YU ISSN 0015—3206
FZKAAA 17 (4) 535 (1985)

LETTER TO THE EDITOR

BOUND STATE OF TWO SPINLESS CHARGED PARTICLES IN QUANTUM ELECTRODYNAMICS

II. LAMB SHIFT (ONE-PHOTON APPROXIMATION)

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Received 26 March 1985

UDC 530.145

Original scientific paper

In the framework Logunov-Tavkhelidze quasipotential approach the Lamb shift in the $(\pi^+ \pi^-)$ atom (one-photon approximation) is calculated.

In our paper[1] we found the expression for the fine structure splitting of $(\pi^+ \pi^-)$ atom in the framework of the quasipotential approach of Logunov-Tavkhelidze. In this paper the formula from the paper[1] will be preceeded by the number 1. In the present work we derive an analytic expression and find numerical values for Lamb shift of the order α $[f\ s]$ in $(\pi^+\pi^-)$ atom. It is well known[2,3], that in the evaluation of Lamb shift it is necessary to eliminate ultraviolet as well as infrared divergencies. In practice, this means that one has to utilize simultaneously two approaches: a nonrelativistic one, in the region of low frequencies and a relativistic one in the region of high frequencies. Consequently, one meets the problem[2] of »matching« the results obtained. At first this problem was resolved[4] in the frame of the covariant Bethe-Salpeter equation. One should note that all effects of »binding« appear only for small moment of virtual fotons. The parameter λ which restricts low frequency (LF) region should be chosen in such a way, that

$$(Z\alpha)^2\,\mu \ll \lambda \ll \mu. \qquad (1)$$

In practice it is convenient to take $\lambda \sim Z \alpha \mu$. Accordingly, we split the integral over momenta of the virtual photon into two parts:

$$\int d^3 k \, d k^0 = \left[\int\limits_{|\vec{k}| < \lambda} d^3 k + \int\limits_{|\vec{k}| < \lambda} d^3 k \right] \int\limits_{-\infty}^{\infty} d k^0. \tag{2}$$

Two-particle Green function G satisfies Bethe-Salpeter equation (1.6). The kernel K of the Bethe-Salpeter equation is given as an infinite sum of two-particle Feynman diagram. In the problem considered, the ordinary irreducible perturbation theory for the construction of the kernel K is not applicable, because it is necessary to take into account the effects of »binding« (interaction) in the intermediate states of two particles system. This means that it is necessary to perform a selective summation of infinite sums of diagrams entering the kernel K. The simplest way to realize it is with the aid of the suitable equation.

It was shown in Ref. 5 that the kernel K satisfies the equation of Dyson-Schwinger type, which in the approximation necessary for this problem has the following form[5] (Fig. 1):

$$G = G_0 + G_0 K G,$$

$$\Delta G = G - G_0 \cong G_0 \left[K_\gamma + K_\Delta \right] G^0. \tag{3}$$

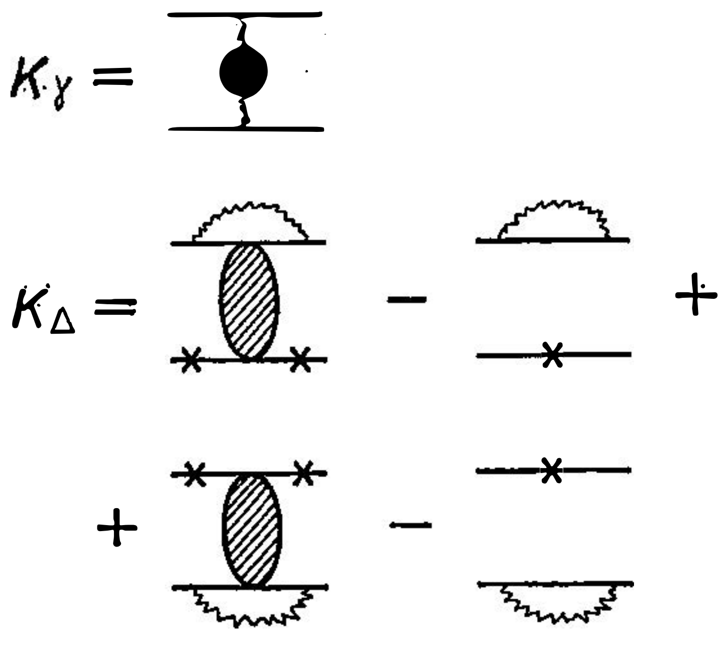

Fig. 1. —×— — $S^{-1}_{a,b} K_\gamma$ - one photon exchange kernel with pointlike vertexes, the dashed circle denotes Green function of two-particles in ladder approximation.

