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# Testing Conformity with Benford's Law in Chaotic Dynamical Systems

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**Abstract:** Benford's law, also known as the first digit law, gives a monotonically decreasing distribution of the first digit in the considered data set. Contrary to our intuition, which suggests that the first digit would appear with a uniform distribution, this is decreasingly logarithmic law, where the digit 1 is appearing with 30% chance and the digit 9 is appearing with 4.58% chance. The purpose of this paper is to consider whether the Benford's law can be applied on different dynamical systems (Lorenz, Hénon and Rössler). As a procedure, we analyze the frequency of the first and the second digit of the coordinates of the trajectories generated by these dynamical systems. We conclude that some trajectories follow Benford's law, while others do not, and some results depend of the choice of the parameters of the model. As the main point is that natural data generally follow Benford's law, we have shown that for some dynamical systems the distinction between the trajectories that follow Benford's law and that do not follow Benford's aw may be very small which may require much more careful consideration.

**Keywords:** Benford's law; dynamical systems; manipulations

## 1 INTRODUCTION

Benford's law is a very complex mathematical tool which can be used to predict the distribution of leading digits across various types of data sets, including financial reports, stock prices as well as survey results. The law specifically describes the relative frequency distribution of leading digits in data, suggests that the distribution of first digits in numerical data is not uniform, as might be expected, but follows a logarithmic pattern. Specifically, the digit 1 appears with probability 0.3, while the digit 9 occurs with probability 0.0458.

Benford's law was initially discovered by the American scientist Newcomb [1], when he noticed that the first pages of logarithmic tables were worn more than the others. This observation led him to realize that the digit 1 appears more frequently than other digits. Later, after almost 60 years, Benford [2] confirmed the law formally in his paper. Nigrini [3] expanded the application of the law to accounting, auditing, and taxation for fraud detection purposes. Following Nigrini's work in forensic accounting, many researchers in their papers have explored Benford's law in this field. In [4] and [5] authors discussed the application of Benford's law in detecting fraud in accounting data. Krakar and Žgela [6] investigated the application of Benford's law in payment systems auditing. Benford's law in auditing and financial fraud detection was analyzed in [7], while Benford's law in forensic accounting was considered in [8]. Rigorous mathematical proof of the law was provided by Hill [9] and he (among all) showed that the combination of two distributions can follow Benford's distribution, even if the individual distributions do not.

The topic of this paper is an application of the Benford's law in different dynamical systems, especially those arising from natural processes. Chaotic dynamical systems, such as a Lorenz, Hénon or Rössler produce trajectories; the coordinates of those trajectories can be analysed using the Benford's law. If we show that the first and second digits follow the Benford's law, this may indicate that the system has a structure that is typical of a wide range of natural and social phenomena.

There is not so much literature on the application of the Benford's law in dynamical systems. Application of Benford's law in dynamical systems was investigated in [10]. In that paper authors confirmed that: Lorenz system's trajectories follow Benford's law; Hénon system's trajectories do not align with either a uniform distribution or Benford's law; Rössler system exhibits a mixture of behaviors, with some parameter settings following the Benford's law and others yielding a uniform distribution. Berger [11] showed that one-dimensional projections of almost all orbits of many multi-dimensional dynamical systems follow the Benford's law. Some dynamical properties of Benford's sequences were investigated in [12], and for non-autonomous difference equations this article presents necessary and sufficient conditions for the sequence to conform to the law in its strongest form. Li et al. [13] used only information from the first digit of considered series and suggested the method based on the Benford's law in order to distinguish noise from chaos. By applying the law to discrete data, they confirmed that chaotic data indeed can be distinguished from noise data, quantitatively and clearly. In Berger and Xu [14] authors provide new rigorous benchmarks against which to evaluate empirical observations regarding the Benford's law.

In this paper we consider the distribution of the first and second digits in the coordinates of trajectories produced by three different dynamical systems Lorenz, Hénon, and Rössler. We confirm some results from the literature and provide some new ones.

The first goal of this paper is to explore how Benford's law can be applied to various chaotic dynamical systems and to analyze the distribution of the first and second digits in the coordinates of these systems to determine whether they conform to Benford's law. The second goal is to assess the impact of different parameter sets on conformity with Benford's law.

This paper contributes to the literature by introducing a new test which is robust to deviations from assumptions and can provide more reliable results compared to other tests. The introduction of the bootstrap test for assessing conformity with Benford's law provides a more robust testing approach, which could be useful in future research on dynamical systems. The paper also provides a detailed analysis of various tests using them to assess conformity with Benford's law. It investigates the distribution of the first and especially of the second digits in the coordinates of chaotic systems, which is not often seen in the literature.

