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ROBUST QUADRATIC STABILITY OF TWO INTERCONNECTED SYSTEMS WITH RESPECT TO THE STRUCTURE OF A LYAPUNOV FUNCTION

Summary

In this paper we consider a system formed by the interconnection of two linear time-invariant systems as the simplest example of networked dynamical systems. The interconnection between a plant and a controller, which is present in virtually any controlled system, is also an example of such a network. We show that if such a network exhibits certain robust stability properties, then there necessarily exists a specifically structured Lyapunov function that certifies the stability of the network. More precisely, the robustness property we consider is stability robustness (quadratic stability) with respect to an uncertainty in the interconnection channels between the systems. This interconnection uncertainty also accounts for the case when the two systems are disconnected. The considered class of Lyapunov functions is characterized by an additive structure in the sense that the Lyapunov function for the network is composed as a sum of two quadratic terms, where each term is a function of a state vector of one system only. Previously, it was shown in the literature that the existence of such structured Lyapunov functions implies the robustness of the network. The main contribution of this paper is a proof of the converse statement: if the network composed of two systems is robustly stable (in the sense of quadratic stability), then it necessarily admits a Lyapunov function with an additive structure.

Key words: *control systems, stability, robustness, Lyapunov theory, dynamical networks, linear matrix inequalities*

1. Introduction

The complexity of man-made systems is ever-increasing. One of the reasons for this is large-scale networking of systems, e.g., the networking of a large amount of heterogeneous renewable energy sources in smart grids, the networking of cars in automated highways (or, in general, in traffic networks), and the Internet of Things, to name but a few. There exists a widely recognized need to better understand and manage complex large-scale dynamical networks, with robustness and efficiency, as well as the trade-offs between the two, playing a central role [1]. Verifying the stability of a large-scale dynamical network in a centralized way is often computationally not scalable. Furthermore, this task is often simply impossible due to the non-existence of the overall network model in one place, e.g., due to confidentiality issues. For example, in a market-based competitive environment of smart grids, energy producers might not be willing to share confidential data about their facilities. An alternative to the centralized approach is a distributed network analysis/synthesis approach, where the overall network properties are verified without central knowledge about the network. One promising approach in this direction is based on the dissipativity theory of dynamical systems; see e.g., [2, 3, 4] and the references therein. In our previous work [5, 6], we have shown a close interconnection between the dissipativity of systems in the network and the existence of structured (with an additive structure) Lyapunov functions that provide certificates of the network stability. This paper continues in this line of research.

It is known that if the network of interconnected systems admits an additive Lyapunov function, then its stability is robust with respect to the disconnection of the systems from the network [5]. In this paper, we consider a simple network of only two interconnected linear time-invariant systems and show that the converse statement is also true. More precisely, we show that if the network is robustly stable with respect to an uncertainty in the interconnection channels between the two systems (which also account for the disconnection of the systems), then the nominal system (i.e., the network without uncertainties) *necessarily* admits a Lyapunov function with an additive structure.

2. Notation and preliminaries

We use **R** to denote the field of real numbers, while $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$ denote, respectively, column vectors with n elements and n by m matrices with elements in **R**. In this paper, I denotes the identity matrix, with a dimension that will always be clear for the context. The transpose of a matrix A is denoted by A^T. For a square matrix M, we use $\mathrm{tr}(M)$ to denote its trace, i.e., the sum of the elements on the main diagonal. The matrix inequalities $A \succ B$ ($A \prec B$) and $A \succeq B$ ($A \preceq B$) mean that A and B are symmetric matrices; $A - B$ is positive definite (negative definite) and positive semidefinite (negative semidefinite), respectively. For matrices A, B we will use $\mathrm{diag}(A, B)$ to denote the matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. All the matrices we use will be real matrices, and for square matrices, we will use the usual inner product defined by $\langle A, B\rangle = \mathrm{tr}(A^T B)$. It can be shown that $\langle A, B\rangle = \langle A^T, B^T\rangle = \langle B, A\rangle = \langle B^T, A^T\rangle$ and $\langle A^T, BC\rangle = \langle B^T, CA\rangle = \langle C^T, AB\rangle$.

In the remainder of the paper, we will make use of the following lemma.

Lemma 1. Let $M = \begin{bmatrix} M_1 & M_{12} \\ M_{12}^T & M_2 \end{bmatrix}$ be a square symmetric matrix. If $M \prec 0$, then $\widetilde{M} := \begin{bmatrix} M_1 & -M_{12} \\ -M_{12}^T & M_2 \end{bmatrix} \prec 0$.

