu_{B,g_B}$ for each bubble B and channel $g_B$, let $\nu'_{B,g_B}$ be the minimal index $\geqslant \nu_{B,g_B}$ such that the number of lines in any division of G with respect to g that crosses B in the channel $g_B$ be $>$ $> r$. (The number of lines to be counted if other bubbles B′ are crossed is $\nu_{B',g_{B'}}$). For each B, we then consider its $\nu'_B$-expansion and the equivalent one with indices of pure irreducibility equal to $r$.

i) Each term obtained by this procedure belongs to $E$. First, being given any internal line $l$ of the new graph G, there always exists a division of G with respect to g with at most $r$ lines including $l$. This is trivial for original lines of G. If $l$ is an internal line arising from the $\nu'_B$-expansion of a bubble B, there is at least one $g_B$ such that the graph $G_B$ can be cut with respect to $g_B$ with less than $\nu'_{B,g_B}$ lines. Thus there is at least one way of dividing G (with respect to g) that crosses B in $g_B$, contains $l$, and has at most $r$ intermediate lines. In fact, if it crosses other bubbles B′, it is necessarily in channels $g_{B'}$, such that $\nu'_{B',g_{B'}} > \nu_{B',g_{B'}}$ by definition of $\nu_{B',g_{B'}}$. Since $\nu_{B',g_{B'}}$ is less than $r$ and is thus an index of reducibility, the $\nu'_{B'}$-expansion of B′ is such that any subgraph $G_{B'}$ can be cut in $g_{B'}$ with $\nu_{B',g_{B'}}$ lines. Hence, the division considered crosses only lines of G (and not the bubbles b associated with its vertices). On the other hand, any division of G (with respect to g) that crosses a new bubble b has more than $r$ intermediate lines since one crosses at least $\nu'_{B,g_B}$ lines in any channel $g_B$, where B is the original bubble containing b.

ii) It finally remains to check that each term of $E$ is obtained once (and only once). This is made as in the proof of Lemma 1 by reconstructing the term of $E'$ it comes from. Skeleton lines are here those such that there exist divisions of G with respect to g with at most $r$ intermediate lines that include $l$ and all belong to $G_1$, resp. $G_2$. This determines in a unique way the skeleton graph G. The index $\nu_{B,g_B}$ is determined in turn for each B, $g_B$ by considering the minimal number of lines in division of B with respect to $g_B$ (across lines or bubbles b). If this number is $< r$, it defines an index $\nu_{B,g_B}$ of reducibility. Otherwise $\nu_{B,g_B}$ is an index of pure irreducibility equal to $r$.

## 5. *Irreducibility, asymptotic completeness and S-matrix discontinuity formulae*

### 5.1. *General background: complements*

It has been shown in Sect. 4 how irreducibility properties yield unitarity relations for the $S$ matrix or Green functions. The purpose of this section is to show similarly how they yield asymptotic completeness and various $S$-matrix discontinuity formulae that we first briefly describe. Unitarity relations are formulae for the discontinuities between specified boundary values $F_+$ and $F_-$ of the Green function F. The formulae discussed here involved other boundary values of various analytic continuations of Green functions or $S$-matrix amplitudes.

i) *Asymptotic completeness relations*

Being given an $N$-point Green function F and a $m \to n$ channel g ($m + n = N$), asymptotic completeness can be characterized in general by a formula (see Ref. 13) for the discontinuity of the analytic Green function F across $\omega_g \cup \sigma_g$ where $\sigma_g$ is the cut $s_g > 4\mu^2$ and $\omega_g$ is the set $s_g = \mu^2$. In terms of boundary values, this formula can be expressed as a set of formulae for the discontinuities between boundary values (at real points) of F from adjacent cells[16] $S_1, S_2$ that differ only by the sign attributed to Im $p_g$, where $p_g$ is the energy-momentum of the channel. There is one formula for each possible pair ($S_1, S_2$), which in the region $s_i < [(r + 1)\mu]^2$ takes the form:

$$F^{S_1} - F^{S_2} = \sum_{r' \leqslant r} F^{\Sigma} \underset{(r')}{*} F^{\Sigma'} \tag{40}$$

where the terms in the right hand side are, for each $r' < r$, on mass shell convolution integrals, over $r'$ internal energy-momenta, of boundary values $F^{\Sigma}$ and $F^{\Sigma'}$ of ($m + r'$)-point and ($n + r'$)-point Green functions, respectively; the cells $\Sigma$ and $\Sigma'$ depend on $S_1, S_2$, but there is always a different sign for Im $p_g$ in $\Sigma$ and $\Sigma'$. Apart from the $2 \to 2$ case, the boundary values $F^{S_1}$, $F^{S_2}$ (as also $F^{\Sigma}$, $F^{\Sigma'}$) do not coincide with the boundary values $F_+$, $F_-$ whose mass-shell restrictions are the connected $S$ matrix and its hermitian conjugate (The latter are obtained from cells that may vary in general with the real region considered).

The extension of analyticity to the complex mass-shell, that cells do *not* intersect, and also to unphysical sheets, is part of the results one aims ultimately to achieve in general in the non linear program. These results should on the other hand follow (in a formal sense), together with the various relevant discontinuity formulae, from the conjectures on series expansions presented in this work and from analyticity properties and discontinuity formulae of individual terms $F_{G,\alpha}$: this will be shown in Sect. 5.3 in the $3 \to 3$ case. We note that in this case, and when external energy-momenta vary in a real neighborhood of the mass-shell, the specification of the cell S for the boundary value function $F^S$ is equivalent to specifications of signs for the imaginary parts Im $s_I$ (= Im $s_{CI}$) = 2 Re $p_I \cdot$ Im $p_I$ of the invariant variables $s_I = p_I^2$. In fact, (leaving aside trivial cases where I contains only one line[17]), Re $p_I$ is known from simple kinematical arguments to belong to $V^+$, resp. $V^-$, or to be space-like in the whole $3 \to 3$ physical region, if I contains only zero or one incoming, resp. outgoing, line (in other cases, Re $p_I$ = —

[16] A cell is a complex off-shell domain characterized by a consistent set of signs attributed to all partial sums Im $p_I = \sum_{i \in I}$ Im $p_i$ of imaginary parts of energy-momentum variables. The sets I are all proper subsets of incoming and outgoing particles (In view of energy-momentum conservation and of sign conventions of field theory $p_I + p_{CI} = 0$ where CI is the complement of I). The sign attributed to Im $p_I$ specifies if Im $p_I$ belongs to the (open) cone $V_+$ or $V_-$. Cell boundary values are known from the linear program to be boundary values of analytic continuations (in the complex off-shell cell domains) of the $N$-point function F, which is analytic in particular in a real neighborhood of the origin in $p$-space. Two cell boundary values that differ only by the signs attributed to Im $p_I$ for some sets I coincide at real points $p$ such that the respective $p_I$ lie outside the spectral regions $p_I^2 = \mu^2$, $p_I^2 \geq 4\mu^2$.

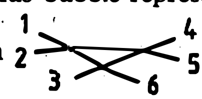
[17] $p_I^2 = \mu^2$, $p_i \in V^+$, resp. $p_i \in V^-$, on the mass-shell if I is composed of one outgoing, resp. one incoming, line i, according to the sign conventions of field theory.

$-\operatorname{Re} p_{CI}$). If $\operatorname{Re} p_I$ is space-like[18], the spectral condition of field theory ensures on the other hand (see Footnote 16) that the boundary values corresponding to opposite signs for $\operatorname{Im} p_I$ coincide (i. e. the specification of this sign is irrelevant). The same remark applies also to functions corresponding to $3 \to l$ or $l \to 3$ processes, also involved in the dicontinuity formula (40) of the $3 \to 3$ case. (In e. g. the $3 \to l$ case, $\operatorname{Re} p_I \in V^+$ or is space-like if I contains outgoing lines and zero or one incoming line. In other cases $\operatorname{Re} p_I = -\operatorname{Re} p_{CI}$ belongs to $V^-$ or is space-like).