In accordance with the representation (3) and on the base of the definitions (1. 2—7) the corresponding quasi-potential is given by

$$V = T = V_\gamma + V_\Delta = V_c + \Delta V_{1\gamma}, \tag{4}$$

where

$$V_{\gamma,\Delta} = [\bar{G}_0^{(+)}]^{-1} [\overline{G_0 K_{\gamma,\Delta} G_0}]^{(+)} [\bar{G}_0^{(+)}]^{-1}. \tag{5}$$

Since Coulomb interaction plays the most important role in »binding« effects, in (LF) region it is possible to use modified Coulomb Green function of the nonrelativistic Schrödinger equation. The positive frequency component of the four--time Green function is sufficient to get the necessary precision[3]:

$$G_c^{(+)} - G_0^+ = G_0^{(+)} [\bar{G}_0^{(+)}]^{-1} [G_c^{NR} - \bar{G}_0^{(+)}] [\bar{G}_0^{(+)}]^{-1} G_0^{(+)}, \tag{6}$$

where

$$G_0^{(+)}(p, q) = \mathrm{i}\,(2\pi)^4\, \delta^4(p - q)\, S_a^{(+)}(p_a)\, S_b^{(+)}(p_b),$$

$$S_{a,b}^{(+)} = \left(p_0 - \sqrt{\vec{p}^{\,2} + m_{a,b}^2}\right)^{-1} = (p_0 - \varepsilon_{a,b}(\vec{p}))^{-1},$$

$$\bar{G}_c^{(+)}(p, q; P) \cong G_c^{NR}(p, q; P) = \sum_n \frac{\psi_{cn}(\vec{p})\, \psi_{cn}^*(\vec{q})}{W - W_n + \mathrm{i}\,0}.$$

Here the summation is over discrete as well as over continuous energy spectra.

By substituting Green's function (6) in the expression corresponding to the diagrams (Fig. 1) and by integrating over k^0, we obtain quasipotential (4) $\Delta V_{1\gamma}$ in (LF) region[3] (in the center of mass frame $\vec{P} = 0$, $\vec{p}_a = \vec{p}$, $\vec{p}_b = -\vec{p}$):

$$\Delta V_{1\gamma}^{<}(\vec{p}, \vec{q}; W) = \frac{e_a e_b}{(2\pi)^3} \int\limits_{|\vec{k}| < \lambda} \frac{\mathrm{d}^3 k}{2k} \left\{ \frac{(2p - k)_\mu}{\sqrt{2 E_a(\vec{p})} \sqrt{2 E_a(\vec{p} - \vec{k})}} \otimes \right.$$

$$\otimes \left[\sum_n \frac{\psi_{cn}(\vec{p} - \eta_b \vec{k})\, \psi_{cn}^*(\vec{q} - \eta_b \vec{k})}{W - W_n - k + \mathrm{i}\,0} - \frac{(2\pi)^3\, \delta^3(\vec{p} - \vec{q})}{E_a(\vec{p}) - E_a(\vec{p} - \vec{k}) - k + \mathrm{i}0} \right] \otimes$$

$$\otimes \frac{(2q - k)^\mu}{\sqrt{E_a(\vec{q})} \sqrt{E_a(\vec{q} - \vec{k})}} +$$

$$+ \frac{Z^2 (2p - k)_\lambda}{\sqrt{2 E_b(\vec{p})} \sqrt{2 E_b(\vec{p} - \vec{k})}} \left[\sum_n \frac{\psi_{cn}(\vec{p} + \eta_a \vec{k})\, \psi_{cn}^*(\vec{q} + \eta_a \vec{k})}{W - W_n - k + \mathrm{i}\,0} - \right.$$

$$- \frac{(2\pi)^3 q^3 (\vec{p} - \vec{q})}{E_b(\vec{p}) - E_b(\vec{p} - \vec{k}) - k + \mathrm{i}\,0}\Bigg] \frac{(2q - k)^\lambda}{\sqrt{2 E_b(\vec{q})}\sqrt{2 E_b(\vec{q} - \vec{k})}}\Bigg\},$$

$$k = |\vec{k}|,\ \eta_{a,b} = \frac{m_{a,b}}{m_a + m_b},\ e_a = -e,\ e_b = Z e,\ \alpha = \frac{e^2}{4\pi}. \qquad (7)$$

