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ON THE STRUCTURE OF SCATTERING DATA AND NONLINEAR FIELD IN THE CONTINUOUS SPECTRUM FOR THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

ASESH ROY CHOWDHURY and MINATI NASKAR

High Energy Physics Division, Department of Physics,

Jadavpur University, Calcuta — 700032, India

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Contrary to the usual procedure of obtaining the nonlinear field in the continuous spectrum we have first determined the explicit structure of the transmission and reflection coefficients from the space part of the IST equation itself. These explicit structures were then used in the Gelfand-Levitan equation to ascertain the oscillatory and similar behaviour of the nonlinear field variable $q(x, t)$. Our methodology is elaborated with derivative nonlinear Schrödinger equation as an example, but the procedure is quite general to be applied to any eigenvalue problem.

1. *Introduction*

After the initial realization that the method of inverse scattering[1] has immense potentiality for the analysis of nonlinear problems, there have been many attempts for the application of the method for the continuous values of the eigenvalue. Important contribution in this direction was done by Ablowitz[2], and Ablowitz and Newell[3]. But it has been a custom to obtain the behaviour of the nonlinear field in the continuous spectrum through a steepest descent integration in the Gelfand-Levitan kernel, avoiding explicit specification of the transmission and reflection coefficients a and b. Our procedure is analogous to that of discrete spectrum, in the sense that we first obtain the explicit form of the transmission and reflection coefficients from the 1st IST (Inverse Scattering Transform) equation and then obtain the nonlinear variable q through Gelfand-Levitan equation. We have reproduced the oscillatory and similarity behaviour of $q(x, t)$ as have been obtained in previous analysis. It can be mentioned at this point that initially such a procedure was suggested by Nonkov et al.[4] for the AKNS system.

2. Formulation

The derivative nonlinear Schrödinger equation reads;

$$q_t + \mathrm{i} q_{xx} + (|q|^2 q)_x = 0. \tag{1}$$

The x-part of the IST[5] equation is the Kaup-Newell spectral problem written as:

$$v_{1x} + \mathrm{i}\,\zeta^2 v_1 = q\,\zeta\, v_2$$

$$v_{2x} - \mathrm{i}\,\zeta^2 v_2 = r\,\zeta\, v_1. \tag{2}$$

Setting

$$\psi_1 = v_1\, \mathrm{e}^{\mathrm{i}\zeta^2 x};\ \psi_2 = v_2\, \mathrm{e}^{-\mathrm{i}\zeta^2 x}$$

the eigenvalue problem (2) can be transformed to the form:

$$\psi_{1x} = q\, \mathrm{e}^{2\mathrm{i}\zeta^2 x}\, \zeta\, \psi_2$$

$$\psi_{2x} = r\, \mathrm{e}^{-2\mathrm{i}\zeta^2 x}\, \zeta\, \psi_1. \tag{3}$$

Let us now put:

$$q = A(x)\, \mathrm{e}^{\mathrm{i}\Phi},\ r = A(x)\, \mathrm{e}^{-\mathrm{i}\Phi}$$

so that Eq. (3) is

$$\psi_{1x} = A\,\zeta \exp\,[\mathrm{i}\,\{2\,\zeta^2\, x + \Phi(x)\}]\,\psi_2 \tag{4a}$$

$$\psi_{2x} = A\,\zeta \exp\,[-\,\mathrm{i}\,\{2\zeta^2\, x + \Phi(x)\}]\,\psi_1. \tag{4b}$$

We now set

$$\widetilde{\varphi} = 2\zeta^2\, x + \Phi(x) \tag{5}$$

so that we get

$$\psi_{1x} = A\,\zeta\, \mathrm{e}^{\mathrm{i}\widetilde{\varphi}(x)}\, \psi_2$$

$$\psi_{2x} = A\,\zeta\, \mathrm{e}^{-\mathrm{i}\widetilde{\varphi}(x)}\, \psi_1. \tag{6}$$

Now as in Ref. 4 we assume that the following restrictions hold:

$$\Phi_{xx} > 0; \left|\frac{\mathrm{d}}{\mathrm{d}x} \ln A\right|^2 \ll \Phi_{xx}, \quad \left|\frac{\mathrm{d}}{\mathrm{d}x} \ln \Phi_{xx}\right|^2 \ll \Phi_{xx}.$$

These conditions mean that the field can be represented as the product of an almost constant amplitude and an exponential function with linearly varying frequency as its argument.