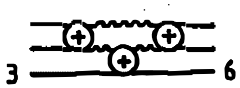
A cell can be defined in these cases by specifying the set S of channels g = = (I, CI) such that $\operatorname{Im} s_g < 0$ (Note that only a limited number of such sets correspond to cells: see (iii)).

Asymptotic completeness also allows one[21] to express each cell boundary value $F^S$ in terms of on mass-shell convolution integrals involving only boundary values $F_+$ and $F_-$: this provides a subset of the relations needed in multiparticle dispersion relations (see (iii)), modulo the extension of analyticity to the complex mass-shell.

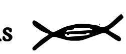
ii) *Local S-matrix discontinuity formulae*

These formulae are a generalization of Cutkosky discontinuity formula for Feynman graphs. If G is a graph with no set of more than one line between any pair of vertices, the discontinuity of the physical (connected) *S* matrix around the $+\alpha$-Landau singularity $L^+(G)$ is obtained locally (if there is e. g. no other singularity in the region considered) as the on-mass-shell convolution integral obtained by replacing each vertex of G by a plus bubble representing a connected *S*-matrix kernel. E. g., if G is the triangle graph  the discontinuity $F_+ - F^{(L)}$



is equal to  .



If there are sets of multiple lines, the formula is to be modified, as already exemplified in the standard $2 \to 2$ case where the discontinuity around the 2--particle threshold is locally equal to . A next example is the discontinuity $F_+ - F^{(L')}$ of the $2 \to 2$ function around the 3-particle threshold $s = (3\mu)^2$ (associated with the graphs , , ...), where $F^{(L')}$ is obtained from $F_+$ by analytic continuation along the path L′ shown in Fig. 2 in complex *s*-space to be distinguished from the path L which would give $F_-$.

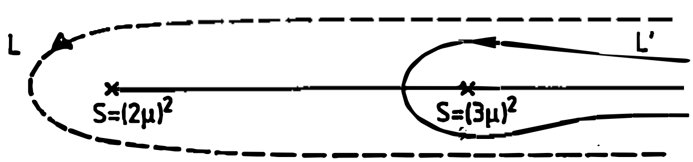




Fig. 2. The paths L and L′ of analytic continuation.

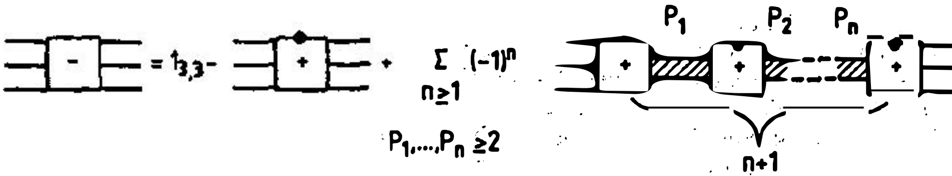
[18] This is always the case if I is composed of one incoming and one outgoing line. If I is composed of two incoming (resp. outgoing) lines and one outgoing (resp. incoming) line, this is the case in parts of the physical region.

$F_+ - F^{(L')}$ is equal, according to Ref. 22, to a double convolution integral:

$F_+ - F^{(L')} =$  (41)

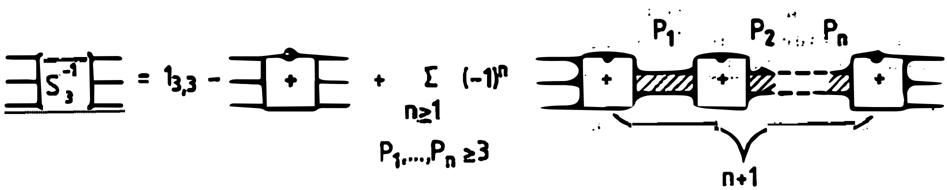


where $S_3^{-1}$ is the inverse of S in the restricted Fock space composed of states with at least three particles. In terms of formal developments,




(42)

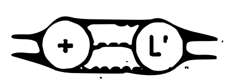
where the boxes  represent kernels of $\dot{S} = S - 1$ and where the sets  represent on mass-shell convolution over $p_i$ internal lines, $i = 1, \ldots, n$. This expansion is analogous to that of $S^{-1} = (1 + \dot{S})^{-1}$:




(43)

except that the $p_i$, $i = 1, \ldots, n$ must be $\geqslant 3$ in (42).

Another form of the above discontinuity, given in Ref. 6, is:

$F_+ - F^{(L')} =$  (44)

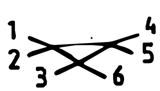


where 

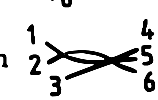

is here an analytic continuation of the physical function around the 3-particle threshold $s = (3\mu)^2$ of the $3 \to 2$ channel and the 2-particle subenergy thresholds associated to two incoming particles (but not around the 2-particle threshold in $s$).

(iii) Global $S$-matrix discontinuity formulae of the type needed in multiparticle dispersion relations.

These formulae are generalizations of the standard unitarity relation (36) of the $2 \to 2$ case for the discontinuity $F_+ - F_-$ between the (on shell) function $F_+$ and its analytic continuation $F^{(L)} = F_-$ below the cut $s \geqslant 4\mu^2$ in Im $s$-space i. e. below the 2, 3, 4, ...-particle thresholds (see Fig. 2). They involve boundary values of analytic continuations of the $S$-matrix (in the complex mass-shell) which differ in general from $F_+$ and $F_-$, but discontinuities are again expressed in terms of on-mass-shell integrals involving only physical (connected) $S$-matrices (or their hermitian conjugates). We present them below in the $3 \to 3$ case[22] ($2 \to 4$ and $4 \to 2$ cases are also given there). There are in this case 16 basic channels $g_1, \ldots, g_{16}$ (each containing at least 2 initial or 2 final lines) which are relevant on-shell, instead of one in the $2 \to 2$ case: the channel $t = (123; 456)$, the three channels denoted $i$, $i = 1, 2, 3$ corresponding to initial subenergies, e. g. $i = 2$ refers to the channel (13; 2456), three analogous channels $f$, $f = 4, 5$ or 6 and nine crossed channels denoted $(if)$, e. g. $i = 3$, $f = 4$ refers to the channel (124; 356). To each set S of channels is associated a function that we shall still denote $F^S$ as in (i) for reasons explained below. For a subset of sets S, called »good sets« (see below), which include the sets corresponding to cells as described in (i) and others, $F^S$ is the boundary value in the physical region of an analytic continuation of $F_+$ in the complex mass-shell : starting from the common analytic function at small values of $s$, $F^S$ is obtained by analytic continuation beneath all cuts associated with channels $g \in S$ and above all those associated with channels $g \notin S$, i. e. with minus $i\varepsilon$ and plus $i\varepsilon$ distortions, respectively, in the space of imaginary parts $\mathrm{Im}\, s_g$ of the variable $s_g$, around the normal threshold singularities $s_g = (l\mu)^2$ associated with g; $F^S$ is equal to $F_+$, resp. $F_-$, if S is the empty set, resp. the set of all channels $g_1, \ldots, g_{16}$. The way analytic continuation has to be made around other Landau singularities is also well determined. In fact, the way in which the function is continued around a normal threshold specifies the way in which it has to be continued around any other singularity that emerges from it or into which it merges: e. g. in the $3 \to 3$ case, if we consider the triangle singularity of the graph



and the 2-particle normal threshold associated with the contracted graph



, the same rule holds in the neigbourhood of the points where the two surfaces are tangent.

The good sets S are, in the $3 \to 3$ case, those for which there is no pair $i, f$ such that either $(if) \in S$, $t \in S$, $i \notin S$, $f \notin S$ or $(if) \notin S$, $t \notin S$, $i \in S$, $f \in S$. They include 26018 sets S (out of the $2^{16} = 56536$ possible sets), among which the 2282 sets corresponding to all cells of field theory. Examples of good sets which do not correspond to cells are all sets S composed of only one channel $(i)$ or one channel $(f)$[19].

It is believed that in the case of sets S corresponding to cells, the boundary value $F^S$ occurring in field theory (and obtained a priori from the complex off-shell cell domain) coincides, when restricted to the mass-shell, with that introduced above.

