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BOUND STATE OF TWO SPINLESS CHARGED PARTICLES IN QUANTUM ELECTRODYNAMICS

I. FINE STRUCTURE

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The fine structure energy levels of $(\pi^+ \pi^-)$ - atom is evaluated in the framework of Logunov-Tavkhelidze quasipotential approach.

1. Introduction

Investigation of the hydrogen-like atoms, in particular energy levels, allows high accuracy testing of quantum electrodynamics and of bound-state relativistic theory. Also it gives the possibility to get supplementary information about the structure of the elementary particles. The reason of such special status of hydrogen-like atoms lies in the fact that relativistic and radiative corrections to the Coulomb energy levels are so small that may be calculated with high accuracy in the framework of the perturbation theory[1,2]. The well-known approach for the description of the bound state problem in quantum field theory based on the covariant Bethe-Salpeter equation[3] used in the investigation of the hydrogen[4] and positronium[5], has essential difficulties connected with the relative time variable (or relative energy).

Almost in all practical applications of the Bethe-Salpeter equation the relative time variable was by some means or other integrated out.

This operation was most consistently carried out in the Logunov-Tavkhelidze quasipotential approach[6]. The quasipotential approach in the field theory is a highly effective method for the description of both, bound states[7] and scattering problem[8].

In recent years there has been a constantly growing interest in the structure of the s. c. exotic atoms, like $(\pi\,\mu)$, $(\pi\,e)$, $(\pi\,\pi)$ atoms. Recently an experiment has been proposed in order to measure the Lamb-shift in the $(\pi\,\mu)$-atom[9] produced in the decay of $K_{L0} \to \pi\,\mu\,\nu$[10]. The energy levels of this atom have been calculated up to the two-photon exchange approximation including the hadronic vacuum polarization effects. Measurement of the Lamb-shift in $(\pi\,\mu)$-atom can provide an independent determination of the pion charge radius[9,11]. In the previous publication[12] it has been shown that in high energy accelerators it would be possible to create $(\pi^+\,\pi^-)$ atomic systems and to measure the life time, $|\psi(0)|^2$ and the Lamb-shift. As mentioned above it would be of great importance to calculate the energy levels in the $(\pi^+\,\pi^-)$ atom more accurately. In the present note we calculate the fine structure in the $(\pi^+\pi^-)$-atom in the framework of the Logunov-Tavkhelidze quasipotential approach.

2. *Calculation of bound states by the quasipotential method*

The quasipotential approach allows to combine simplicity and clearness of the three-dimensional description of the nonrelativistic quantum mechanics with the covariant apparatus of the quantum field theory.

Common scheme of the quasipotential approach is presented in detail in the reviews[7,8]. Therefore, below we shall rewrite only the necessary formulae. Further we shall consider the bound state of two spinless particles with mass m_a and m_b.

The quasipotential equation in the center-of-mass system is given by:

$$\left(M - \sqrt{\vec{p}^2 + m_a^2} - \sqrt{\vec{p}^2 + m_b^2}\right)\psi_M(\vec{p}) = \int \frac{d^3q}{(2\pi)^3} V(\vec{p}, \vec{q}, P)\,\psi_M(\vec{q}), \tag{1}$$

where M is mass of the bound state, $\vec{p}$ and $\vec{q}$ are the relative momenta of the initial and final states. The quasipotential $V(\vec{p}, \vec{q}, P)$ is expressed in terms of the scattering amplitude off mass-shell:

$$V = T(1 + \bar{G}_0^{(+)}\,T)^{-1} = T - T\,\bar{G}_0^{(+)}\,T + \ldots, \tag{2}$$

where

$$\bar{G}_0^{(+)}(\vec{p}, \vec{q}, P) = \frac{(2\pi)^3\,\delta^3(\vec{p} - \vec{q})}{M - \sqrt{\vec{p}^2 + m_a^2} - \sqrt{\vec{p}^2 + m_b^2}}. \tag{3}$$

$P = (M, \vec{O})$ is the momentum associated with the center of mass and the operator product denotes the integration over three-dimensional relative momenta.

The scattering amplitude

$$T = [\bar{G}_0^{(+)}]^{-1} \, \Delta\bar{G}^{(+)} [\bar{G}_0^{(+)}]^{-1},$$

$$\Delta\bar{G}_0^{(+)} = \bar{G}^{(+)} - G_0^{(+)}, \tag{4}$$

is defined by the Fourier transform of two-time Green's function in the momentum space:

$$\bar{G}(\vec{p}, \vec{q}, P) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dp^0 \, dq^0 \, G(p, q P),$$

$$p = (p^0, \vec{p}), \; q = (q^0, \vec{q}),$$

projected on the positive frequency states.