Proof. Using the Schur complement rule, $M \prec 0$ is equivalent to $M_2 \prec 0$, $M_1 - M_{12}M_2^{-1}M_{12}^T \prec 0$, see e.g., [7]. The latter two inequalities are further equivalent to $M_2 \prec 0$, $M_1 - (-M_{12})M_2^{-1}(-M_{12}^T) \prec 0$, which implies $\widetilde{M} \prec 0$.

2.1 Stability

Consider an autonomous linear time invariant system

$$\dot{x} = Ax \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector. The system (1) is said to be *uniformly exponentially stable* if there exist real constants $a > 0$ and K such that for every system trajectory $x(t)$ we have

$$\|x(t)\| \leq K\, e^{-a(t-t_0)}\|x(t_0)\|$$

for all $t \geq t_0 \geq 0$.

It is well known, see e.g., [7], that the system (1) is uniformly exponentially stable if and only if it admits a quadratic Lyapunov function of the form $V(x) = x^T P x$, where P is a positive definite symmetric matrix of appropriate dimensions. Furthermore, this is equivalent to the existence of P which satisfies the following matrix inequalities:

$$P \succ 0, \quad A^T P + PA \prec 0. \tag{2}$$

2.2 Robust quadratic stability

Consider a time varying system

$$\dot{x} = A(t)x \tag{3}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, and the time varying matrix $A(t)$ is at any time t a convex combination of two known matrices, A_+ and A_-, that is

$$A(t) = \lambda(t)A_+ + \big(1 - \lambda(t)\big)A_-, \qquad \lambda(t) \in [0,1] \text{ for all } t. \tag{4}$$

Definition 1. We say that the system (3) is quadratically stable if it admits a common quadratic Lyapunov function $V(x) = x^T P x$ for all possible system trajectories.

It is well known, see e.g., [7, 8], that the quadratic stability is equivalent to the feasibility of the following linear matrix inequalities:

$$P \succ 0, \quad A_+^T P + PA_+ \prec 0, \quad A_-^T P + PA_- \prec 0. \tag{5}$$

The notion of quadratic stability is important because if the system (3),(4) is shown to be quadratically stable, then it is also necessarily *uniformly exponentially stable* for all possible trajectories of the parameter $\lambda(t)$ such that $\lambda(t) \in [0,1]$.

2.3 Theorem of alternatives

In the proof of our main result, we will make use of a theorem of alternatives for linear matrix inequalities (LMIs) on several occasions. The LMI alternatives can be directly derived using the Lagrange duality [9]. Here, we briefly recall the main notions, present some basic results, and provide the interested reader with references for more details.

Consider a linear matrix inequality in the following rather general form:

$$G_0 + G(x) \prec 0, \tag{6}$$

where G_0 is a fixed Hermitian matrix and $G(\cdot)$ is a linear Hermitian-valued map from the decision variables x. Note that x is, in general, a vector valued variable, but in practice, it is also often a matrix, as for example in (2) where P is a matrix valued variable. We are interested in deriving a test that systematically offers us a certificate of infeasibility if the LMI (6) is infeasible. Here, by the term infeasibility, we mean that there does not exist an x that satisfies (6). Such infeasibility tests are also often called alternatives [9, 10]. For the completeness of the presentation, here we only briefly recall a path for deriving the LMI alternatives, while the interested reader is referred to the excellent tutorial paper [11] and to [10] for more details.

The inequality (6) is infeasible if and only if the optimal value of the following optimization problem (with $t \in \mathbf{R}$ and x as decision variables)

$$\text{infimize} \quad t \quad \text{subject to} \quad G_0 + G(x) \preceq tI \tag{7}$$

is larger than or equal to zero. Note that the optimization problem (7) is a convex optimization problem (since $G(\cdot)$ is linear) and is always strictly feasible as we can always take t to be sufficiently large. Therefore, Slater's constraint qualification is satisfied, and the strong

Lagrange duality holds [9, 11]. By strong duality, the infimum in (7) is larger than or equal to zero if and only if there exists a (Hermitian) Lagrange multiplier $Y \succeq 0$ such that the following holds:

$$0 \leq \inf_{t,x} \; t + \langle Y, G_0 + G(x) - tI \rangle = \inf_{t,x} t(1 - \langle Y, I \rangle) + \langle Y, G_0 \rangle + \langle G^*(Y), x \rangle. \quad (8)$$

The right-hand side from the inequality sign in (8) is the Lagrange dual function for (7), and it presents a lower bound for the optimal value of (7). By strong duality, the largest lower bound (attained for some $Y \succeq 0$) is equal to the optimal value of (7).