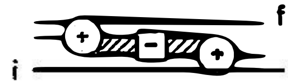
The analysis of Ref. 22, provides the formulae:

$$F^S = \sum_{S' \subset S} (-1)^{S'} F_{S'} \tag{45}$$

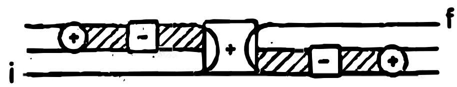
[19] If e. g. $S = \{(6)\}$, the conditions $\mathrm{Im}\, s_{(6)} < 0$, $\mathrm{Im}\, s = \mathrm{Im}\, s_t > 0$, $\mathrm{Im}\, s_{(6i)} > 0$, $i = 1, 2, 3$ would yield in the field theory framework $\mathrm{Im}\, p_{45} \in V^-$, $\mathrm{Im}\, p_{45i} \in V^+$, $\mathrm{Im}\, p_{123} = -\mathrm{Im}\, p_{456} \in V^-$, since near physical points $\mathrm{Re}\, p_{45} \in V^+$, $\mathrm{Re}\, p_{456} \in V^+$, $\mathrm{Re}\, p_{45i} \in V^+$. However, these relations are not consistent: for at least one $i$, $\mathrm{Im}\, p_i \in V^-$, hence $\mathrm{Im}\, p_{45i} = \mathrm{Im}\, p_{45} + \mathrm{Im}\, p_i \in V^-$.

with explicit formulae for all functions $F_S$, e. g.:

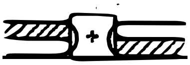
$F_t\, (= F_+ - F^{(t)}) =$  (46)

$F_{(if)} =$  (47)



$F_{(i),(f)} =$  (48)



where boxes refer to non connected amplitudes, the terms in the right hand sides are sums of on mass-shell convolution integrals correspoding, for each set  , to all possible sets of intermediate particles in the energy region considered, and  in (48) denotes the contribution to the full non connected $S$ matrix in which lines $i$ and $f$ cannot go straight through (i. e. are connected to non trivial bubbles).

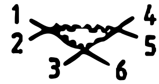
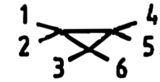
On the other hand, $F_S \equiv 0$ whenever two channels $g_i, g_j \in S$ are overlapping[20]. This last fact ensures that generalized Steimann relations are satisfied (see Ref. 22) and limits non zero functions $F_S$ to those associaed with the empty set, the 16 sets S with a single channel $g_i$, the sets S with two channels of the form $\{(i), (f)\}$, $\{(i), t\}$, $\{t, (f)\}$, $\{(i), (if)\}$, $\{(if), (f)\}$ and finally the sets S with three channels of the form $\{(i), t, (f)\}$, $\{(i), (if), (f)\}$.

All results above are derived in Ref. 22 from an analysis which uses as already mentioned in Sect. 1 a number of ad hoc assumptions. One of our purposes in Sect. 5.2, 5.3. will be to get a complementary understanding, in the framework of the series expansion formalism of Sect. 3, of the origin of these results (see further discussion in Sect. 6).

### 5.2. *Discontinuity formulae for individual graphs*

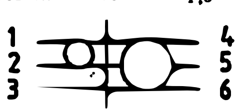
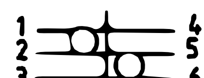
We now present new conjectures on individual Feynman-type integrals $F_G$, e. g. the terms $F_{G,\alpha}$ occurring in the expansion of F in a given energy region. As in Sect. 4, it is generally conjectured that, in view of the irreducibility properties of the bubbles, discontinuity formulae are analogous in the region $s < [(r + 1)\,\mu]^2$ to those that should hold for Feynman integrals, with constants at each vertex replaced by the corresponding irreducible kernels, whenever Property (P) stated in Conjecture 1 applies.

[20] Two channels $g_i$, $g_j$ are overlapping if each one of the two sets of lines that define $g_i$ intersects each one of the sets that define $g_j$. E. g. the channels t and (34) are overlapping.

Some discontinuity formulae for Feynman integrals $I(G)$ of graphs G are well known, in particular Cutkosky formula for the discontinuity $I_+(G) - I^{(L)}(G)$ around the main Landau singularity $L_+(G)$ of G. E. g. the discontinuity around the triangle singularity is equal to  for the graph .
However, we shall state below further conjectures on the terms $F_G$ which have not been proved or even systematically considered to our knowledge for Feynman integrals and that we also present implicitly as conjectures in this case.

We mainly consider the $3 \to 3$ case in view of the application in Sect. 5.3, but a large part of the discussion applies more generally. It is useful to give some preliminary definitions and results. Leaving the indices $\underline{\nu}_b$ of each bubble and the index $r$ implicit and being given a channel g (= the channel (123 ; 456) in the $3 \to 3$ case) we first define $(F_G)^S$ for each set S of channels by the formula:

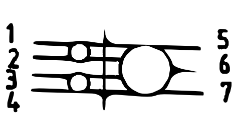
$$(F_G)^S = \sum_L (-1)^l (F_{G_1})_+ * (F_{G_2})_+ \dots * (F_{G_{l+1}})_+ \tag{49}$$

where the sum $\Sigma$ runs over all ways $L = (L_1, \dots, L_l)$ of dividing G with respect to the channel g considered into successive (connected or non connected) subgraphs $G_1, \dots G_{l+1}$, $l \geqslant 1$, such that each division $L_\sigma$, $\sigma = 1, \dots, l$, has at most $r$ intermediate lines and be consistent with S : namely, being given any division of G into two *connected* subgraphs $G_{1,\sigma}$, $G_{2,\sigma}$ obtained by cutting lines of $L_\sigma$ that are *internal* lines of G, the external lines of $G_{1,\sigma}$ and $G_{2,\sigma}$ must be those of a channel $g' \in S$. E. g. the division , resp. , with respect to the channel g = (123 ; 456), is consistent with the channel (3) = = (12 ; 346), resp. (34) = (124 ; 356)[21]. In view of this definition, $(F_G)^S$ can be also be written:

$$(F_G)^S = \sum_{S' \subset S} (-1)^{|S'|} (F_G)_{S'} \tag{50}$$

where $(-1)^S (F_G)_S$ is defined as in (49), but with a sum $\Sigma'$ in the right hand side running only over sets L such that each channel of S be obtained for at least one set of internal lines of one $L_\sigma$. As appears in the examples given below, the sum $\Sigma'$ is moreover limited by:

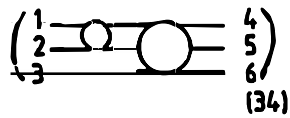
[21] In a more general case, the division



is consistent with S if S contains the channels (12; 34567) and (34; 12567).

*Lemma* 3. $(F_G)_S \equiv 0$ in the following situations:

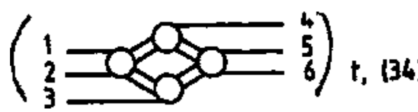
a) S contains a channel $g'$ that cannot be obtained from any division $L_\sigma$ of G with respect to g. E. g.




$\equiv 0$ (51)

(b) S contains two overlapping channels[20] $g_1, g_2$

In case b), one cannot in fact find divisions $L = (L_1, \dots, L_l)$ into *successive* subgraphs such that both $g_1$ and $g_2$ be obtained from divisions $L_{\sigma 1}, L_{\sigma 2}$ of L. E. g.:

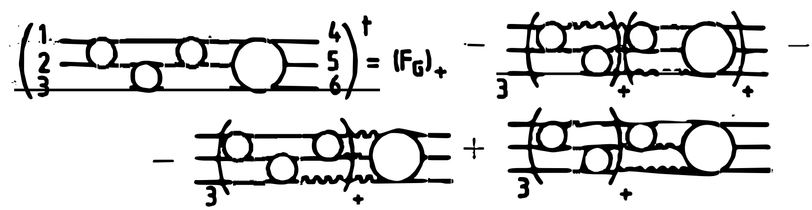



$\equiv 0$ (52)

Condition a) depends on the graph G. Condition b) is independent of it. In the $3 \to 3$ case, it restricts sets S such that $(F_G)_S$ is not identically zero to those already mentioned in Sect. 5.1 iii). On the other hand, various contributions to the terms $(F_G)_{S'}$ in the right hand side of (50) may coincide apart possibly from multiplicative coefficients. They can then be regrouped together, and in some cases cancel among themselves (see examples below).