### 1.1 Properties of Benford's Law

The primary purpose of Benford's law is to suggest that data set may be inaccurate. If data do not align with the law, further analysis is needed to investigate potential errors. However, even when data follow the law, this does not guarantee the absence of fraud. A valuable aspect of Benford's law is that deviations in the expected frequency of certain digits can indicate not only fraud but also possible data entry errors or biases in data presentation. It is important to note that Benford's law treats both positive and negative values equally, as well as decimal numbers. However, there are limitations to its applicability. The law may not be reliable with small sample sizes, coded data, perfectly uniform distributions, psychologically rounded numbers, and certain mathematical sequences such as square roots or the reciprocals of consecutive positive integers, etc. Here, we give two standard definitions from the literature.

Definition 1: (Significand) For any positive number $x > 0$ and base $B$, $x$ is represented in notation as $x = S_B(x) \cdot B^{k(x)}$, where $S_B(x) \in [1, B)$ is called the significand of $x$ and the integer $k(x)$ (necessarily unique) represents the exponent. For a negative number $x$, it holds that $S_B(x) = S_B(-x)$ and $S_B(0) = 0$.

Definition 2: (Benford's law) A random variable $X$ obeys Benford's law in base $B$ if for any $s \in [1, B)$ holds

$$P\left(S_B(x) \le s\right) = \log_B(s) \tag{1}$$

or

$$P\left(D_1(x) = d_1\right) = \log_B\left(1 + \frac{1}{d_1}\right) \tag{2}$$

where $d_1 \in \{1, 2, ..., B-1\}$ is a digit in the first position in the number that takes value in the set of all possible outcomes and $D_1(X)$ is a random variable representing the first left digit of $S_B(x)$, that is first significant digit of $X$.

If data obeys Benford's law, we can also say that it follows Benford distribution. Further, we will consider decimal system, where $B = 10$. Fig. 1 represents the Benford distribution graphically for the first digit, using a probability diagram.

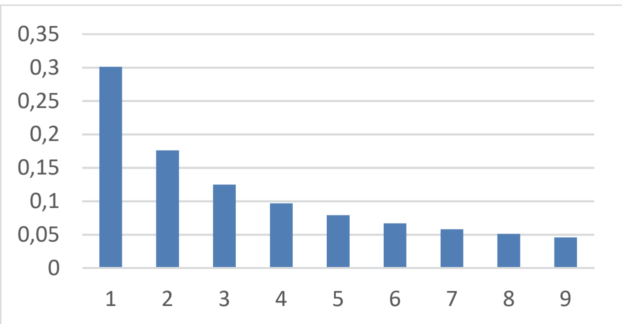


**Figure 1** Benford's distribution for the first digit

The probability of digits appearing on other positions is further given by:

$$P\left(D_k(X) = d_k\right) = \\ = \sum_{d_1=1}^{9} \sum_{d_2=0}^{9} \cdots \sum_{d_{k-1}=0}^{9} \log\left(1 + \frac{1}{\sum_{i=1}^{k} 10^{k-i} d_i}\right) \tag{3}$$

where $D_k(X)$ is a random variable representing the digit in the $k$-th position of the random variable $X$, and $d_k$ is a digit in that position, $d_k \in \{0, 1, ..., 9\}$. The probability of digits appearing in higher positions is uniformly distributed and equals 0.1.

Specially, the probability of an occurrence of the second digit is obtained by the following formula:

$$P\left(D_2(X) = d_2\right) = \\ = \sum_{d_1=1}^{9} \log\left(1 + \frac{1}{10d_1 + d_i}\right), d_2 \in \{0, 1, ..., 9\} \tag{4}$$

where $D_2(X)$ is a random variable modeling the second order significant digit of a number $X$, and $d_2$ is a digit in the second position.

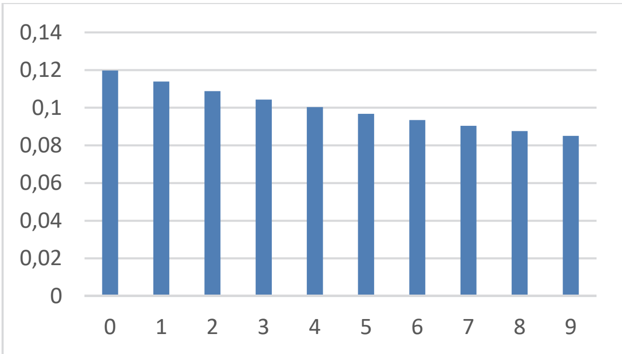


**Figure 2** Benford's distribution for the second digit

Fig. 2 represents the Benford distribution graphically for the second digit, using a probability diagram.