Further, the condition in (8) is equivalent to the feasibility of the following set of inequalities/equalities:

$$Y \succeq 0, \quad (9a)$$

$$\langle Y, I \rangle - 1 = 0, \quad (9b)$$

$$G^*(Y) = 0, \quad (9c)$$

$$\langle Y, G_0 \rangle \succeq 0. \quad (9d)$$

The mapping $G^*(\cdot)$ is the adjoint to $G(\cdot)$, which is defined by the requirement $\langle Y, G(x) \rangle = \langle G^*(Y), x \rangle$ for all admissible Y and x. Indeed, if (9b) and (9c) do not hold, one can always find such (t, x) that make the considered infimum arbitrarily small. More precisely, if (9b) does not hold, we can select t so that the term $t(1 - \langle Y, I \rangle)$ in (8) is arbitrarily small. An analogous statement holds for (9c), in which case the term $\langle G^*(Y), x \rangle$ in (8) can be made arbitrarily small by an appropriate selection of x. However, with (9b) and (9c) satisfied, the infimum in (8) is larger than or equal to zero if and only if (9d) holds.

In the remainder of the paper, when necessary, the adjoints $G^*(\cdot)$ will be directly computed for specific given forms of $G(\cdot)$. For this purpose, the rules for the standard inner product, presented at the beginning of this section, will be sufficient.

Finally, note that the condition (9b) can be replaced by the condition $Y \neq 0$, as typically done in [10], since Y enters homogenously in (9a), (9c), and (9d). Therefore, if for some $Y \neq 0$ (9a), the equality and the inequality in (9c) and (9d), respectively, are satisfied, we can always scale this Y with some real scalar so that (9b) holds as well. In this paper we will, however, prefer the use of the corresponding condition in the form (9b).

The above is, in fact, a proof of the following theorem of alternatives that states that exactly one of the following statements is true:

1) There exists x such that (6) holds.
2) There exists Y such that (9) holds.

2.4 Certificate of instability

In the remainder of the paper, we will make use of the following lemma, the proof of which follows the path from subsection 2.3.

***Lemma* 2.** The matrix inequalities (5) are infeasible, i.e., the system (3),(4) is not quadratically stable if and only if there exist symmetric matrices $Z_0 \succeq 0$, $Z_1 \succeq 0$, $Z_2 \succeq 0$ such that

$$Z_1 A_+^T + A_+ Z_1 + Z_2 A_-^T + A_- Z_2 = Z_0, \quad (10a)$$

$$\langle Z_0 + Z_1 + Z_2, I \rangle = 1. \quad (10b)$$

Proof. The LMIs (5) are infeasible if and only if the optimal value of the optimization problem

$$\text{infimize } t \quad \text{subject to } \; -P \prec tI, \quad A_+^T P + PA_+ \prec tI, \quad A_-^T P + PA_- \prec tI,$$

is larger than or equal to zero. In turn, this is so if and only if there exist $Z_0 \succeq 0, Z_1 \succeq 0, Z_2 \succeq 0$ such that

$$0 \le \inf_{t,P} t + \langle Z_0, -P - tI\rangle + \langle Z_1, A_+^T P + PA_+ - tI\rangle + \langle Z_2, A_-^T P + PA_- - tI\rangle$$

$$= \inf_{t,P} \; t\,(1 - \langle Z_0 + Z_1 + Z_2, I\rangle) + \langle Z_1 A_+^T + A_+ Z_1 + Z_2 A_-^T + A_- Z_2 - Z_0, P\rangle.$$

The infimum in the last expression is equal to 0 if and only if (10) holds. Indeed, if (10) does not hold for some matrices $Z_0 \succeq 0, Z_1 \succeq 0, Z_2 \succeq 0$, the infimum is $-\infty$.