Let us consider a stationary point of the phase function given by

$$\Phi_x(x_0) + 2\zeta^2 = 0. \tag{7}$$

If we denote the root of this equation as $x = x_0$ then we expand the phase in the exponential in equation (6) in Taylor series around $x = x_0$ keeping terms up to second power in $x - x_0 = y$, so that we get

$$\frac{\partial \psi_1}{\partial y} - A(x_0)\,\zeta \exp\left[\mathrm{i}\left\{\tilde{\varphi}(x_0) + \frac{y^2}{2} f_0\right\}\psi_2\right] = 0$$

$$\frac{\partial \psi_2}{\partial y} - A(x_0)\,\zeta \exp\left[-\mathrm{i}\left\{\tilde{\varphi}(x_0) + \frac{y^2}{2} f_0\right\}\psi_1\right] = 0 \tag{8}$$

where

$$f_0 = \Phi_{xx}(x_0).$$

These equations yield:

$$\psi_{zz} - \mathrm{i}z\,\psi_{1z} = \eta(\zeta)\,\psi_1 \tag{9}$$

a second order ordinary differential equation for ψ_1, where;

$$\eta(\zeta) = \frac{A^2(x_0)}{\Phi_{xx}(x_0(\zeta))} \tag{10}$$

and z is defined through $z = (x - x_0) f_0^{1/2}$ whose solution can be written explicitly as:

$$\psi_1 = \mathrm{e}^{\mathrm{i}z^2/4}\,[d_2\, D_{-\mathrm{i}n-1}(-z/\sqrt{\mathrm{i}})] \tag{11}$$

with

$$d_2 = \frac{\Gamma(1 + \mathrm{i}n)}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{\pi n}{4}} (f_0)^{-\frac{\mathrm{i}n}{2}}\, b(\zeta) \exp\left[\mathrm{i} \arg a(x_0(\zeta)) - \mathrm{i}\,\Phi(x_0)\right].$$

On the other hand ψ_2 can be represented as

$$\psi_2 = \mathrm{e}^{-\mathrm{i}z^2/4}\,[a_1\, D_{\mathrm{i}n-1}(\sqrt{\mathrm{i}}\,y) + a_2\, D_{\mathrm{i}n-1}(-\sqrt{\mathrm{i}}\,y)]. \tag{12}$$

In these expressions D_α is the parabolic cylindrical function as discussed in Ref. 4. In Eq. (12) we have

$$a_1 = \frac{\Gamma(1-\mathrm{i}n)}{\sqrt{2\pi}} \mathrm{e}^{-\frac{\pi n}{4}} (f_0)^{\frac{\mathrm{i}n}{2}} \mathrm{e}^{\mathrm{i}\Phi(\zeta)}$$

$$a_2 = \frac{\Gamma(1-\mathrm{i}n)}{\sqrt{2\pi}} \mathrm{e}^{-\frac{\pi n}{4}} (f_0)^{\frac{\mathrm{i}n}{2}} \mathrm{e}^{\mathrm{i}\Phi(\zeta)} \; |a(\zeta)| \tag{13}$$

along with

$$\Phi(\zeta) = \frac{1}{\pi} \ln |2\zeta| \ln |a(\zeta)| + \frac{1}{\pi} \int_{-\zeta}^{\zeta} \ln |\zeta' - \zeta| \frac{\mathrm{d}}{\mathrm{d}\zeta'} \ln |a(\zeta')| \, \mathrm{d}\zeta'. \tag{14}$$

The solution of the IST equation explored in our above calculation through the stationary phase approximation is known as »resonance region«.

At a point far from the resonance region x_0 we approximate the set of Eqs. (2) as:

$$\begin{aligned} \mathrm{i}\,\tilde{\varphi}_x \psi_{1x} + A^2 \psi_1 &= 0 \\ -\,\mathrm{i}\,\tilde{\varphi}_x \psi_{2x} + A^2 \psi_2 &= 0 \end{aligned} \tag{15}$$

whose explicit solution can be written as

$$\psi_2 = \exp\left[-\,\mathrm{i}\zeta^2 \int \frac{A^2}{\tilde{\varphi}_x} \mathrm{d}x\right]. \tag{16}$$

At a general point these two solutions should match.