## 2 THE TESTS OF CONFORMITY WITH THE BENFORD'S LAW

For testing conformity with Benford's law, the null hypothesis claims that the data follow Benford's distribution, while the alternative hypothesis suggests they do not. All decisions are made at a 1% significance level. Proposed tests in the literature for testing compliance with Benford's law are four tests: z-test, chi-square test, Kolmogorov-Smirnov (KS) test, and Mean Absolute Deviation (MAD) test. These tests vary in their methodology; while z-test examines each digit individually, the other three tests evaluate all digits together. A further distinction is that the MAD test is independent of sample size, unlike the other three tests, allowing it to be used effectively even with large datasets. In this paper we use z-test, chi-square test, MAD test and for the first time for testing conformity with the law we suggest using the bootstrap test.

$Z$-test checks whether the distribution of each digit significantly differs from the expected Benford's distribution. The test statistic is of the form:

$$Z_i = \frac{\left|p_{oi} - p_i\right| - \frac{1}{2n}}{\sqrt{\frac{p_i\left(1 - p_i\right)}{n}}} \tag{5}$$

where $Z_i$ is $Z$-statistic for the digit $i$($i$ = 1, 2, …, 9) for the first digit and $i$ = 1, 2, …, 9 for the second digit), $p_{oi}$ is the observed frequency proportion of the digit $i$, $p_i$ is the expected frequency proportion of the digit $i$ according to the Benford's law, $n$ is the number of observations of the examined variable, the term $1/2n$ is Yates' correction factor and it is used when it is smaller than the absolute difference $\left|p_{oi} - p_i\right|$ in the numerator. If the value of $Z$-statistic exceeds the critical value 2.575, the null hypothesis is rejected at the 1% significance level.

The chi-square test is used to check the difference between the overall distribution of observed first and second digit frequencies and the expected frequencies according to Benford's law. Compared to the $z$-test, the key advantage of this test is that it evaluates all digits simultaneously. If the test leads to the rejection of the null hypothesis, this may indicate potential data manipulation, warranting a more detailed investigation of the dataset. The chi-square statistic is calculated as shown in the following formula:

$$\chi^2 = \sum_{i=m}^{9} \frac{\left(O_i - E_i\right)^2}{E_i} = n \sum_{i=m}^{9} \frac{\left(p_{oi} - p_i\right)^2}{p_i} \tag{6}$$

for testing the first digit ($m$ = 1) and the second digit ($m$ = 0), where $O_i$ is the observed frequency of the digit $i$, $E_i$ is the expected frequency of the digit $i$ implied by the Benford's law ($E_i = np_i$). The number of degrees of freedem for statistic $\chi^2$ is 8 for testing the first digit and 9 for testing the second digit. Observed values of chi-square test statistics are compared to the critical values, which are 20.090 and 21.666.

As previous tests considered the dataset size, the third test Mean Absolute Deviation (*MAD*) test ignores it, and because of that it can be applied to large databases. The test statistic can be calculated with the next formula:

$$MAD = \frac{1}{n} \frac{\sum_{i=m}^{9} \left|O_i - E_i\right|}{K} = \frac{\sum_{i=m}^{9} \left|p_{oi} - p_i\right|}{K} \tag{7}$$

for the first digit ($m$ = 1, $K$ = 9) and the second digit ($m$ = 0, $K$ = 10), where $O_i$, $E_i$, $p_{oi}$, $p_i$, $i$ and $n$ were introduced earlier. Based on personal experience, Nigrini [15] suggested critical scores for conformity, for nonconformity, and for some in-between categories.

**Table 1** Range of MAD critical values for the first and second digit

| Range | First digit | Second digit |
|---|---|---|
| Conformity | 0.000 - 0.006 | 0.000 - 0.008 |
| Acceptable conformity | 0.006 - 0.012 | 0.008 - 0.010 |
| Marginally acceptable conformity | 0.012 - 0.015 | 0.010 - 0.012 |
| Nonconformity | above 0.015 | above 0.012 |

There are no objective critical scores for the MAD test and that is the lack of this test. Range of MAD critical values for testing the first and the second digits are given in Tab. 1 (see [15]).

Last test which we use here is bootstrap test. Traditional tests (which are used to check conformity with Benford's law) can be sensitive to sample size and initial assumptions. By introducing the bootstrap test, this paper offers a method that is more robust and can be applied to a wider range of data, including large datasets. In a two-sample problem, we have samples $z = (z_1, z_2, \ldots, z_n)$ and $y = (y_1, y_2, \ldots, y_m)$ from possibly different probability distributions $F$ and $G$. The null hypothesis which is tested is in agreement with Benford's law i.e. $H_0$:$F = G$. Let $x$ be the combined sample which consists of all $n + m$ observations. Test statistic can be of the formula:

$$t\left(x\right) = \frac{\bar{z} - \bar{y}}{\sqrt{\frac{Var\left(z\right)}{n} + \frac{Var\left(y\right)}{m}}} \tag{8}$$