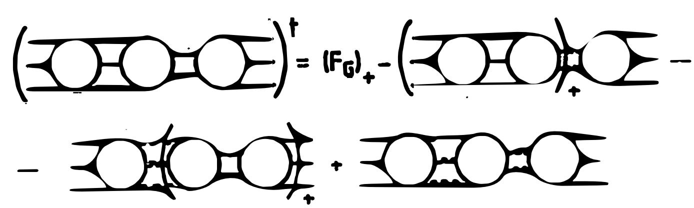
*Examples* (in all cases $g = (123\,;456) \equiv t$)

a)



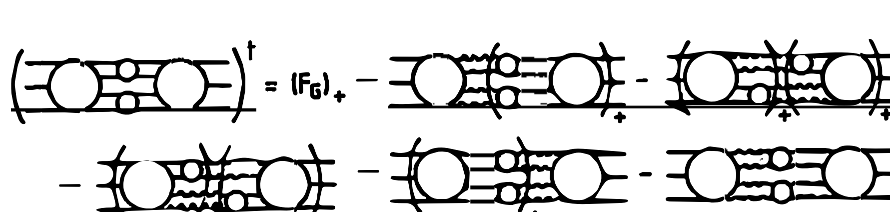

(53)

b)

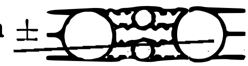
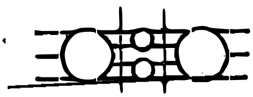
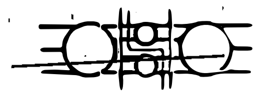
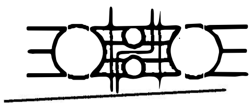



(54)

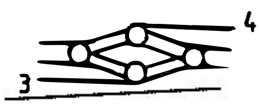
c)

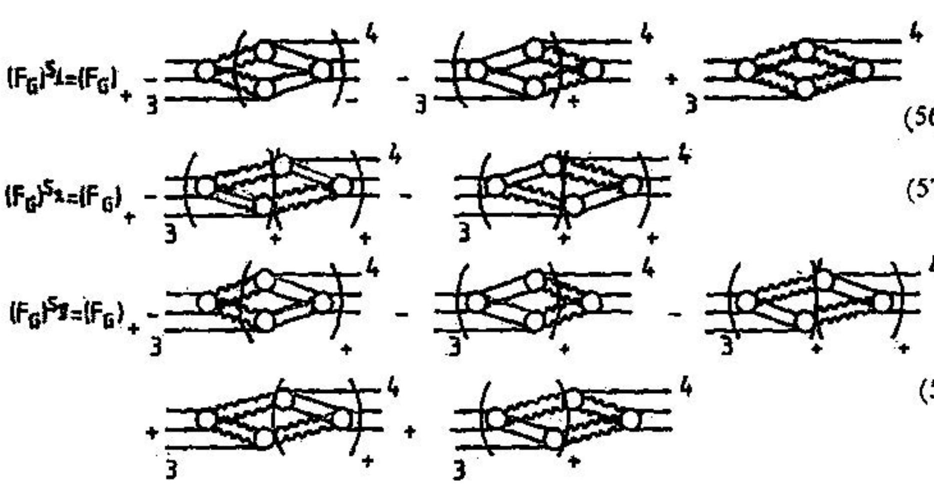



(55)

In fact, there are several contributions of the form $\pm$ , corresponding to the divisions ,  and , which add up to give the last term in (55).

d)

$F_G =$ , $S_1 = \{3, 4\}, S_2 = \{t, (34)\}, S_3 = \{3, 4, t\}$.




(56)

(57)

(58)

The three terms in the right hand side of (56), apart from $(F_G)_+$, are, respectively, $(F_G)_3$, $(F_G)_4$ and $(F_G)_{3,4}$. The two terms in the right hand side of (57), apart from $(F_G)_+$, are $(F_G)_t$ and $(F_G)_{(34)}$: the term $(F_G)_{t,(34)}$ is identically zero in view of Lemma 3 (see (52)). In (58), one encounters the terms $(F_G)_3$, $(F_G)_4$, $(F_G)_t$, $(F_G)_{3,t}$, $(F_G)_{t,4}$: the terms $(F_G)_{3,4}$ and $-(F_G)_{3,4,t}$, which correspond, respectively, to $L = (L_1, L_2)$ and $L = (L_1, L_3, L_2)$, where $L_1, L_2, L_3$ are shown in Fig. 3:

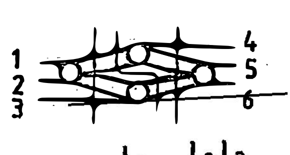
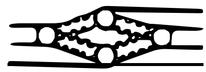



Fig. 3. The divisions $L_1, L_2, L_3$.

are both of the form $\pm$ , but have opposite signs and thus cancel each other.

It is finally useful to define $(\overline{F}_G)^S$ in a way analogous to $(F_G)^S$ but with minus signs ($(\overline{F}_G)^S = (F_G)_-$ if S is the empty set):

$$(\overline{F}_G)^S = \sum_L (-1)^{l+N_c+1} (F_{G_1})_- * \ldots * (F_{G_{l+1}})_- \tag{59}$$

where $N_c$ is the total number non trivial connected components of the graphs $G_1, \ldots, G_{l+1}$.

The following result can be checked from Conjecture 5 (graph by graph unitarity) in a real neighbourhood of the physical region, at $s < [(r+1)\,\mu]^2$:

*Lemma* 4. $$(F_G)^S = (\overline{F}_G)^{\overline{S}},\ \forall S \tag{60}$$

where $\overline{S} = E/S$ is the complement of S in the set E of all channels. In particular:

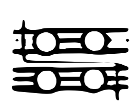
$$(F_G)^E = (F_G)_-. \tag{61}$$

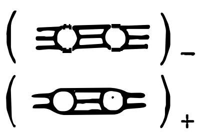
*Proof*

The case S = E (Eq. (61)) is proved by a repeated application of Conjecture 5. The latter gives in the physical region, possibly for non connected functions $F_{G'}$ (encountered in the recurrent procedure):

$$(-1)^{N_c(G')+1} (F_{G'})_- = (F_{G'})_+ + \sum (F_{G'_1})_+ * (F_{G'_2})_- (-1)^{N_c(G'_2)} \tag{62}$$

where $N_c(G)$ is the number of non trivial connected components of G, and the sum $\sum$ runs over all possible divisions of G′ (with respect to g) into two subgraphs $G'_1$, $G'_2$ that both include non trivial connected components[22]. Eq. (62) can be obtained as follows if G′ is not connected. Any division of G, and thus of G′, with respect to g that crosses one or more bubbles includes at least $r+1$ lines. If G′ is composed of several connected components $G_c'^{(i)}$, one checks that the conditions

[22] Divisions including only external lines such as  are allowed if G′ is not connected. In this example, $G'_1$ = , $G'_2$ = , and

$(F_{G'_1})_+ * (F_{G'_2})_- =$  (no integration).

of application of Conjecture 5 are then satisfied for each one in the region $s_i < [(r_i + 1)\,\mu]^2$ where $r_i = r - \sum_{i \neq j} N_j$ and $N_j$ is the minimum of the numbers of incoming and outgoing lines of $G_c'^{(j)}$. Eq (62) is then obtained by multiplying the relations provided by Conjecture 5[23], written for each connected component in the form:

$$(\mathrm{F}_{G_c'})_+ - (\mathrm{F}_{G_c'})_- + \sum (-1)^{N_c(G'_{c,2})} (\mathrm{F}_{G'_{c,1}})_+ * (\mathrm{F}_{G'_{c,1}})_+ * (\mathrm{F}_{G'_{c,2}})_- = 0 \tag{63}$$

(each one of these relations depending on its own external energy-momenta variables).