According to the general formalism, the contribution of this point of the potential to the energy level shift (1.11) is given by

$$\Delta W_n^< = \frac{1}{(2\pi)^6} \int \mathrm{d}^3p\, \mathrm{d}^3q\, \psi_{cn}^*(\vec{p})\, \Delta V_{1\gamma}^<(\vec{p}, \vec{q}; W)\, \psi_{cn}(\vec{q}). \qquad (8)$$

The corresponding real part of the correction to the energy levels is expressible with the aid of well known Bethe logarithm[2,3]:

$$\mathrm{Re}\, W_n^< = \frac{4\alpha (Z\alpha)^4 \mu^3}{3\pi n^3} \left[\frac{1}{m_a^2} + \frac{Z^2}{m_b^2}\right] \left[\ln \frac{(Z\alpha)^2 \mu}{2 W_{nl}^{av}} + \delta_{l0} \ln \frac{2\lambda}{(Z\alpha)^2 \mu}\right], \qquad (9)$$

where W_{nl}^{av} is the average value of the excitation energy which takes into account the reduced mass.

In the evaluation of the energy spectra in high frequency (HF) region it is possible, as it was indicated above, to neglect the »binding« effects and to use quasipotential (1.17) obtained in the scattering approximation:

$$\Delta V_{1\gamma}^> = \frac{Z e_a e_b}{\vec{k}^2} [\varrho_a(k^2)\, \varrho_b(k^2)\, d(k^2) - 1],\ k^2 = -\vec{k}^2. \qquad (10)$$

The form of invariant functions are known[6]:

$$\varrho_{a,b}(k^2) \cong 1 + \frac{\alpha}{3\pi} \frac{k^2}{m_{a,b}^2} \left(\ln \frac{m_{a,b}}{2\lambda} + \frac{1}{12}\right) + \frac{k^2}{6} \langle r_\pi^2 \rangle, \qquad (11)$$

$$d(k^2) \cong 1 - \Pi(e^+ e^-),\ \Pi(k^2) = k^2 \int_{4m_e^2}^{\infty} \frac{\mathrm{d}m^2\, \varrho(m^2)}{m^2 (m^2 - k^2 - \mathrm{i}\,0)} \qquad (12)$$

where λ is a parameter of the infrared cut off.

In passing to the evaluation of the contribution of potential, we should note, that the principal part of the energy shift in $(\pi^+ \pi^-)$ atom comes from the electron--positron vacuum polarization effects.

In this case one should not use nonrelativistic expression for polarization operator $\Pi(e^+ e^-)$ (12)

$$\Pi^{NR}(e^+ e^-) = \frac{\alpha}{15\pi}\frac{k^2}{m_e^2}. \tag{13}$$

Partly because the characteristic momenta of pions $|\vec{p}_\pi| \sim \alpha m_\pi$ in $(\pi^+ \pi^-)$ atom are nonrelativistic, but with respect to electrons they are relativistic.

Consequently, quasipotential $\Delta V^>_{1\gamma}$ in the approximation required splits into two parts:

$$\Delta V^>_{1\gamma} = \Delta V^>_{rez} + \Delta V^>(e^+ e^-),$$

where

$$\Delta V^>_{rez} = \frac{4(Z\alpha)\alpha}{3}\left\{\frac{1}{m_a^2}\left(\ln\frac{m_a}{2\lambda} + \frac{1}{12}\right) + \right.$$

$$\left. + \frac{1}{m_b^2}\left(\ln\frac{m_b}{2\lambda} + \frac{1}{12}\right)\right\} + \frac{4\pi(Z\alpha)}{6}\langle r_\pi^2\rangle, \tag{14}$$

$$\Delta V^>(e^+ e^-) = -\frac{4\pi \cdot Z\alpha}{\vec{k}^2}\Pi^>(e^+ e^-). \tag{15}$$

Corresponding energy shift is determined by equation (8) and has the form

$$\Delta W^>_h = \frac{4\alpha(Z\alpha)^4\mu^3}{3\pi n^3}\delta_{l0}\left\{\frac{1}{m_a^2}\left(\ln\frac{m_a}{2\lambda} + \frac{1}{12}\right) + \right.$$

$$\left. + \frac{1}{m_b^2}\left(\ln\frac{m_b}{2\lambda} + \frac{1}{12}\right)\right\} + \frac{4(Z\alpha)^4\mu^3}{3n^3}\langle r_\pi^2\rangle\delta_{l0}. \tag{16}$$