3. The main result

Consider two interconnected systems shown in Fig. 1, where G_1 and G_2 are linear time-invariant systems given by state-space realizations:

$$G_1: \begin{cases} \dot{x}_1 = A_1 x_1 + B_1 w \\ z = C_1 x_1 \end{cases}, \qquad G_2: \begin{cases} \dot{x}_2 = A_2 x_2 + B_2 z \\ w = C_2 x_2 \end{cases}, \tag{11}$$

where $x_1(t) \in \mathbf{R}^{n_1}$, $x_2(t) \in \mathbf{R}^{n_2}$ are the state vectors, while $z(t) = \mathbf{R}^{n_z}$ and $w(t) = \mathbf{R}^{n_w}$ are the interconnection signals (see Fig. 1).

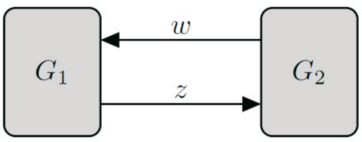

Fig. 1 Nominal system

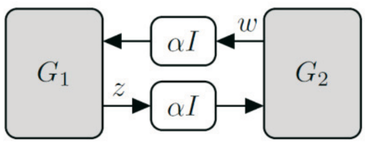

Fig. 2 Uncertain system

In Fig. 2, the interconnected systems are modified in the sense that interconnection signals are affected by an uncertain parameter $\alpha \in \mathbf{R}$. The overall interconnected system G_α, which is shown in Fig. 2, is given by

$$G_\alpha: \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & \alpha B_1 C_2 \\ \alpha B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{12}$$

For convenience, we introduce the abbreviations $A_{12} \coloneqq B_1 C_2$, $A_{21} \coloneqq B_2 C_1$, $A(\alpha) \coloneqq \begin{bmatrix} A_1 & \alpha A_{12} \\ \alpha A_{21} & A_2 \end{bmatrix}$ and $x \coloneqq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, so the dynamics of the interconnected system G_α can be written in a more compact form:

$$G_\alpha: \quad \dot{x} = A(\alpha)x. \tag{13}$$

Finally, we introduce the following definitions: $A_+ \coloneqq A(1)$, $A_- \coloneqq A(-1)$. Note that the dynamics of the interconnected system shown in Fig. 1 is defined by the matrix A_+. For future reference, we will use the symbol G to denote the system from Fig. 1. We have

$$G: \quad \dot{x} = A_+ x. \tag{14}$$

We can think of the system G as a nominal system, while G_α represents a set of uncertain systems in which the parameter α defines deviations from the nominal system.

Next, we present the main result of the paper.

Theorem 1. The system G_α, given by (13), is quadratically stable for $\alpha \in [-1,1]$ *if and only if* the system G, given by (14), admits a structured quadratic Lyapunov function of the form

$$V(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^T P_1 x_1 + x_2^T P_2 x_2. \tag{15}$$

Note that the function $V(x)$ has an additive structure in the sense that it is a sum of two quadratic terms where each term is a function of the states of one system (G_1 or G_2 from (11)) only.

Proof. The proof follows in two steps. In step 1, we show that the existence of a structured Lyapunov function (15) for the system G implies the quadratic stability of G_α for $\alpha \in [-1,1]$. In the second step, we prove that the nonexistence of a structured Lyapunov function (15) for the system G implies that the system G_α is not quadratically stable for $\alpha \in [-1,1]$.

Step 1. Suppose the system G is stable and that it admits a Lyapunov function of the form (15). With the abbreviation $P_D \coloneqq \mathrm{diag}(P_1, P_2)$, this means that the following matrix inequalities hold:

$$P_D \succ 0, \tag{16a}$$

$$A_+^T P_D + P_D A_+ \prec 0. \tag{16b}$$

Using the definitions of A_+ and P_D, the inequality (16b) reads as

$$\begin{bmatrix} A_1^T P_1 + P_1 A_1 & A_{21}^T P_2 + P_1 A_{12} \\ A_{12}^T P_1 + P_2 A_{21} & A_2^T P_2 + P_2 A_2 \end{bmatrix} \prec 0. \tag{17}$$

From Lemma 1, it follows that the inequality (17) also holds when A_{12} and A_{21} are replaced by $-A_{12}$ and $-A_{21}$, respectively. The inequality obtained by such replacement is, in fact, the inequality which states that

$$A_-^T P_D + P_D A_- \prec 0. \tag{18}$$

Based on the results presented in section 2.2., the inequalities (16) and (18) together imply that the system G_α is quadratically stable for $\alpha \in [-1,1]$. The structured Lyapunov function (15) acts as a common Lyapunov function for all $\alpha \in [-1,1]$.