Eq. (62) is first applied to G, then to the graph $G_2$ that occurs in Eq. (32) and so forth. This procedure, applied possibly to cases when G itself is not connected, gives:

$$(\mathrm{F_G})_- = (-1)^{N_c(G)+1} (\mathrm{F_G})^{\mathrm{E}} \tag{64}$$

which reduces to Eq. (61) when G is connected.

The general case (60) can be checked by replacing each function $(\mathrm{F}_{G_l})_-$ occurring in $(\overline{\mathrm{F}}_G)^{\overline{S}}$ by $(\mathrm{F}_{G_l})^{\mathrm{E}}$ : $(\overline{\mathrm{F}}_G)^{\overline{S}}$ is transformed in that way into a sum of multiple on mass-shell convolutions involving only functions $(\mathrm{F}_{G_k})_+$. The sum includes all terms occurring in $(\mathrm{F}_G)^{S}$, as part of the terms arising from the contribution $(\mathrm{F}_G)_-$ to $(\overline{\mathrm{F}}_G)^{\overline{S}}$, and a priori manu other terms. Each one of these remaining terms corresponds to a set L of divisions that include (i) a subset of $p \geqslant 0$ divisions relative to S and (ii) a subset of $q \geqslant 1$ divisions relative to $\overline{S}$: among the latter $q'$ arise from the operations $*$ occurring in the expansion of $(\overline{\mathrm{F}}_G)^{\overline{S}}$, $0 \leqslant q' \leqslant q$, and $q - q'$ arise from the expansions of the terms $(\mathrm{F}_{G_l})_-$. A multiplicative sign $(-1)^{p+q'}$, $\times (-1)^{q'}$ is obtained from the signs that occur in the relations (59), (64) and (49). For each L, the total contribution is thus obtained with the multiplicative factor $(-1)^{p+q} \sum_{q'=0}^{q} C_q^{q'} (-1)^{q'} = (-1)^{p+q} (1-1)^q \equiv 0$, since $q \geqslant 1$. I. e. there is exact cancellation of these contributions. Q.E.D.

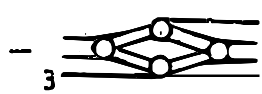
Another result, useful below, says that if S is a good set (no pair $i, f$ such that $(if) \in S$, $t \in S$, $i \notin S$, $f \notin S$ or $(if) \notin S$, $t \notin S$, $i \in S$, $f \in S$ in the $3 \to 3$ case: see Sect. 5.1. iii), there is no »interference« between divisions occuring in the expansion of $(\mathrm{F}_G)^{S}$ and $(\overline{\mathrm{F}}_G)^{\overline{S}}$:

*Lemma* 5. ($3 \to 3$ case)

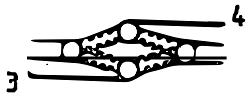
If S is a good set, then being given any set L of divisions $L_1, \ldots, L_l$ giving rise to a non zero term in the expansion (49) of $(\mathrm{F}_G)^{S}$ (resp. in the expansion of $(\overline{\mathrm{F}}_G)^{\overline{S}}$), there exists no division consistent with $\overline{S}$ (resp. with S) whose internal lines all belong to the set of internal lines determined by the division $L_1, \ldots, L_l$.

[23] The condition $s < [(r+1)\,\mu]^2$ yields $s_i < [(r_i+1)\,\mu]^2$ for each $i$, in view of mass-shell conditions.

The content of this lemma can be illustrated on the example of the term $F_G =$




(see Example d) above). First consider a »bad« set, e. g. $S_1 = \{3,4\}$, and the set $L = (L_1, L_2)$ that gives rise to the non zero contribution




to $(F_G)^{S_1}$ (see Eq. (56). $L_1, L_2$ are the divisions shown in Fig. 3). Then the division $L_3$ of Fig. 3 which is consistent with $t \in \bar{S}_1$ or the analogous division consistent with the channel (34) of $\bar{S}_1$, have internal lines that all belong to those of L. But if we consider a good set, such as $S_3 = \{3, 4, t\}$, then this same contribution is absent as explained below (58) and Lemma 5 is thus checked easily.

Lemma 5 is checked more generally by direct inspection. We now state:

*Conjecture* 6

If the irreducible kernels involved at each vertex of a (regularized) Feynman-type integral $(F_G)$ satisfy the analyticity properties of Conjecture 2 and if Property (P) as stated in Conjecture 1 applies, then, for every good set S, $(F_G)^S$ is on-shell, in the physical region, analytic at $s < [(r + 1)\,\mu]^2$ except on some branches (depending on S) of (modified) Landau surfaces of G and of graphs obtained from G by contraction[24]. Apart from exceptional points, good functions $(F_G)^S$ are moreover boundary values of analytic continuations, on the complex mass-shell, of a unique analytic function $F_G$ : starting from small values of *s*, analytic continuation is made beneath, resp. above, normal thresholds associated with S, resp. with $\bar{S}$ (i. e. with minus $i\varepsilon$ and plus $i\varepsilon$ rules, respectively). Analytic continuation around other singularities is also well specified and agrees with previous rules at points where singularities occur.

Similar results apply in the neighbourhood of the physical region, off-shell. If the set S corresponds to a cell, $(F_G)^S$ is also the boundary value of $F_G$ from the (off-shell) directions of the cell (which are in this case independent, for each S, from the real point considered).

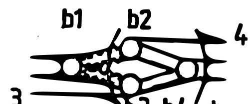
As in Sect. 4, this conjecture is purely mathematical. Its proof should provide more precise and more general information e. g. on on-shell or off-shell analyticity domains. We only give below some arguments that support the plus and minus $i\varepsilon$ rules of analytic continuation, in the physical region, around normal thresholds associated with S or $\bar{S}$. These arguments are related to some of those used in Ref. 22 for the *S* matrix but can be analyzed in a more precise and more detailed way in the present framework. They are based on Lemmas 4 and 5, on Conjecture 3 and on structure theorems[28,25,26)] on the (micro-)analytic structure of on mass-

[24] Singular branches are for $F_+$ and $F_-$ the $+\,\alpha$-parts of the Landau surfaces.

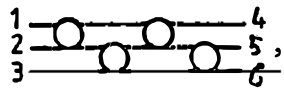
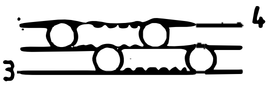
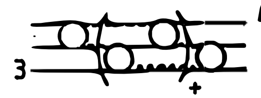
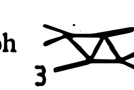
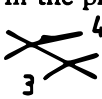
-shell convolution integrals. These theorems give information on the essential support (= singular spectrum) of these integrals in terms of the essential supports of individual factors, and hence by duality on possible directions of analyticity with respect to imaginary parts of the variables. External energy-momenta can either be strictly restricted to the mass-shell or be allowed to vary in a real neighbourhood. From Conjecture 3, the essential support of each term $(F_{G_k})_+$, resp. $(F_{G_k})_-$, at any physical point, is known to be associated with relative configurations of external trajectories of classical space-time diagrams $(D_k)_+$, resp. of opposite diagrams $(D_k)_-$, whose topological structure is $G_k$, or with relative configurations of points in space-time (one for each initial and final particle) that lie at corresponding external vertices of such diagrams. Structure theorems assert that the essential support of the integral is associated with relative configurations of external trajectories (or external points) of space-time diagrams $D$ whose topological structure is G and which are collections of subdiagrams $(D_k)_+$, resp. $(D_k)_-$, that »fit together«. Some limiting procedures may also have to be considered in »$u = 0$« cases.