Bootstrap algorithm for calculation of $p$-value is of the form:

1. Draw $B$ samples of size $n + m$ with replacement from $x$. Let us denote the first $n$ observations $z^*$ and the remaining $m$ observations $y^*$.
2. Evaluate $t\left(\cdot\right)$ on each bootstrap sample:

$$t\left(x^{*b}\right) = \frac{\bar{z}^* - \bar{y}^*}{\sqrt{\frac{Var\left(z^*\right)}{n} + \frac{Var\left(y^*\right)}{m}}}, \ b = 1, 2, ..., B \tag{9}$$

3. Approximate $p$-value with:

$$\#\left\{t\left(x^{*b}\right) \geq t_{obs}\right\} \Big/ B \tag{10}$$

where $t_{obs} = t(x)$ is the observed value of the statistic.

## 3 APPLICATION TO DYNAMICAL SYSTEMS

In this section we examine an application of Benford's law to the low-dimensional chaotic models of dynamical systems with different sets of parameters.

As dynamical systems form the basis for the investigation of real world phenomena, and have broad application in fields as engineering, space exploration, environmental monitoring, artificial intelligence, and also in mathematics, physics, biology, chemistry, economics, medicine and social science, the importance of their study is clear. They are a fundamental part of chaos theory, logistic map dynamics, and bifurcation theory. They are models for any situation or system that changes over time and examine general patterns that emerge in the solutions of linear and nonlinear equations. Without such systems, predicting or solving real-world problems would be almost impossible. Thus, understanding the dynamical systems perspective is crucial for improving the world around us. Dynamical systems are classified according to time evolution into discrete and continuous types, and based on their randomness or behavioral factors, into deterministic and stochastic systems. The common is that all these types of dynamical systems will change and evolve over time through differential equations or iterative processes. For more about dynamical systems see [16] and [17].

### 3.1 Lorenz Dynamical System

The Lorenz system is a set of ordinary differential equations introduced by Edward Lorenz in 1963 (see [18]). It demonstrates how small changes can lead to vastly different outcomes. Chaotic solutions are produced under certain parameters and initial conditions. This phenomenon known as the "butterfly effect," suggests that minor changes could potentially alter large systems. The Lorenz attractor, which appears as a butterfly-like shape in phase space, visually represents this chaotic behavior, highlighting the deterministic yet unpredictable nature of chaotic systems over extended periods.

Lorenz system is of the form:

$$\dot{x}(t) = \sigma\left(y(t) - x(t)\right) \tag{11}$$

$$\dot{y}(t) = rx(t) - y(t) - x(t)z(t) \tag{12}$$

$$\dot{z}(t) = x(t)y(t) - bz(t) \tag{13}$$

We use the set of parameters ($\sigma$, $r$, $b$) that has been used in [10] and check whether the first and second digits of the $x$-coordinate follow Benford's law. The whole approach can be applied to any different set of parameters. We simulated 4000 states in Python with (0, 1, 2) as initial state. The outcomes of statistical tests are given in Tab. 2.