Step 2. We first derive the certificates of the infeasibly of (16). Indeed, the alternative to (16) then presents a statement that the nominal system G does not admit a Lyapunov function of the form (15).

It will be instrumental to parametrize the block diagonal matrices of the form $P_D = \mathrm{diag}(P_1, P_2)$ in the following way. Let $P = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}$ be an unstructured symmetric matrix with the same P_1 and P_2 as in P_D. Let $E_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ be square matrices of the same size as P, where the sizes of the identity matrices in E_1 and E_2 are the same as those of P_1 and P_2, respectively. Then, we have the following relation

$$P_D = P - (E_1 P E_2 + E_2 P E_1). \tag{19}$$

For convenience, we will introduce the following definition: $\mathcal{S}_D(P) := P - (E_1PE_2 + E_2PE_1)$. The set of block diagonal symmetric matrices P_D is now parametrized by the of unstructured symmetric matrices P by the relation $P_D = \mathcal{S}_D(P)$. The inequalities (16) can now be formulated in terms of an unstructured P as follows

$$\mathcal{S}_D(P) \succ 0, \quad A_+^T\mathcal{S}_D(P) + \mathcal{S}_D(P)A_+ \prec 0. \tag{20}$$

The inequalities (20) are infeasible if and only if the optimal value of the optimization problem

$$\text{infimize} \quad t \ \text{ subject to } \ -\mathcal{S}_D(P) \prec tI, \ A_+^T\mathcal{S}_D(P) + \mathcal{S}_D(P)A_+ \prec tI \tag{21}$$

is larger than or equal to zero. In turn, this is so if and only if there exist $X \succeq 0$, $Y \succeq 0$ such that

$$0 \le \inf_{P,t} \ t + \langle X, -\mathcal{S}_D(P) - tI\rangle + \langle Y, A_+^T\mathcal{S}_D(P) + \mathcal{S}_D(P)A_+ - tI\rangle =: t^\star. \tag{22}$$

We have

$$\begin{aligned} t^\star &= \inf_{P,t} t + \langle X, -P + (E_1PE_2 + E_2PE_1) - tI\rangle + \\ &+\langle Y,\ A_+^T\big(P - (E_1PE_2 + E_2PE_1)\big) + \big(P - (E_1PE_2 + E_2PE_1)\big)A_+ - tI\rangle \\ &= \inf_{P,t} \ t\,(1 - \langle X + Y, I\rangle) + \\ &+\langle A_+Y + YA_+^T - (E_1(A_+Y + YA_+^T)E_2 + E_2(A_+Y + YA_+^T)E_1) - X + (E_1XE_2 + \\ &+E_2XE_1), P\rangle. \end{aligned}$$

The above expression can be written in a more compact form as

$$t^\star = \inf_{P,t} \ t\,(1 - \langle X + Y, I\rangle) + \langle \mathcal{S}_D(A_+Y + YA_+^T) - \mathcal{S}_D(X), P\rangle. \tag{23}$$

We can conclude that $t^\star \ge 0$ if and only if there exist matrices X and Y such that the following inequalities are satisfied

$$X \succeq 0, \ Y \succeq 0, \tag{24a}$$

$$\langle X + Y, I\rangle = 1, \tag{24b}$$

$$\mathcal{S}_D(A_+Y + YA_+^T) = \mathcal{S}_D(X). \tag{24c}$$

Indeed, if the above inequalities are satisfied, $t^* = 0$, while otherwise $t^* = -\infty$. The inequalities (24), if feasible, provide a certificate of infeasibly of (16).

Next, we consider the certificates of the statement that the system G_α is *not* quadratically stable for $\alpha \in [-1,1]$. From Lemma 2, it directly follows that this statement is true if and only if there exist symmetric matrices $Z_0 \succeq 0$, $Z_1 \succeq 0$, $Z_2 \succeq 0$ such that

$$Z_1A_+^T + A_+Z_1 + Z_2A_-^T + A_-Z_2 = Z_0, \tag{25a}$$

$$\langle Z_0 + Z_1 + Z_2, I\rangle = 1. \tag{25b}$$

For the remainder, it will be convenient to represent the matrix A_+ as follows

$$A_+ = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}}_{A_D} + \underbrace{\begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}}_{\Delta}. \tag{26}$$

Then we have $A_{-} = A_D - \Delta$, and the inequality (25a) reads as

$$Z_1(A_D+\Delta)^T + (A_D+\Delta)Z_1 + Z_2(A_D-\Delta)^T + (A_D-\Delta)Z_2 = Z_0, \tag{27}$$

which, after some restructuring and with the substitutions $Q \coloneqq Z_1 + Z_2$, $W \coloneqq Z_1 - Z_2$, becomes

$$QA_D^T + A_DQ + W\Delta^T + \Delta W = Z_0. \tag{28}$$

Note that with $Z_1 \succeq 0, Z_2 \succeq 0$, we have $Q \succeq 0$.