The $p$-particle normal threshold associated with a channel $g_l$ is the singularity of the basic graph $G_{g_l,p}$ =  ,
with two vertices whose sets of external lines are those of the channel $g_l$ and p internal lines between these two vertices. The typical case where analytic continuation around it is blocked occurs when the set of internal lines over which there is on-mass-shell integration includes a set associated with a division of G having p internal lines and consistent with $g_l$ : the graph G can in this case be reduced by contraction to $G_{g_l,p}$, and the essential support of the integral at real points of the threshold includes the two opposite directions corresponding to the (relative) external configurations of space-time diagrams $(D_{g_l,p})_+$ and $(D_{g_l,p})_-$, respectively: the two vertices of $G_{g_l,p}$ are represented in these diagrams by points a, b such that $b - a = \lambda k(g_l)$, where $k(g_l)$ is the energy-momentum of the channel, with $b - a \in V_+$, resp. $b - a \in V_-$. By duality, these two directions correspond, respectively, to directions of analyticity $\mathrm{Im}\, s_{g_l} >$ and $\mathrm{Im}\, s_{g_l} < 0$ which are conflicting.

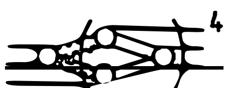
To show that $(F_G)^S$ continues above normal thresholds associated with any channel $g_l$ in $\overline{S}$, one may use the expansion (49). If S is a good set, Lemma 5 ensures in fact that the typical situation just mentioned in which analytic continuation is blocked does not occur for any individual term in this expansion. On the other hand, cases in which a diagram $D_{b_1}$ reduces to $(D_{g_l,p})_+$ do occur: let us consider e. g. the term




in the expansion (58) of $(F_G)^{S_3}$, its four bubbles being denoted here $b_1, \ldots, b_4$, and corresponding diagrams with respective space-time vertices $a_1, \ldots, a_4$. Then the situation just mentioned is obtained for $g_l = (34) \in \overline{S}_3$ and $p = 4$, by contracting lines between $a_1$, $a_2$ and between $a_3$, $a_4$ with $a_3 - a_1 \equiv a_4 - a_2 \in V_+$, a condition consistent with the requirement that the two lines between $a_2, a_4$ are associated with internal lines of a term $(F_{G_k})_+$. Thus one does obtain by duality a plus $i\varepsilon$ rule of analytic continuation corresponding to Im $s_{g_i} > 0$, unless *external configurations* of other relevant diagrams $D$ (different from $D_{g_i,p}$) lead to some conflict, e. g. coincide with the external configurations of $(D_{g_i,p})_-$.

Whether such a situation may arise or not requires a further analysis. If we consider e. g. the term $F_G =$  , $S = \{3, 4, t\}$ and $g_i = (45) \in \in \bar{S}$, terms such as  or  are encountered in the expansion of $(F_G)^S$. These terms are individually singular along a mixed-$\alpha$ branch of the Landau surface of the graph  which contains in the physical region the one-particle threshold singularity of the graph $G_{g_i,1} =$  ; moreover external configurations of corresponding diagrams $D$ do coincide with those of the diagrams $(D_{g_i,1})_-$. Thus, the above argument cannot be used to show that $(F_G)^S$ has a plus $i\varepsilon$ analytic continuation around the one-particle threshold of the channel (34). However, the problem does not occur[25] in this case since the channel (34) is not a relevant channel: $(F_G)^S$ is in fact equal to $(\bar{F}_G)^{(34)} \equiv (F_G)_-$, which is not singular along the one-particle threshold of $G_{g_i,1}$ (see in this connection Footnote 24). We shall assume that a more complete analysis leads to the same conclusion, apart possibly from exceptional situations.

The arguments given above cannot be applied to channels $g_i \in S$, where analytic continuation is indeed blocked in the typical way for some of the individual on-mass-shell integrals occurring in the expansion of $(F_G)^S$, e. g. for the term

 in the expansion (58), if $g_i = (3)$. However a fully analogous argument can now be applied to the terms occuring in the expansion of $(\bar{F}_G)^{\bar{S}}$, which is equal to $(F_G)^S$ by Lemma 4, with plus signs replaced by minus signs. It now leads to expect minus $i\varepsilon$ rules of analytic continuation. Q. E. D.

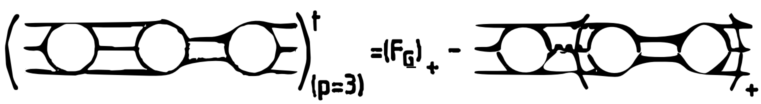
According to Conjecture 6, differences between terms $(F_G)^S$ and $(F_G)^{S'}$ are (if S and S′ are good sets) discontinuities between corresponding boundary values of analytic continuations of a common analytic function $F_G$. These discontinuities can be evaluated from Eq. (50). E. g.:

$$(F_G)^{S_1} - (F_G)^{S_1 \cup S_2} = \sum_{\substack{S' \subset S_1 \cup S_2 \\ S' \not\subset S_1}} (-1)^{|S'|+1} (F_G)_{S'}, \qquad (65)$$

[25] This problem does arise at the level of the complete $S$ matrix in a pure $S$-matrix approach where diagrams are not a priori separated. It is treated implicitly in Ref. 22 by the ad hoc assumption of separation of singularities or mixed-$\alpha$ cancellation.

where $S_1$ and $S_2$ are disjoint. The terms $(F_G)_S$ represent on the other hand multiple discontinuities.

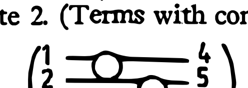
We conclude this section with the following remark on a possible extension of the previous analysis. Being given a set S of channels $g_i$ and for each i, a positive integer p $(g_i)$, $(F_G)^S{}_{\{pi\}}$ can be defined in a way analogous to (49) but with the supplementary condition that each set of internal lines of a division $L_\sigma$ corresponding to a channel $g_i$ ($\in$ S) has at least $p\,(g_i)$ lines. E. g. if S has the unique channel $t$ and if $p\,(t) = 3$, let $F_G$ be the term considered in Eq. (54). Then:

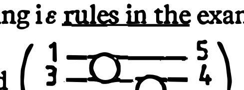


(66)

In this case a conjecture analogous to Conjecture 6 can be stated but with analytic continuation above (not beneath) the $p$-particle normal thresholds associated with S if $p < p\,(g_i)$. Discontinuity formulae follow in the same way as above.

### 5.3 *Applications: Formal derivation of asymptotic completeness relations and S-matrix discontinuity formulae*

We now give some examples of applications of the formulae and conjectures of Sect. 5.2 in the 3 → 3 case (in any energy region $s < [(r+1)\,\mu]^2$). In all the following, $F^S$ is identified formally with $\sum\limits_{G,\alpha} (F_{G,\alpha})^S$ in the region considered. Similarly $F_S = \sum\limits_{G,\alpha} (F_{G,\alpha})_S$. Rules of analytic continuation are derived formally for $F^S$ from those conjectured in Sect. 5.2 for individual terms, at least if there is a common domain of analytic continuation for all terms $(F_{G,\alpha})^S$. If S corresponds to a cell, there are always common directions of analyticity, namely those of the (complex, off-shell) domain associated with S. We shall admit that this is also the case more generally and on the complex mass-shell, apart from exceptional points, such as those already mentioned for $F_+$ that lie on several Landau surfaces with conflicting $i\varepsilon$ rules: see Footnote 2. (Terms with conflicting $i\varepsilon$ rules in the example given there are e. g. the terms 

and 

).

The discontinuity formulae of the $S$ matrix and Green functions will follow in all cases from (i) an algebraic transformation of the relevant discontinuity formulae for individual terms (ii) a formal resummation over G, $\alpha$ and (iii) an algebraic analysis analogous to that used in the proof of Theorem 1 in Sect. 4.2, which will be omitted.

i) *Asymptotic completeness*

We start from formula (65), written in the case of two adjacent cells $S_1$ and $S_1 \cup$ $\cup\, t$, $t \notin S_1$. The sum in the right-hand side of (65) is taken over all subsets S' containing $t$ and possibly channels of $S_1$. We then consider the expansions described below (50) of the terms $(F_G)_{S'}$ and regroup all terms of these expansions that have the same first division $\underline{L}$ consistent with $t$. $\underline{L}$ divides G into two connected subgraphs G′, G″. For each $\underline{L}$ one has to consider all possible further divisions of G′ and of G″ that are divisions of G with respect to $t$ and are, respectively, consistent with $S_1$ and $S_1 \cup t$. Let $t'$, resp. $t''$, be the channel relative to the external lines of G′, resp. G″, determined by the $r'$ ($< r$) lines of $\underline{L}$. The above divisions of G′, resp. G″, are divisions with respect to $t'$, resp. $t''$, which are consistent with channels of a set $\Sigma'$, resp. $\Sigma''$. $\Sigma'$ is the set of channels g′ that divide the external lines of G′ into two subsets—one of which contains two of the three incoming lines of G that are also those of one of the subsets determined by a channel in $S_1$; it may also contain lines of $\underline{L}$. $\Sigma''$ is similarly the set of channels g″ that divide the external lines of G″ into two subsets one of which contains either two of the three outgoing lines of G that are also those of one of the subsets determined by a channel in $S_1$, or the three outgoing lines of G, plus possibly in either case lines of $\underline{L}$.