**Table 2** Tests of the first and second digits of the x-coordinate in Lorenz system

| | Parameters ($\sigma$, $r$, $b$) | | | | | |
|---|---|---|---|---|---|---|
| | (16, 45.92, 6) | (16, 45.92, 5) | (16, 45.92, 4) | (16, 45.92, 3) | (16, 45.92, 2) | (16, 45.92, 1) |
| Digit | Z statistic for the first digit-(for the second digit) | | | | | |
| 0 | / (2.547471) | / (1.232333) | / (0.258157) | / (0.11203) | / (0.30687) | / (1.39794) |
| 1 | 2.22335 (0.841111) | 8.738284 (0.194103) | 21.80262 (0.303596) | 15.56345 (0.004977) | 11.42699 (0.602216) | 14.25357 (2.23467) |
| 2 | 18.51763 (0.218342) | 13.32887 (0.167564) | 1.36568 (2.300203) | 6.845004 (1.660411) | 8.25634 (1.101863) | 4.06383 (4.43284) |
| 3 | 10.71771 (0.998398) | 12.10465 (0.325902) | 13.53941 (1.670895) | 8.852509 (0.91563) | 5.55255 (1.25705) | 4.21343 (1.36051) |
| 4 | 8.450299 (0.226328) | 7.702012 (0.857943) | 8.289952 (1.121115) | 6.205438 (0.721093) | 0.796391 (0.857943) | 4.38817 (0.45792) |
| 5 | 4.291694 (1.428407) | 4.642992 (0.946922) | 5.989635 (0.732928) | 0.310313 (1.107417) | 0.01171 (2.123886) | 4.467343 (1.08602) |
| 6 | 2.3715 (1.841995) | 2.43474 (1.635518) | 3.193619 (1.146492) | 0.34782 (0.918281) | 0.28458 (0.266247) | 3.0039 (2.88525) |
| 7 | 1.386706 (1.71483) | 1.995504 (1.428106) | 0.507331 (1.71483) | 0.777908 (0.4356) | 2.130792 (0.270182) | 1.72493 (3.03817) |
| 8 | 0.164997 (0.173375) | 1.169325 (0.06152) | 0.738899 (1.616304) | 0.337167 (1.571562) | 0.667161 (0.777392) | 1.02585 (2.73485) |
| 9 | 0.476495 (1.559131) | 0.884919 (1.218957) | 0.854665 (1.105566) | 0.703397 (0.368522) | 0.325227 (2.069392) | 1.53537 (0.02835) |
| | Other tests for the first digit-(for the second digit) | | | | | |
| Chi-square test (test statistic) | 477.29957 (17.72169) | 414.93451 (15.15528) | 602.50214 (31.49497) | 314.06886 (16.69652) | 180.92056 (17.36726) | 222.97647 (100.54553) |
| MAD test (test statistic) | 0.02881 (0.00561) | 0.03199 (0.00391) | 0.03583 (0.00574) | 0.02662 (0.00379) | 0.019867 (0.00464) | 0.024167 (0.00947) |
| Bootstrap test (p value) | 0.450 (0.494) | 0.495 (0.479) | 0.524 (0.501) | 0.545 (0.518) | 0.530 (0.501) | 0.535 (0.524) |

*Source:* Authors' calculations

From Tab. 2 we can generally conclude:
- if parameter $b$ is greater than 1, the "individual" distribution of the second digit follows Benford's law in all considered cases;
- if parameter $b$ is greater than 1 and different from 4, the "overall" distribution of the second digit follows Benford's law in all considered cases;
- *MAD* test shows conformity with Benford's law for the second digit, but for the first digit shows nonconformity;
- bootstrap test shows conformity with Benford's law for both digits;
- the "individual" distribution of the first digit in some cases follows Benford's law, but in some other cases does not follow.

The observed deviations in the first-digit conformity in the Lorenz system across specific parameters can be used to identify how nonlinear interactions among various factors impact first-digit distributions. In natural systems with complex interactions the nature of these interactions might lead to non-conformity with Benford's law. Exploring how the Lorenz system deviates from Benford's law could help researchers refine their understanding of these interactions in non-chaotic systems.

### 3.2 Rössler Dynamical System

The Rössler system is a set of three nonlinear ordinary differential equations. It is a continuous-time dynamical system with chaotic behavior, characterized by the fractal structure of its attractor, the Rössler attractor. That attractor was introduced in 1976 by Otto Rössler (see [19]), but the originally theoretical equations were later found to be useful in modelling equilibrium in chemical reactions. According to the original Rössler paper, the Rössler attractor was designed to resemble the chaotic behavior of the Lorenz attractor but in a simpler form. While some aspects of the system can be understood using linear techniques such as eigenvectors, nonlinear approaches like bifurcation diagrams and Poincaré maps are needed for understanding its complex dynamics.

Rössler system is of the form:

$$\dot{x}(t) = -y(t) - z(t) \tag{14}$$

$$\dot{y}(t) = x(t) + ay(t) \tag{15}$$

$$\dot{z}(t) = b + z(t)\big(x(t) - c\big) \tag{16}$$

We use the set of parameters ($a$, $b$, $c$) from [10] and check whether the first and second digits of the $x$-coordinate follow Benford's law. The whole approach can be applied to any different set of parameters. We simulated 10000 states in Python with (0, 1, 2) as initial state.

