

## LETTER TO THE EDITOR

### HOLSTEIN-PRIMAKOFF REPRESENTATION OF SPIN-OPERATORS AND BOUND STATES

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The Holstein-Primakoff representation is tested on the problem of bound-states of spin waves. It was shown that the result obtained by the use of this representation essentially differs from the corresponding ones obtained in Pauli representation of spin operator as well as in other boson representation of spin operators, such as Dyson's representation, Marumori's representation and the exact boson representation.

The problem of bound states of two spin waves was analysed by many authors<sup>1-3)</sup>. The existence of bound states was proved in all mentioned papers, but the unique example where the energies as well as the wave functions can be explicitly expressed is the one-dimensional Heisenberg ferromagnet with spin  $S = 1/2$ . The mentioned system will be analysed here. The analysis will be carried out using Holstein-Primakoff representation of spin-operators<sup>4)</sup> and the results will be compared to analogous results obtained using other boson representations of spin-operators.

The convenience using different boson representations of spin-operators was tested in Ref. 3, where bound states energies as well as the corresponding wave functions were calculated by the use of the exact boson representation of spin operators<sup>5)</sup> as well as using Dyson-Maleev representation<sup>1,6)</sup> and Marumori's

representation<sup>7)</sup>. There was shown that all mentioned representations, as well as the direct calculation with spin operators<sup>2)</sup> lead to the following expression for bound states energies:

$$E(Q) = 2\Delta - 2I \left( 1 - \frac{1}{2} \sin^2 \frac{Qa}{2} \right); \quad \Delta = \mu \mathcal{H} + I. \quad (1)$$

In the formula (1)  $\mu$  is the magnetic moment of the atom,  $\mathcal{H}$  is the external magnetic field,  $I$  is the exchange integral for the nearest neighbours in one-dimensional lattice,  $Q$  is the wave vector characterizing the moving of the centrum of mass of two bound spin waves and  $a$  is the lattice constant.

It was also shown, in Ref. 3, that the coupling of spins at different lattice points leads to formation of bound states with the energies (1). This was proved by the use of representation from Ref. 5. The binding at one lattice point corresponds to zero value of energy. In the representation of Ref. 7 the energy  $E = \frac{1}{2} \Delta$  corresponds to the binding of two excitations at one lattice point. According to the representation in Refs. 1 and 6 the bound states arise from the binding of excitations at one lattice point and also due to the binding at different lattice points. Result obtained by the use of the boson representation from Ref. 5 is compatible with the result from Ref. 2 only. The Holstein-Primakoff representation of spin-operators has not yet been examined. According to Ref. 4 the spin-operators for  $S = 1/2$  are expressed in terms of Bose-operators  $B$  in the following way:

$$S^+ = B - \frac{1}{2} B^+ B B; \quad S^- = B^+ - \frac{1}{2} B^+ B^+ B; \quad \frac{1}{2} - S^z = B^+ B. \quad (2)$$

The Hamiltonian of one-dimensional Heisenberg ferromagnet, taken in the nearest neighbours approximation, can be written, using (2), in the form:

$$H = \Delta \sum_n B_n^+ B_n - \frac{I}{2} \sum_{n,\lambda} B_n^+ B_{n+\lambda} + \frac{I}{4} \sum_{n,\lambda} B_n^+ B_n^+ B_n B_{n+\lambda} + \\ + \frac{I}{4} \sum_{n,\lambda} B^+ B_{n+\lambda}^+ B_{n+\lambda} B_{n+\lambda} - \frac{I}{2} \sum_{n,\lambda} B_n^+ B_{n+\lambda}^+ B_n B_{n+\lambda}; \quad \lambda = \pm 1. \quad (3)$$

The wave function of the system with two excitations can be expressed as:

$$|2\rangle = \sum_{f,g} A_{fg} B_f B_g^+ |0\rangle. \quad (4)$$

The eigen-problem of the Hamiltonian for the state (4) is the following:

$$H |2\rangle = E_2 |2\rangle, \quad (5)$$

i. e.

$$\sum_{f,g} A_{fg} H B_f^+ B_g^+ |0\rangle = E_2 \sum_{f,g} A_{fg} B_f^+ B_g^+ |0\rangle. \quad (6)$$