For two scalar particles system the projected Green's function is given by:

$$\bar{G}^{(+)}(\vec{p}, \vec{q}, P) = \frac{1}{\sqrt{2E_a(\vec{p})}\sqrt{2E_b(\vec{p})}} \bar{G}(p, q, P) \frac{1}{\sqrt{2E_a(\vec{q})}\sqrt{2E\sigma_b(\vec{q})}}, \tag{5}$$

$$E_{a,b}(\vec{p}) = \sqrt{\vec{p}^2 + m_{a,b}^2}.$$

Four-time Green's function satisfies Bethe-Salpeter equation

$$G - G_0 = G_0 K G = G K G_0 = G_0 K G_0 + \ldots. \tag{6}$$

In the framework of quantum electrodynamics the quasipotential may be constructed using the perturbation theory:

$$V = V_{1\gamma} + V_{2\gamma} + \ldots,$$

$$V_{1\gamma} = T_{1\gamma}, \; V_{2\gamma} = T_{\gamma 2} - V_{1\gamma} \bar{G}_0^{(+)} V_{1\gamma} + \ldots, \tag{7}$$

where $T_{1\gamma}$ and $T_{2\gamma}$ are one-photon exchange amplitude, including radiative corrections and two-photon exchange amplitude, respectively.

It is convenient to separate explicitly the Coulomb potential

$$V_{1\gamma} = V_c + \Delta V_{1\gamma}, \; V_c(\vec{K}^2) = -\frac{4\pi Z a}{\vec{K}^2}, \; a = \frac{e^2}{4\pi}, \; e_a = -e, \; e_b = Z e, \tag{8}$$

where Z is the charge of the particle b (*nuclei* of the atom). The equation (1) in the nonrelativistic limit reduces to the Schrödinger equation

$$\left(W - \frac{\vec{p}^2}{2\mu}\right)\psi(\vec{p}) = \frac{1}{(2\pi)^3}\int d^3q\, V(\vec{p},\vec{q};W)\,\psi(\vec{q}),$$

$$\mu = \frac{m_a \cdot m_b}{m_a + m_b},\quad W = M - m_a - m_b. \tag{9}$$

For the first approximation one takes the exact solution of the equation (9) with the Coulomb potential

$$W_{cn} = -\frac{(Z\alpha)^2}{2n^2}\mu,\quad n = 1, 2, 3, \ldots \tag{10}$$

$\psi_{cn}(\vec{p})$ is Coulomb wave function. The corrections to the Coulomb levels are obtained with the aid of ordinary perturbation theory:

$$W = W_{cn} + \Delta W$$

$$\Delta W \cong \langle n|\Delta V|n\rangle = \frac{1}{(2\pi)^6}\int d^3p\, d^3q\, \psi^*_{cn}(\vec{p})\,\Delta V(\vec{p},\vec{q};W)\,\psi_{cn}(\vec{q}). \tag{11}$$

In the following, we need to use the value of the quantity

$$|\psi_{cn}(\vec{r}=0)|^2 = \frac{(Z\alpha\mu)^3}{\pi n^3}\delta_{l0}, \tag{12}$$

where l is orbital quantum number.

3. *Fine structure levels of the* $(\pi^+\pi^-)$*-atom*

The general expression for the one-photon exchange quasipotential in the scattering approximation has the following form (in the center of mass $\vec{p}_a = -\vec{p}_b = \vec{p}$, $\vec{q}_a = -\vec{q}_b = \vec{q}$, $m_a = m_{\pi^-}$, $m_b = m_{\pi^+}$):

$$V_{1\gamma}(\vec{p},\vec{q}) = \frac{Ze^2\,\Gamma_{a\mu}(p_a,q_a)}{\sqrt{2E_a(\vec{p})}\sqrt{2E_a(\vec{q})}}\,D^{\mu\nu}(K)\,\frac{\Gamma_{b\nu}(p_b,q_b)}{\sqrt{2E_b(\vec{p})}\sqrt{2E_b(\vec{q})}}, \tag{13}$$

where

$$\Gamma_{a\mu}(p_a, q_a) = (p_a + q_a)_\mu \varrho_a(k^2),$$

$$\Gamma_{b\mu}(p_b, q_b) = (p_b + q_b)_\mu \varrho_b(k^2),$$

$$k = (0, \vec{p} - \vec{q}), \quad p^0_{a,b} = \varepsilon^0_{a,b}(\vec{p}), \quad q^0_{a,b} = \varepsilon_{a,b}(\vec{q}),$$

$$\varrho_{a,b}(k^2 = 0) = 1.$$

For the $(\pi^+\pi^-)$-atom $Z = 1$. But we shall keep Z as a variable in order to take into account the difference between radiative and relativistic corrections.