**Table 3** Tests of the first and second digits of the $x$-coordinate in Rössler system

| | Parameters ($a$, $b$, $c$) | | | |
|---|---|---|---|---|
| | (0.23, 0.23, 5.7) | (0.2, 0.2, 5.7) | (0.17, 0.17, 5.7) | (0.14, 0.14, 5.7) |
| Digit | $Z$ statistic for the first digit-(for the second digit) | | | |
| 0 | / (0.231046) | / (4.35907) | / (1.67894) | / (3.52731) |
| 1 | 15.44607 (3.54119) | 21.921 (2.53392) | 20.28591 (1.05449) | 18.60723 (1.62108) |
| 2 | 1.32579 (0.40143) | 4.37117 (0.20874) | 7.52156 (2.90634) | 4.975 (0.81891) |
| 3 | 4.85473 (2.96091) | 4.55225 (1.39048) | 0.13611 (0.14723) | 2.34418 (0.80157) |
| 4 | 1.97754 (2.97937) | 4.41144 (2.31359) | 8.43414 (2.24701) | 3.02547 (2.0114) |
| 5 | 5.35085 (0.69363) | 10.01664 (0.79513) | 12.86796 (0.38911) | 10.34991 (0.49061) |
| 6 | 4.17963 (0.36083) | 11.819 (3.07568) | 10.65906 (2.69767) | 11.05903 (0.12028) |
| 7 | 4.68462 (0.50566) | 9.39064 (2.63292) | 8.14996 (1.238) | 11.44417 (2.94678) |
| 8 | 6.37462 (1.14958) | 2.01901 (1.07884) | 10.04966 (3.59022) | 8.05334 (0.76049) |
| 9 | 3.03754 (0.26893) | 2.75052 (3.81882) | 3.13321 (2.31281) | 1.31547 (0.55579) |
| | Other tests | | | |
| Chi-square test (test statistic) | 304.46374 (59.02505) | 706.27404 (92.12672) | 827.23155 (71.85739) | 676.9675 (54.29494) |
| MAD test (test statistic) | 0.01717 (0.00403) | 0.02606 (0.00671) | 0.02852 (0.00547) | 0.02492 (0.00421) |
| Bootstrap test (p value) | 0.472 (0.513) | 0.493 (0.518) | 0.497 (0.514) | 0.474 (0.495) |
| | Parameters ($a$, $b$, $c$) | | | |
| | (0.11, 0.11, 5.7) | (0.08, 0.08, 5.7) | (0.05, 0.05, 5.7) | (0.2, 0.14, 5.7) |
| Digit | $Z$ statistic for the first digit-(for the second digit) | | | |
| 0 | / (2.81876) | / (2.07942) | / (1.12443) | / (2.07942) |
| 1 | 12.82994 (2.59687) | 0.38152 (3.03756) | 2.57343 (0.11017) | 18.41102 (1.14892) |
| 2 | 0.87948 (1.20428) | 6.31391 (0.91526) | 1.3783 (0.52989) | 1.29954 (1.686) |
| 3 | 0.34785 (0.2781) | 1.46701 (0.50712) | 1.07379 (1.32505) | 7.09305 (4.69492) |
| 4 | 1.40287 (2.74634) | 1.87613 (1.71438) | 1.09863 (1.04860) | 6.57491 (2.97936) |
| 5 | 3.16607 (0.21993) | 1.46269 (1.16732) | 0.12961 (0.38911) | 6.53581 (0.52445) |
| 6 | 7.13937 (2.04473) | 0.89992 (0.32647) | 0.21998 (0.73885) | 7.53934 (3.04132) |
| 7 | 9.51898 (1.16825) | 6.01086 (3.81860) | 0.49199 (0.43591) | 1.43319 (5.94586) |
| 8 | 5.83016 (0.22992) | 4.28755 (1.39718) | 0.93010 (0.54826) | 3.24402 (0.79586) |
| 9 | 1.17196 (1.05779) | 3.13321 (0.26893) | 0.98062 (0.51993) | 0.35876 (0.62751) |
| | Other tests | | | |
| Chi-square test (test statistic) | 294.61533 (53.26464) | 102.47334 (64.84679) | 10.60766 (10.4046) | 427.2237 (154.59839) |
| MAD test (test statistic) | 0.014655 (0.00443) | 0.0081 (0.00465) | 0.00339 (0.00221) | 0.02006 (0.00709) |
| Bootstrap test (p value) | 0.477 (0.476) | 0.497 (0.484) | 0.495 (0.508) | 0.491 (0.514) |

The results are given in Tab. 3. From Tab. 3 we can generally conclude:
- the "individual" distribution of the second digit follows Benford's law in allmost all considered cases; we have some sporadic deviations;
- if ($a$, $b$, $c$) = (0.05, 0.05, 5.7) we have conformity with Benford's law for both digits (see Fig. 3 and Fig. 4);
- *MAD* test shows conformity with Benford's law for the second digit, but for the first digit it shows nonconformity;
- bootstrap test shows conformity with Benford's law for both digits.

The mixed results for the Rössler system indicate that it is difficult to distinguish deterministic chaos from stochastic processes. Although chaotic systems may deviate from expected distributions, this does not always mean that the system is random or stochastic. The system's sensitivity to initial conditions and parameter settings could be the cause of this. To distinguish between stochasticity and deterministic chaos, additional statistical analyses (such as Lyapunov exponents, phase space analysis, and entropy metrics) are necessary.