Applying the operator  $\sum_{f,g} A_{fg} B_f^+ B_g^+$  to the ground state eigen-problem:  $H |0\rangle = E_0 |0\rangle$  one obtains the following relation:

$$\sum_{f,g} A_{fg} B_f^+ B_g^+ H |0\rangle = E_0 \sum_{f,g} A_{fg} B_f^+ B_g^+ |0\rangle. \quad (7)$$

Subtracting (7) from (6) we arrive to the fundamental relation:

$$\sum_{f,g} A_{fg} [H, B_f^+ B_g^+] |0\rangle = E \sum_{f,g} A_{fg} B_f^+ B_g^+ |0\rangle; \quad E = E_2 - E_0, \quad (8)$$

which defines the excitation energies  $E$  as well as the symmetrical coefficients  $A_{fg} = A_{gf}$ . After calculating the commutator  $[H, B_f^+ B_g^+]$ , the relation (8) becomes:

$$\begin{aligned} \sum_{f,g} \left\{ 2\Delta A_{fg} - \frac{I}{2} \sum_{\lambda} (A_{f,g+\lambda} + A_{f+\lambda,g}) + \frac{I}{4} \sum_{\lambda} (A_{f,g+\lambda} + A_{f+\lambda,g}) \delta_{f,g} + \right. \\ \left. + \frac{I}{2} \sum_{\lambda} A_{f+\lambda,g} \delta_{f+\lambda,g} - \frac{I}{2} \sum_{\lambda} A_{fg} (\delta_{f+\lambda,g} + \delta_{f,g+\lambda}) \right\} B_f^+ B_g^+ |0\rangle = \\ = E \sum_{f,g} A_{fg} B_f^+ B_g^+ |0\rangle. \end{aligned} \quad (9)$$

Going over to the centrum of mass system:  $\frac{f+g}{2} = R$  and  $f-g = r$  and using the Fourier's transformation:

$$\begin{aligned} A_{fg} = \frac{1}{N^2} \sum_{Q,q} a_Q(q) e^{ia \left[ \frac{f+g}{2} Q + (f-g)q \right]} \\ a_Q(q) = A_{\frac{Q}{2} + q, \frac{Q}{2} - q}; \quad a_Q(-q) = a_Q(q), \end{aligned} \quad (10)$$

one reduces (9) to the following integral equation with degenerate kernel:

$$a_Q(q) = -\frac{1}{2} \frac{G_Q \cos qa}{\varepsilon - G_Q \cos qa} \frac{1}{N} \sum_{q'} a_Q(q') + \frac{\cos qa - \frac{1}{2} G_Q}{\varepsilon - G_Q \cos qa} \frac{1}{N} \sum_{q'} a_Q(q') \cos q'a \quad (11)$$

where:

$$G_Q \equiv \cos \frac{Qa}{2}; \quad \varepsilon \equiv \frac{2\Delta - E}{2I}. \quad (12)$$

Going over from sums to integrals  $\frac{1}{N} \sum_q \rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} dx$ ,  $x \equiv qa$ , we reduce (11) to the following form:

$$a_Q(x) = -\frac{1}{2} \frac{G_Q \cos x}{\varepsilon - G_Q \cos x} C_1(Q) + \frac{\cos x - \frac{1}{2} G_Q}{\varepsilon - G_Q \cos x} C_2(Q), \quad (13)$$

where:

$$C_1(Q) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx a_Q(x); \quad C_2(Q) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx a_Q(x) \cos x. \quad (14)$$

After successive application of the operators  $\frac{1}{2\pi} \int_{-\pi}^{\pi} dx (\dots)$  and  $\frac{1}{2\pi} \int_{-\pi}^{\pi} dx (\dots) \cos x$  to (13) one obtains the following system of homogeneous algebraic equations defining  $C_1$  and  $C_2$ :

$$M_{11}(Q) C_1(Q) + M_{12}(Q) C_2(Q) = 0, \quad (15)$$

$$M_{21}(Q) C_1(Q) + M_{22}(Q) C_2(Q) = 0,$$

where

$$M_{11}(Q) = \frac{1}{2} \left( 1 + \frac{\varepsilon}{G_Q} A_Q \right); \quad M_{12}(Q) = \frac{1}{G_Q} \left( 1 - \frac{2\varepsilon - G_Q^2}{2G_Q} A_Q \right);$$