It is convenient to chose the photon propagator in the Coulomb gauge

$$D^{00}(k) = -\frac{d(k^2)}{\vec{k}^2}, \quad D^{0i}(k) = D^{i0}(k) = 0,$$

$$D^{ij}(k) = -\frac{d(k^2)}{\vec{k}^2}\left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2}\right), \quad i, j = 1, 2, 3,$$

$$d(k^2 = 0) = 1. \tag{14}$$

The quantity $d(k^2)$ describes the vacuum polarization effects and $\varrho_{a,b}(k^2)$ describes the structure of the particles. As the average values of the momenta in the atom $\langle\vec{p}^2\rangle \cong 2\mu\,|W_{cn}|$ are small, it is possible to expand the quasipotential $V_{1\gamma}$ (13) in the small parameter $\vec{p}^2/m^2$. Keeping the first order terms in this parameter we obtain:

$$V_{1\gamma} = V_c + \Delta V_{1\gamma}, \quad \Delta V_{1\gamma} = V_0 + V_S, \tag{15}$$

$$V_0 = -\frac{Z e^2}{\vec{k}^2}\varrho_a(k^2)\,\varrho_b(k^2)\,d(k^2)\left\{\frac{\vec{p}^2 + \vec{q}^2}{2 m_a m_b} - \frac{(\vec{p}^2 - \vec{q}^2)^2}{4 m_a m_b \vec{k}^2}\right\}, \tag{16}$$

$$V_S = -\frac{Z e^2}{\vec{k}^2}\left[\varrho_a(k^2)\,\varrho_b(k^2)\,d(k^2) - 1\right], \quad k^2 = -\vec{k}^2. \tag{17}$$

Besides, there is a correction to the kinetic energy associated with the expansion in $\vec{p}^2/m^2$ of the square root in the equation (1):

$$V_{kin} = -\frac{\vec{p}^4}{8\mu^3}\left(1 - \frac{3\mu}{m_a + m_b}\right)(2\pi)^3\,\varrho(\vec{p} - \vec{q}). \tag{18}$$

Now we shall discuss the physical meaning of the different terms in the expression (15). Sum of the potentials V_0 and V_{kln} gives the fine structure splitting of the Coulomb energy levels (10). The potential V_0, together with second term in the correction (18) give rise not only to the general shift, depending on the principal quantum number n, but also the S-shift characteristic for the scalar electrodynamics, mentioned for the first time by Todorov[13].

The potential V_S reflects the correction associated with the structure of the particles and with vacuum polarization and gives rise to Lamb-shift.

Let as now calculate the fine structure. At first as a potential in the equation (1) we will take $V_c + V_0$ without the terms which take into account the radiative corrections, vacuum polarization effects and the structure of the particles. This means that we will put:

$$d(k^2) = 1, \; \varrho_a(k^2) = \varrho_b(k^2) = 1.$$

The equation obtained in this approximation is identical with Breit equation[2,13] (in the center of mass system):

$$\left[W - \frac{\vec{p}^2}{2\mu} + \frac{\vec{p}^4}{8\mu^3}\left(1 - \frac{3\mu}{m_a + m_b}\right)\right]\psi(p) = \frac{1}{(2\pi)^3}\int d^3q\,(V_c + V_0)_{\vec{p},\vec{q}}\,\psi(\vec{q}). \quad (19)$$

Equation (19) gives the correct result only in first order perturbation.

Let us rewrite equation (19) in the configuration space, using explicit expressions for potentials.

$$\left\{W + \left(\frac{Z\alpha}{r} - \frac{\vec{p}^2}{2\mu}\right) + \frac{\vec{p}^4}{8\mu^3}\left(1 - \frac{3\mu}{m_a + m_b}\right) + \frac{Z\alpha}{r\,m_a m_b}\left[\vec{p}^2 + \right.\right.$$

$$\left.\left. + \frac{i\vec{r}\vec{p}}{r^2} - \frac{\vec{L}^2}{2r^2}\right]\right\}\psi(\vec{r}) = 0, \quad (20)$$

$$\vec{p} = -i\vec{\nabla}, \; \vec{L} = \vec{r} \times \vec{p}.$$

Using explicit expressions for average values of powers of r and of the operator p in Coulomb wave functions, we obtain the corrections to Coulomb levels, i. e. the fine structure

$$\Delta W_{fs} = \frac{(Z\alpha)^4\,\mu}{n^3}\left[\frac{3}{8n} - \frac{1}{2l+1}\right] + \frac{(Z\alpha)^4\,\mu^2}{(m_a+m_b)n^3}\left[\delta_{l0} - \frac{1}{8n}\right], \; \mu = \frac{m_a m_b}{m_a + m_b}. \quad (21)$$

Finally, we note that the limiting form of (21) when one mass (say m_b) approaches infinity agrees with the energy level formula for the Klein-Gordon particle in the Coulomb field.

$$\Delta W_{fs}(m_b \to \infty) = \frac{(Z\alpha)^4 m_a}{n^3}\left[\frac{3}{8n} - \frac{1}{2l+1}\right]. \quad (22)$$

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VEZANO STANJE DVIJU NABIJENIH ČESTICA BEZ SPINA U KVANTNOJ ELEKTRODINAMICI
I. FINA STRUKTURA

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Fina struktura energetskih nivoa ($\pi^+ \pi^-$) atoma izračunata je u okviru kvazipotencijalnog pristupa Logunova i Tavkhelidzea.