By summing the results from (HF) region (16) and from (LF) region (9) we obtain an expression, independent of λ, $Z = 1$:

$$\Delta W(2S) = \frac{\alpha^5\mu^3}{6\pi}\left\{\frac{1}{m_a^2}\left(\ln\frac{m_a}{2W_{20}^{av}} + \frac{1}{12}\right) + \frac{1}{m_b^2}\left(\ln\frac{m_b}{2W_{20}^{av}} + \frac{1}{12}\right)\right\}, \tag{17}$$

$$\Delta W(2p) = \frac{\alpha^5\mu^3}{6\pi}\left[\frac{1}{m_a^2}\ln\frac{R_y}{W_{2,1}^{av}} + \frac{1}{m_b^2}\ln\frac{R_y}{W_{2,1}^{av}}\right], \tag{18}$$

where

$$R_y = \frac{\alpha^2\mu}{2}.$$

By using the exact expression[6,5] for $\Pi^{>}(e^{+}\,{}^{-}e)$ in (15) we obtain[8]

$$[\Delta W(2S) - \Delta W(2p)]_{e^+e^-} = -Ry \int_0^1 \frac{c^2}{(\sqrt{x}+c)^4}\,\varrho\left(\frac{c^2}{x}\right) \mathrm{d}x, \tag{19}$$

where

$$c = \frac{2\,m_e}{\alpha\,\mu}, \quad \mu = \frac{m_a\,m_b}{m_a + m_b}.$$

Spectral function $\varrho\left(\frac{c^2}{x}\right)$ in second order approximation equals:

$$\varrho\left(\frac{c^2}{x}\right) = \frac{\alpha}{3\pi}\left(1 + \frac{1}{2}x\right)\sqrt{1-x}. \tag{20}$$

Apart from this there is a correction due to the expression (1.21)

$$[\Delta W(2S) - \Delta W(2p)]_{[fs]} = -\frac{\alpha^4\,\mu}{12}\left(1 - \frac{3\mu}{2\,(m_a + m_b)}\right). \tag{21}$$

Summing the expressions (17), (18), (19), (21) we obtain, for Lamb shift of order α $[fs]$ in one-foton approximation, the following expression:

$$\begin{aligned} \Delta W(2S) - \Delta W(2p) = &-\frac{\alpha^4\,\mu}{12}\left[1 - \frac{3\mu}{2\,(m_a+m_b)}\right] + \\ &+ \frac{\alpha^5\,\mu^3}{72\pi}\left[\frac{1}{m_a^2} + \frac{1}{m_b^2}\right] + \frac{\alpha^5\,\mu^3}{6\pi}\left[\frac{1}{m_a^2}\ln\frac{m_a}{2\,W_{20}^{av}} + \frac{1}{m_b^2}\ln\frac{m_b}{2\,W_{20}^{av}} - \right. \\ &\left. - \left(\frac{1}{m_a^2} + \frac{1}{m_b^2}\right)\ln\frac{R_y}{W_{21}^{av}}\right] - \frac{\alpha\,R_y}{3\pi}\int_0^1 \frac{c^2\sqrt{1-x}\left(1+\frac{x}{2}\right)\mathrm{d}x}{(\sqrt{x}+c)^4}, \end{aligned}$$

where

$$m_a = m_{\pi^-}, \quad m_b = m_{\pi^+}, \quad c = \frac{2\,m_e}{\alpha\,\mu}, \quad R_y = \frac{\alpha^2\,\mu}{2}. \tag{22}$$

The contributions from strong interactions coming from $\langle r_\pi^2\rangle$ in (16) and from hadronic vacuum polarization are not included in the above expression.

We will come back to this problem another time. Contributions from strong interactions are determined by $\pi-\pi$ scattering data. Therefore, the measurement of Lamb shift in the $(\pi^+\,\pi^-)$ atom would give more precise values of $\pi\pi$-scattering length.

Numerical value of Lamb shift given by the expression (22) for

$$m_a = m_b = m_\pi$$

equals:

$$\Delta W(2\varrho) - \Delta W(2p) \simeq -112.4 \cdot 10^{-3}\,\text{eV}.$$

Acknowledgment

We are greatful to Dr B. Dragović, Dr M. M. Musakhanov and Dr R. N. Faustov for their interest, help and discussions. One of us (K. A) is thankful to the Institute of Physics for the hospitality and support.

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VEZANO STANJE DVIJU NABIJENIH ČESTICA BEZ SPINA U KVANTNOJ ELEKTRODINAMICI

II. LAMBOV POMAK

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UDK 530.145

Originalni znanstveni rad

U okviru Logunov-Tavkhelidzeovog kvazipotencijalnog pristupa izračunat je Lambov pomak u $(\pi^+ \pi^-)$ atomu u aproksimaciji jednog fotona.