$$M_{21}(Q) = -\frac{\varepsilon}{2G_Q} \left( 1 - \frac{\varepsilon}{G_Q} A_Q \right); \quad M_{22}(Q) = \frac{2\varepsilon + G_Q^2}{2G_Q^2} - \frac{2\varepsilon - G_Q^2}{2G_Q^2} \frac{\varepsilon}{G_Q} A_Q; \quad (16)$$

$$A_Q = \left( \frac{\varepsilon^2}{G_Q^2} - 1 \right)^{-1/2}.$$

The system (15) has non-trivial solutions if the determinant of the system is equal to zero. According to (15) and (16) this condition leads to the following cubic equation:

$$y^3 - \frac{G_Q^2 + 2}{4G_Q} y^2 - \frac{1}{4} y - \frac{G_Q}{32} = 0 \quad (17)$$

where  $y = \varepsilon G_Q^{-1}$ .

Eq. (17) has one real root ( $y_1$ ) and two complex conjugate roots ( $y_2, y_3$ ), where from:

$$\varepsilon_1(Q) = -\frac{1}{12} [2 + F_1(Q) + F_2(Q) + G_Q^3], \quad (18)$$

$$\varepsilon_{2,3}(Q) = \frac{1}{24} \{4 + F_1(Q) + F_2(Q) + 2G_Q^2 \pm i\sqrt{3} [F_1(Q) - F_2(Q)]\}. \quad (19)$$

The notations used in (18) and (19) are the following:

$$\begin{aligned} F_{1,2}(Q) &= [P_1(Q) \pm P_2(Q)]^{1/3}, \\ P_1(Q) &= G_Q^6 + 51 G_Q^4 + 48 G_Q^2 + 8, \\ P_2(Q) &= G_Q^3 [27 (2 G_Q^4 + 71 G_Q^2 + 16)]^{1/2}. \end{aligned} \quad (20)$$

The expression for  $\varepsilon(Q)$  which can be easily obtained from the formula (1) is the following:

$$\varepsilon(Q) = -\frac{1}{2} (G_Q^2 + 1). \quad (21)$$

Comparing (18) to (21) we can conclude that the energies of bound states in Holstein-Primakoff representation differ from the standard values obtained in Refs. 2 and 3. Besides, the application of Holstein-Primakoff representation gives two additional solutions with finite life-time which is given by the following expression:

$$\tau(Q) = \frac{\hbar}{2I} \frac{|F_1(Q) - F_2(Q)|^{-1}}{\sqrt{3}}. \quad (22)$$

These solutions can be considered as parasitic and are the consequence of insufficient elimination of «nonphysical» states in HP representation.

The expressions  $\varepsilon_1(Q)$ ,  $\text{Re } \varepsilon_{2,3}(Q)$ ,  $\varepsilon(Q)$ ,  $10^{14} \tau(Q)$  ( $I$  was taken of the order  $5 \times 10^{-21}$  J) and also the expression

$$\varepsilon(Q, k) = -G_Q \cos ka, \quad (23)$$

corresponding to the energy of two free spin waves are graphically expressed in Fig.1.

As seen from Fig. 1 the curve  $\varepsilon_1(Q)$  is very close to  $\varepsilon(Q)$ . The deviations of  $\varepsilon_1(Q)$  with respect to  $\varepsilon(Q)$  are also the consequence of insufficiently correct elimination of «nonphysical» states in the Holstein-Primakoff representation. The appearance of two additional complex energies can be also considered as a defect of this representation, resulting from above mentioned reason.

Finally, we shall calculate the wave function corresponding to the energies:

$$E_{HP}^{(1)}(Q) = 2\Delta + 2I \varepsilon_1(Q). \quad (24)$$

Taking into account Eqs. (13) and (15) it can be easily calculated

$$\alpha_Q^{(1)}(q) = C(Q) \frac{1 - \left[ \frac{M_{11}^{(1)}(Q)}{M_{12}^{(1)}(Q)} + \frac{2}{G_Q} \right] \cos qa}{\varepsilon_1(Q) - G_Q \cos qa}, \quad (25)$$