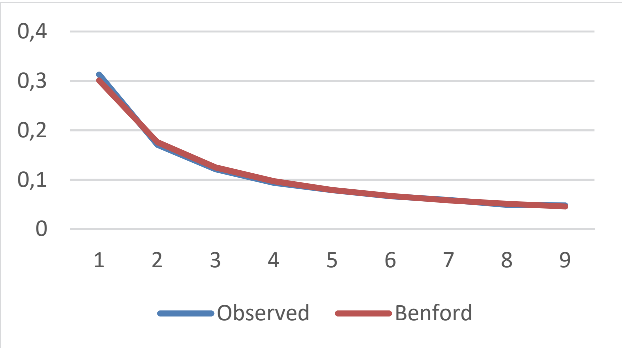


**Figure 3** Observed and Benford's distribution for the first digit in Rössler Dynamical System

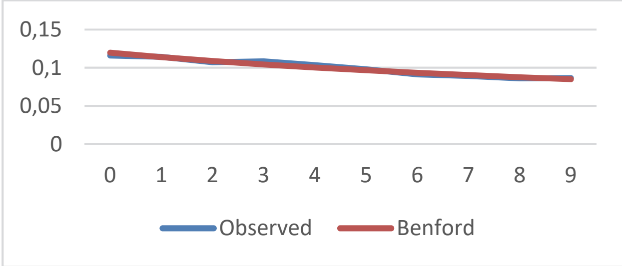


**Figure 4** Observed and Benford's distribution for the first digit in Rössler Dynamical System

## 3.3 Hénon Dynamical System

The Hénon map is a well-known discrete-time dynamical system and one of the most extensively studied examples exhibiting chaotic behavior. This map transforms a point ($x_n$, $y_n$) in the plane to a new point, illustrating complex dynamics that make it the main model for exploring chaos in dynamical systems. Michel Hénon (see [20]), introduced the Hénon map as a simplified model of the Poincaré section of the Lorenz system. For the classical Hénon map, an initial point in the plane will either approach a fractal set of points, known as the Hénon attractor, or diverge to infinity. The Hénon attractor itself is fractal, exhibiting smoothness in one direction and forming a Cantor set structure in the other. Hénon system is of the form:

$$x_{n+1} = y_n + 1 - a x_n^2 \tag{17}$$

$$y_{n+1} = b x_n \tag{18}$$

where $a$ and $b$ are two parameters. With different values of those parameters the map may exhibit chaotic dynamics, intermittency, or convergence to a periodic orbit.

**Table 4** Tests of the first and second digits of the x-coordinate in Hénon system

| | Parameters ($a$, $b$) | | | | |
|---|---|---|---|---|---|
| | (1.3, 0.3) | (1.3, 0.26) | (1.3, 0.22) | (1.3, 0.18) | (1.3, 0.14) |
| Digit | $Z$ statistic for the first digit (for the second digit) | | | | |
| 0 | /<br>(50.9996) | /<br>(22.31906) | /<br>(1.52491) | /<br>(11.53691) | /<br>(25.83097) |
| 1 | 58.45962<br>(35.42764) | 0.57773<br>(9.52187) | 16.77594<br>(19.12243) | 14.66123<br>(25.95299) | 0.66493<br>(21.5147) |
| 2 | 8.83422<br>(11.03125) | 22.0396<br>(4.76897) | 24.95371<br>(3.2596) | 31.72705<br>(5.89297) | 27.08023<br>(11.44873) |
| 3 | 5.33869<br>(33.87869) | 10.20854<br>(2.27385) | 9.42210<br>(1.65222) | 18.67785<br>(6.75611) | 14.68518<br>(2.60102) |
| 4 | 32.4689<br>(60.90221) | 1.94374<br>(6.77431) | 0.11832<br>(5.64249) | 2.28178<br>(4.84355) | 8.16371<br>(7.07391) |
| 5 | 28.97604<br>(32.46502) | 2.83280<br>(10.67506) | 7.31344<br>(3.63730) | 24.64352<br>(4.55086) | 0.09256<br>(4.14483) |
| 6 | 30.13735<br>(31.94244) | 6.8194<br>(11.01404) | 14.09876<br>(4.20974) | 30.77729<br>(2.11346) | 41.13638<br>(8.16173) |
| 7 | 24.49267<br>(18.11658) | 18.50319<br>(9.88652) | 28.51417<br>(0.61028) | 31.42334<br>(7.02693) | 0.49199<br>(8.03825) |
| 8 | 23.11649<br>(30.68491) | 1.51993<br>(6.10161) | 26.3832<br>(1.68015) | 3.96996<br>(7.90556) | 16.58307<br>(4.79286) |
| 9 | 21.50192<br>(20.38499) | 27.95967<br>(6.79499) | 10.88251<br>(1.38051) | 8.20374<br>(1.45223) | 18.44047<br>(5.21727) |
| | Other tests | | | | |
| Chi-square test<br>(test statistic) | 6571.24835<br>(21838.2137) | 1619.72388<br>(1846.58225) | 2566.1093<br>(797.51681) | 3747.1716<br>(1867.75098) | 3022.2195<br>(2647.55037) |
| MAD test<br>(test statistic) | 0.0803<br>(0.09839) | 0.02848<br>(0.02735) | 0.04517<br>(0.01317) | 0.05554<br>(0.02401) | 0.03986<br>(0.03047) |
| Bootstrap test<br>($p$ value) | 0.524<br>(0.515) | 0.422<br>(0.522) | 0.475<br>(0.534) | 0.503<br>(0.556) | 0.489<br>(0.540) |