$\Sigma'$ or $\Sigma''$ may include »initial« or »final« channels, respectively, (one subset composed of two incoming, resp. two outgoing, lines of G), and also crossed channels; $\Sigma''$ includes also the channel $t''$ and all channels with one subset composed only of lines of $\underline{L}$ (incoming lines in G″).

*Example*: Let $F_G$ and $L_1$, $\underline{L} = L_2$, $L_3$ be the term and divisions shown in Fig. 4

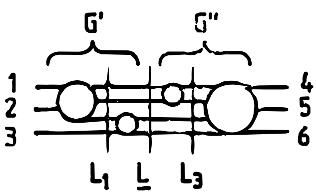



Fig. 4.

$L_1$ is consistent with $S_1$ if $S_1$ contains the channel (3) = (12;3456). It corresponds to a crossed channel in $\Sigma'$. $L_3$ is consistent with $t$ and corresponds to a channel in $\Sigma''$ one subset of which is composed of two of the four incoming lines of G″.

In view of the definitions of $(F_{G'})^{\Sigma'}$ and $(F_{G''})^{\Sigma''}$, the sum of all terms corresponding to the same division $\underline{L}$ is thus equal to $(F_{G'})^{\Sigma'} * (F_{G''})^{\Sigma''}$ and the following formula is obtained:

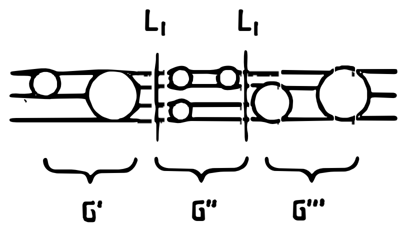
$$(F_G)^{S_1} - (F_G)^{S_1 \cup t} = \sum (F_{G'})^{\Sigma'} * (F_{G''})^{\Sigma''} \tag{67}$$

where the sum $\Sigma$ runs over all divisions $\underline{L}$ of G with respect to $t$ into two *connected* subgraphs G′, G″, these divisions having moreover a number $r' < r$ of intermediate lines. $F_{G'}$ and $F_{G''}$ have $r' + 3$ external lines and $\Sigma'$, $\Sigma''$ are the sets defined above ($t'' \in \Sigma''$, $t' \notin \Sigma'$). By formal resummation and an algebraic analysis analogous to that of Sect. 4.2 (omitted here), one thus reobtains Eqs. (40) in the case $m = n = 3$. The result is a priori obtained in a neighbourhood of the physical region and can then be extended by analytic continuation. It can be checked that the sets $\Sigma'$, $\Sigma''$ introduced here correspond to cells for the $(r' + 3)$ point functions involved and coincide with those introduced (in a different way) in the axiomatic framework.

ii) *S-matrix discontinuity formulae*

We start below with the derivation of discontinuity formulae needed in multiparticle dispersion relations (see Sect. 5.1 (iii)), and will then briefly comment the derivation of formulae given in Sect. 5.1 (ii).

As an example of the first formulae, we wish to derive below formula (46), and consider to that purpose any graph G that can be cut with respect to the channel $t$ in one or several ways. $(F_G)_t$ is according to Sect. 5.2 a sum of terms associated with sets L of divisions $L_1, \ldots, L_l$, that are all consistent with $t$. Let us divide the sum into a sum over sets L with only one division $L_1$ ($l = 1$) and a sum over remaining sets, and let us regroup the latter by subclasses that have the same external divisions $L_1, L_1'$ ($= L_l$). The graph G is divided by $L_1$ in the first case into two connected successive subgraphs G′, G″ and it is divided by $L_1, L_1'$ in the second case into three successive subgraphs G′, G″, G‴ where G′ and G‴ are again connected but G″ may be non connected, as in the example:




The terms of a class $(L_1, L_1')$ are obtained by considering all possible sets of zero, one or more *intermediate* divisions of G with respect to $t$; the latter are all possible divisions of G″ (with $< r$ intermediate lines) into successive, connected or not connected, subgraphs with respect to the channel $t''$ (which divides the set of external lines of G″ into the two subsets determined by the lines of $L_1$ and $L_1'$). These divisions are not necessarily consistent with $t''$, and may be consistent with any other channel: e. g. the division $L_2$ in Fig. 5, which is consistent with $t$ in G, is not consistent with $t''$ but with the crossed channel $(\alpha\beta\alpha';\gamma\beta'\gamma')$.

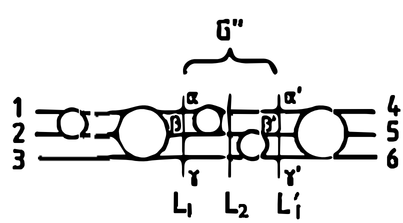


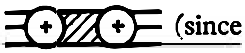
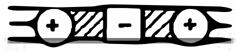

Fig. 5.

By definition of $(F_G'')^E$ (extended possibly to non connected graphs G″), the sum of all terms in the class $(L_1, L_1')$ is thus equal to $(F_{G'})_+ * (F_{G''})^E * (F_{G'''})_+$. $(F_{G''})^E$ is also equal by Eq. (64) to $(-1)^{N_c(G'')+1} (F_{G''})_-$. Hence, the following formula is obtained:

$$(F_G)_t = \sum_{L_1} (F_{G'})_+ * (F_{G''})_+ + \sum_{L_1, L_1'} (-1)^{N_c(G'')+1} (F_G')_+ * \\ * (F_{G''})_- * (F_{G'''})_+ . \quad (68)$$

The formal resummation over G, $a$, together with an algebraic analysis analogous to that of Sect. 4.2, then gives Eq. (46):

$F_t =$  ,

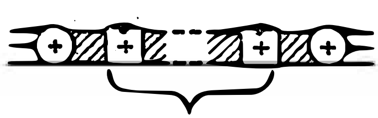
In fact, the summation over G, $a$ of the terms in the first sum of (68) gives  (since G′ and G″ are connected), which corresponds to the identity part of  , while the summation of the terms in the second sum gives the remaining contribution to  corresponding to all non trivial parts of  (at least one non trivial connected component). Since G′, G‴ have to be connected, one finds again connected terms on the left and right sides of the discontinuity formula, while this is not the case for the minus box.

Other discontinuity (or multiple discontinuity) formulae such as (47) and (48) and other formulae involved in multiparticle dispersion relations are derived similarly. In e. g. (48), the fact that line $i$ (or $f$) cannot go straight through corresponds to the fact that the only allowed divisions in the terms $(F_G)_{i,f}$ have to be consistent with $i$ or $f$ but *not* with the crossed channel ($if$).

Eq. (68) can be alternatively derived without regrouping the terms in the class $L_1$ or $(L_1, L_1')$. One first obtains:

$$(F_G)_t = \sum_{\substack{G_1,\dots,G_{n+1} \\ G_1, G_{n+1} \text{ connected} \\ n \geqslant 1}} (-1)^{n-1} (F_{G_1})_+ * (F_{G_2})_+ \dots * (F_{G_{n+1}})_+ \tag{69}$$

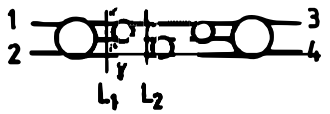
where the sum $\Sigma$ runs over all ways of dividing G into successive subgraphs $G_1, \dots, G_{n+1}$, $n = 1, 2, \dots$, where $G_1$ and $G_{n+1}$ are connected but $G_2, \dots, G_n$ are connected or not connected. (Each division has a set of $< r$ intermediate lines). By formal resummation (and a further algebraic analysis), this allows in turn to express $F_t$ as the formal infinite sum:

$F_t = \sum_{n>1} (-1)^{n-1}$  (70)

n-1 factors

from which Eq. (68) is reobtained, in view of Eq. (43).