Now let us recapitulate that the Jost function is defined by the asymptotic limit

$$\xi(x,\zeta) = \tilde{\psi}(x,\zeta) \to \binom{0}{1} \mathrm{e}^{-\mathrm{i}\zeta^2 x}, \quad x \to \infty$$

so that combining this with Eqs. (9) and (10) we obtain:

$$\xi_1 = 0$$

$$\xi_2 = [\Phi_{xx}(x_0)(x - x_0)]^{-\mathrm{i}\zeta^2 n(\zeta)} \exp [\mathrm{i}\,\zeta^2 \int_{x_0}^{\infty} p(x)\,\mathrm{d}x]$$

where

$$p(x) = \ln \tilde{\varphi}_x \frac{\mathrm{d}}{\mathrm{d}x} \frac{A^2}{\tilde{\varphi}_{xx}}.$$

Similarly the other asymptotes may be obtained from (11). Now the transmission and reflection coefficients required for the Gelfand-Levitan equation are defined as:

$$\bar{a}(\zeta) = \xi_2 (x, \zeta) e^{i\zeta^2 x}$$

$$b(\zeta) = - \xi_1 (x, \zeta) e^{-i\zeta^2 x} \tag{17}$$

which yields

$$a(\zeta) = \exp \left[- \pi n + i \int_{-\infty}^{\infty} \frac{A^2}{\tilde{\varphi}_x} dx \right] \tag{18}$$

$$b(\zeta) = \sqrt{\frac{2\pi i}{n \Gamma(in)}} e^{-\frac{\pi n}{2}} \exp [- i \{\tilde{\varphi} + L_1 - L_2 + n \ln \tilde{\varphi}_{xx})\}]$$

where

$$L_1(\zeta) = \int_{\zeta}^{\infty} \ln (\zeta' - \zeta) \frac{d}{d\zeta'} n(\zeta') d\zeta'$$

$$L_2(\zeta) = \int_{-\infty}^{\zeta} \ln (\zeta' - \zeta) \frac{d}{d\zeta'} n(\zeta') d\zeta'.$$

Now for obtaining the nonlinear field we use the information that

$$q(x, t) = F(x, x, t) \tag{19}$$

with

$$F(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b(\zeta, t)}{a(\zeta, t)} e^{i\zeta^2 x} d\zeta.$$

Inserting the expressions for a, b, given in (18) we obtain:

$$F(x, t) = \sqrt{\frac{i}{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi} \Gamma(in)} e^{-\frac{3\pi}{2} n} \exp \{-iy\} d\zeta. \tag{20}$$

y is given by the equation

$$y = \tilde{\varphi} + L_1 - L_2 + n \ln \Phi_{xx} - \int_{-\infty}^{\infty} \frac{A^2}{\tilde{\varphi}_x} dx - (\zeta^2 + 4\zeta^4 t). \tag{21}$$

The remaining integration can be done only with the help of steepest descent method. The point of stationary phase is obtained to be:

$$\zeta' = \pm \frac{i}{2}\sqrt{\frac{3x}{2t}}. \tag{22}$$

Equation (14) yields

$$q = F(x, x, t) = \frac{1}{4}\sqrt{\frac{i}{x}}\,\frac{e^{-\frac{3}{2}\pi n}\left(\frac{i}{2}\sqrt{\frac{3x}{2t}}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{i}{2}\sqrt{\frac{3x}{2t}}\right)}\,e^{-it\left[\frac{\tilde{\Theta}}{t}\left(i\sqrt{\frac{2x}{t}}\right)\right]} \tag{23}$$

which displays the oscillatory behaviour of the nonlinear field q and also its dependence on the similarity variable through its functional form.

In our above analysis we have outlined a procedure for the analysis of nonlinear equations in the region of continuous spectrum through an explicit determination of the transmission and reflection coefficient for the Koup-Newell eigenvalue problem.

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STRUKTURA PODATAKA RASPRŠENJA I NELINEARNOG POLJA ZA KONTINUIRANI SPEKTAR NELINEARNE SCHRÖDINGEROVE JEDNADŽBE S DERIVACIJOM

ASESH ROY CHOWDHURY i MINATI NASKAR

High Energy Physics Division, Department of Physics,

Jadavpur University, Calcutta — 700032, India

UDK 530.182

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Suprotno uobičajenoj proceduri, da bismo dobili nelinearno polje u kontinuiranom spektru, mi smo prvo odredili eksplicitnu strukturu koeficijenta transmisije i refleksije. Te su se strukture koristile u Gelfand-Levitanovoj jednadžbi da bi se odredilo oscilatorno i slično ponašanje nelinearnog polja q. Naša metodologija razrađena je na nelinearnoj Schrödingerovoj jednadžbi s derivacijom kao primjer. Procedura je sasvim općenita i može se primijeniti na bilo koji problem vlastitih vrijednosti.