*Source*: Authors' calculations

The map's behavior across various parameter values can be analyzed using its orbit diagram, which provides an overview of these dynamic changes. We use the set of parameters ($a$, $b$) from [10] and check whether the first and second digits of the $x$-coordinate follow Benford's law. The whole approach can be applied to any different set of parameters. We simulated 10000 states in Python with (0, 0) as initial state. The results are given in Tab. 4.

From Tab. 4 we can generally conclude:
- the "individual" distribution of the first and second digit does not follow Benford's law, actually in most cases we have deviations;
- *MAD* test shows non conformity with Benford's law for both digits;
- bootstrap test shows conformity with Benford's law for both digits, which is expected because this test is robust to deviation of assumptions.

## 4 CONCLUSION

In this paper, we considered the distribution of the first and second digits in the coordinates of trajectories produced by different chaotic dynamical systems: Lorenz, Rössler, and Hénon, using various parameter sets.

The results suggest that the Lorenz system conforms to Benford's law for the second digit in almost all parameter settings (i.e. for $b > 1$). For the first digit, the Lorenz system in some cases follows Benford's law, but in some other cases does not follow, which means that deeper analysis sholud be conducted.

In the case of the Rössler system, we found that the first digit exhibits mixed behavior: some parameter settings follow Benford's law, while others produce different distributions, which is consistent with previous literature. For the second digit, the MAD test, which is not sensitive to sample size, indicated conformity with the law for all chosen parameters, although the chi-square test did not confirm this result in most cases. Additionally, the $z$-test confirmed the law for only some digits, for both the first and second digits.

Our results for the Hénon system align with the literature for the first digit, showing no conformity with Benford's law, as demonstrated by the chi-square and MAD tests. Similarly, we showed that the second digit does not conform to the law, with chi-square and MAD tests, which support this finding. The z-test only confirmed the law for some digits, for both the first and second digits.

Only bootstrap test showed conformity with the law, but also for its implementation deeper analysis should be conducted.

The practical and theoretical implications of the results are as follows. In a theoretical sense, our results indicate that Benford's law can provide new insight into the analysis of data from chaotic systems. These systems often show complex patterns that cannot be identified by classical methods. Our results suggest that conformity with Benford's law can help recognize structures within the data as well as identify potential anomalies.

Practically, the application of Benford's law in the analysis of dynamical systems can be useful for detecting irregularities in real data, such as financial reports or data on natural phenomena. The results gained in the work may help in the development of methods that will more precisely analyze data and enable a better understanding of chaotic systems.

A limitation of this paper is that only three chaotic dynamical systems (Lorenz, Rössler and Hénon) with specific parameter sets were analyzed. This may not be sufficient to draw general conclusions about the compliance of chaotic systems with Benford's law, given the variability in the behavior of different systems and parameters.

Further research could include other chaotic systems, such as multivariable systems, as well as non-chaotic systems with complex dynamical behavior, to test hypotheses about compliance with Benford's law. The application of Benford's law could be extended to high-dimensional dynamical systems, but these systems often show complex behavior due to the large number of variables (with different ranges and scales) interacting in nonlinear ways. High-dimensional chaos can result in intricate attractors that make it difficult to apply Benford's law in a consistent way across all dimensions. The statistical tests may not be robust enough due to the complexity of the noise and sample variations. Therefore, a combination of tests should be made and adapted to the high-dimensional nature of the system.

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List of Symbols

| Symbol | Name | Unit |
|---|---|---|
| Lorenz System | | |
| $\sigma$ | Prandtl number (related to the velocity of turbulence or the vertical current in the fluid) | - |
| $r$ | Rayleigh number (related to the temperature gradient between the upper and lower fluid layers) | - |
| $b$ | Relative dimension of the fluid (geometric parameter) | - |
| Rössler System | | |
| $a$ | Oscillation parameter ($y$ - component parameter) | - |
| $b$ | Growth parameter (z - component parameter) | - |
| $c$ | Interaction parameter (between $x$ and $z$) | - |
| Hénon System | | |
| $a$ | Nonlinearity parameter (parameter controlling the quadratic component) | - |
| $b$ | Linear interaction parameter (controlling the relationship between $x$ and $y$) | - |

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