In view of the comments made and the end of Sect. 5.2, the same method allows one to reobtain e. g. the discontinuity formula (41) around the three-particle threshold $s = (3\mu)^2$ in the $2 \to 2$ case. In fact, for any graph G, $(F_G)_t^{(\rho=3)}$ is given by a formula analogous to (69) but with on-mass-shell convolutions over at least three particles. The analogue of (70), with sets  of three particles, follows and hence (41), in view of (42).

The alternative form (44) of this same discontinuity can be obtained by considering the expansion of $(F_G)_t$, and by grouping now together all terms that have the same first division $L_1$, which divides G into two connected subgraphs G′, G″. Possible further divisions $L_2, \dots, L_{n+1}$ of G″ consistent with $t$ in G are all possible divisions of G″ with respect to the channel $t''$ (determined by the lines of $L_1$ and the outgoing lines of G), which are consistent with $t''$ or with any other channel $g_1^{(In)}, \dots, g_\varrho^{(In)}$ that divides the set of external lines of G″ into two subsets, one of which is composed of lines of $L_1$ only ($\varrho = 3$ in the region $s < (4\mu)^2$). E. g. in the case 

, $L_2$ (which is consistent with $t$ in G) is consistent in G″ with the channel $(\alpha\beta;\gamma 34)$. By resummation, one obtains:

$$(F_G)_t = \sum_{L_1} (F_{G_1})_+ * (F_{G_2})^{t, g_1^{(In)}, \dots, g_\varrho^{(In)}} \tag{71}$$

which gives in turn formula (44).

The triangle discontinuity formula recalled at the beginning of Sect. 5.1 (ii) can be recovered from an adaptation of Cutkosky discontinuity for Feynman integrals. More general cases should be treated by a combination of previous arguments.

## 6. *Conclusion*

In this paper, a general formalism of formal expansions of multiparticle momentum-space Green functions in terms of Feynman-type convolution integrals involving irreducible kernels at each vertex has been presented for theories with massive particles. If there are convergence problems, corresponding to theories in which renormalization is needed, these integrals are regularized, i. e. an analytic factor equal to one on-mass-shell and with sufficient decrease at infinity in euclidean directions, is included on internal lines: the irreducible kernels then depend on the choice of this factor and are not expected to have a simple perturbative content. However, they are expected to obey regularized Bethe-Salpeter type equations and to satisfy the analyticity properties associated with their degrees of irreducibility.

The above expansions are conjectured on the basis of perturbative arguments if no regularization is needed and are believed to hold in the regularized case in particular on the basis of results of axiomatic field theory for $2 \to 2$ and $3 \to 3$ processes in the low energy region. The more general case involves, however, new features and more work will be needed to confirm the conjecture.

Purely mathematical conjectures on the analytic structure of the Feynman-type integrals involved in these expansions have then been presented. They rely on the idea that, in view of their analyticity properties, the irreducible kernels involved at each vertex should not modify the basic analytic and monodromic structure of the integral in the energy region considered. These integrals should be in particular regular holonomic in that region as the corresponding (regularized or renormalized) Feynman integral, and the above expansions thus provide (formal) decompositions of the (generally non holonomic) $S$-matrix and Green functions into elementary (regular holonomic) contributions with specified Landau singularities and corresponding analyticity and monodromy properties. The decompositions are minimal in a well specified sense. Various discontinuity formulae of the Feynman-type integrals have moreover been conjectured. (Some of them are known for Feynman integrals, while others are also conjectured for the latter). Together with the previous expansions, they have been shown to yield (in a formal sense) corresponding discontinuity formulae of the $S$-matrix and Green functions, such as unitarity relations, discontinuity formulae characterizing asymptotic completeness in field theory and those needed for the $S$-matrix (in the $3 \to 3$ case) in multiparticle dispersion relations.

Although this was not our primary purpose, the analysis may be adapted to a pure perturbative approach and may allow one to complete in that framework previous results, if the discontinuity formulae that have been conjectured can be established for (regularized or renormalized) Feynman integrals. We shall now discuss its possible relevance in axiomatic field theory, constructive theory and $S$-matrix theory.

In axiomatic field theory, the formal expansions should be ultimately established from asymptotic completeness and regularity assumptions, as already achieved so far in an even theory for $2 \to 2$ and $3 \to 3$ processes in the low energy region, through Neumann series expansions in relevant Bethe-Salpeter type equations. The present analysis may be helpful in the development of this program, by providing a more general, although formal, understanding of the links between asymptotic completeness and irreducibility, of the type of irreducible kernels to be considered and possibly of the type of Bethe-Salpeter equations that would have to be introduced in general in the multiparticle case. It also yields information on the analyticity properties of multiparticle Green functions and collision amplitudes that can be expected.

The same remarks apply also to constructive field theory, where irreducibility properties should be directly established, as achieved so far in some models for $2 \to 2$ and $3 \to 3$ processes in the low energy region. Our analysis may provide in this framework a general understanding of the way they might yield asymptotic completeness, at least as a first step in the sense of formal expansions in terms of irreducible kernels. More precisely, one may proceed in successive regions: if analyticity properties are established for the irreducible kernels involved in the expansion of a Green function in a given energy region, then asymptotic completeness would follow in that region. Our analysis is, however, limited from the constructive viewpoint for the following reasons. First, the derivation of analyticity properties of irreducible kernels is a preliminary non trivial work: in particular, our analysis shows that the type of irreducible kernels to be considered is more general than that considered and partly treated so far[36,37] in the general case in constructive theory. Secondly, our analysis is formal and further non trivial estimates would be needed to transform it into actual (non formal) results. Finally, it not clear that the regularized formalism used in this work is the appropriate one in theories that require renormalization. An alternative approach is to introduce »renormalized« Bethe-Salpeter type kernels and to extend to this case the analysis of the links between irreducibility and asymptotic completeness: see Footnote 9 in this connection.

From the viewpoint of $S$-matrix theory, the analysis may provide a complementary understanding of the origin of various results and of various aspects of the assumptions previously made in the derivation of discontinuity formulae, such as »separation of singularities«. While these assumptions have the advantage of eliminating difficult problems at one go, a more detailed understanding, at a more fundamental level, should follow from this work, so far in a field-theoretical framework. It might be possible to produce a localized version of the analysis, in the neighbourhood of the mass-shell which would be more satisfactory from an $S$-matrix viewpoint. While the formal expansions that have been conjectured have been shown to be natural solutions of unitarity equations in each energy region, we do not know if expansions of this type can be actually derived from unitarity (and further assumptions) in a pure $S$-matrix approach, as achieved in the simplified theory of the $m$-particle threshold where the expansions obtained (in terms of different irreducible kernels) are moreover convergent: see Sect. 1.1 and Refs. 5,26. It has been conjectured in Ref. 26 that there should also exist in the non simplified theory convergent expansions of the S-matrix, in the neighbourhood of singular points, in terms of regular holonomic contributions having the same characteristic varieties as those of relevant Feynman integrals. However, new features arise in the non simplified theory, where this conjecture is in our opinion questionable in the absence of more precise arguments that would support it. It has not been discussed in this work where all expansions are considered only from a formal viewpoint.

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# IRREDUCIBILNOST, ANALITIČNOST I UNITARNOST ILI ASIMPTOTSKA POTPUNOST U MASENOJ KVANTNOJ TEORIJI POLJA: NEKE OPĆENITE PRETPOSTAVKE I REZULTATI

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Prikazane su općenite pretpostavke i s njima povezani rezultati za analitičku strukturu prostora impulsa višečestićnih Greenovih funkcija i amplituda raspršenja u kvantnoj teoriji polja s masom.