Finally, we can now summarize the (reformulated) infeasibility conditions as follows. The system G_α is not quadratically stable for $\alpha \in [-1,1]$ if and only if the following inequalities are feasible (with Q, Z_0 and W as variables):

$$Q \succeq 0, \quad Z_0 \succeq 0\,, \tag{29a}$$

$$\langle Z_0 + Q, I\rangle = 1, \tag{29b}$$

$$QA_D^T + A_DQ + W\Delta^T + \Delta W = Z_0. \tag{29c}$$

Finally, our goal is to show that the feasibility of (24) implies the feasibility of (29). To relate the two sets of inequalities, we first consider (24c). With the partitions $Y = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12}^T & Y_2 \end{bmatrix}$, $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}^T & X_2 \end{bmatrix}$, where the sizes of block matrices are in conformity with the partition of (26), the inequality (24c) reads as

$$\begin{bmatrix} A_1Y_1 + Y_1A_1^T + A_{12}Y_{12}^T + Y_{12}A_{12}^T & 0 \\ 0 & A_2Y_2 + Y_2A_2^T + A_{21}Y_{12} + Y_{12}^TA_{21}^T \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}. \tag{30}$$

With the abbreviations $Y_Q \coloneqq \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}$, $Y_W \coloneqq \begin{bmatrix} 0 & Y_{12} \\ Y_{12}^T & 0 \end{bmatrix}$, $X_Z \coloneqq \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$, the equality (30) is equivalently written in the following form:

$$Y_QA_D^T + A_DY_Q + Y_W\Delta^T + \Delta Y_W = X_Z. \tag{31}$$

Note also that (24a) implies $Y_Q \succeq 0$ and $X_Z \succeq 0$, while (24b) implies $\langle Y_Q + X_Z, I\rangle = 1$. Finally, we can conclude that by choosing $Q = Y_Q$, $W = Y_W$ and $Z_0 = X_Z$, the inequalities (29) are satisfied. This concludes the proof, as we have shown that the feasibility of (24) implies the feasibility of (29), i.e., the nonexistence of a structured Lyapunov function (15) for the system G implies that the system G_α is not quadratically stable for $\alpha \in [-1,1]$.

4. Discussion and conclusions

We have considered the interconnection of two LTI systems with an uncertainty in the interconnection channels between the two systems. The uncertainty is described using an uncertain, possibly time-varying, scalar parameter $\alpha(t)$ that models the strength in the interconnection channels between the systems. We have considered the case when $\alpha(t) \in [-1,1]$; therefore, the situation with $\alpha(t) = 0$ is also included. This means that our setting also captures the case when the systems are disconnected. The derived results show that if the system is quadratically stable with respect to all admissible values of $\alpha(t)$, then there *necessarily exists* a Lyapunov function for a nominal system (i.e. the system without uncertainties) with an additive structure.

To put it in a more practical setting, consider a case of two vehicles (e.g., automated flying vehicles) that are mutually interacting (are dynamically “connected”) when the distance between them is smaller than some predefined value, but act autonomously when the distance is larger than the predefined value (are “disconnected”). Then the results of this paper show us that, in the case of linear systems, there is no conservatism introduced if the robustness analysis, and possibly robust control design, of such a dynamical network is based on additive Lyapunov functions for the nominal system. It would also be interesting to see whether the obtained results could be of use when considering the interaction of a robotic system with its environment; see, e.g., [12] and the references therein for examples of such systems.

We also note that quadratic stability is not a necessary requirement for the robustness of an uncertain system in general; see, e.g., [13], where polyhedral Lyapunov functions have been considered because they can lead to less conservative results when compared to quadratic Lyapunov functions.

Finally, one might wonder whether some similar results can be obtained when the uncertainty interval is reduced from $[-1,1]$ to $[0,1]$, which seems to be more convenient for some applications. This is the area of future research. Furthermore, future research will include extending the existing results to networks with an arbitrary number of systems.

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