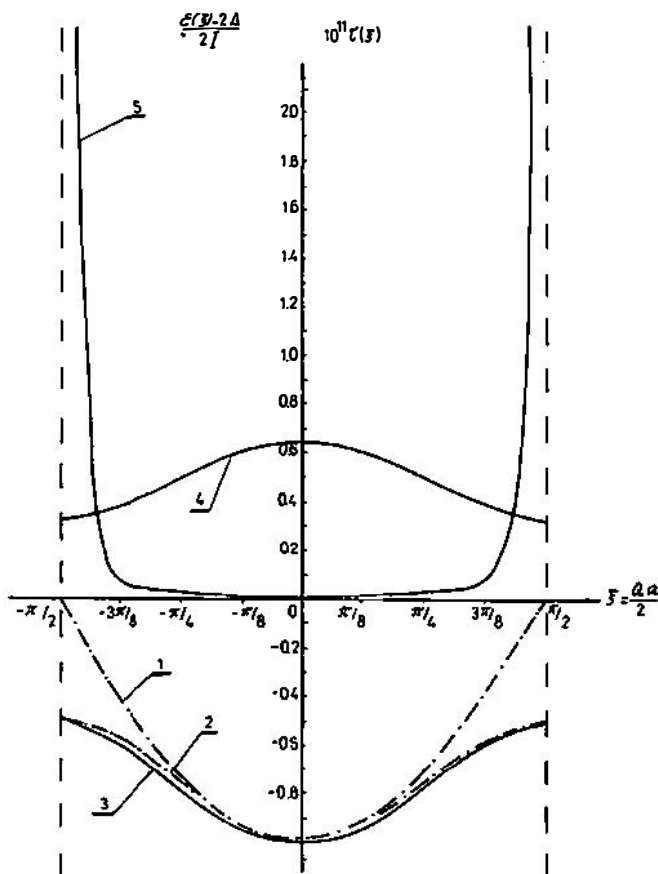


Fig. 1.

- Curve 1  $e(k, Q)$ ; two free wave energies.  
 Curve 2  $\varepsilon(Q)$ ; exact energies of bound spins.  
 Curve 3  $\varepsilon_1(Q)$ ; bound states energies in HP representation.  
 Curve 4  $\text{Re } \varepsilon_{2,3}(Q)$ ; real-parts of bound states energies obtained in HP representation.  
 Curve 5  $10^{11} \tau(Q)$ ; life-time of complex solutions obtained in HP representations.

where  $M_{11}^{(1)}(Q)$  and  $M_{12}^{(1)}(Q)$  are obtained from (16) by substituting  $\varepsilon$  into  $\varepsilon_1(Q)$ . The coefficients  $A_{fg}$  can be calculated by the formula (10) after transition to continual variables  $Qa \rightarrow \xi$  and  $qa \rightarrow \eta$ , i. e.

$$A_{fg}^{(1)} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} d\xi \int_{-\pi}^{\pi} d\eta \alpha_{\xi}^{(1)}(\eta) \exp \left\{ i \left[ \frac{f+g}{2} \xi + (f-g) \eta \right] \right\}. \quad (26)$$

Substituting (25) into (26) we obtain the following expressions for the coefficients  $A_{fg}^{(1)}$ :

$$A_{ff}^{(1)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\xi D_1(\xi) e^{if\xi},$$

$$D_1(\xi) = -C(\xi) \left\{ \left[ 1 - \frac{\varepsilon_1(\xi)}{G\xi} A^{(1)}(\xi) \right] \left[ \frac{M_{11}^{(1)}(\xi)}{M_{12}^{(1)}(\xi)} + \frac{2}{G\xi} \right] + A^{(1)}(-\xi) \right\} \quad (27)$$

and

$$A_{f,f\pm 1}^{(1)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\xi D_2(\xi) \exp \left\{ i \left( f \pm \frac{1}{2} \right) \xi \right\}. \quad (28)$$

$$D_2(\xi) = C(\xi) \left[ 1 - \frac{\varepsilon_1(\xi)}{G\xi} A^{(1)}(\xi) \right] \left\{ 1 - \frac{\varepsilon_1(\xi)}{G\xi} \left[ \frac{M_{11}^{(1)}(\xi)}{M_{12}^{(1)}(\xi)} + \frac{2}{G\xi} \right] \right\}.$$

The coefficients  $A_{fg}^{(1)}$  for  $f \approx g$  are calculated for nearest neighbours, only.

Analysing (27) one can easily conclude that the function  $D_1(\xi)$  is identically different from zero. According to Holstein-Primakoff representation, it follows that the binding at the some lattice point contribute to the energies of bound state. As was mentioned earlier the analogous result gives Dyson-Maleev representation. This is also the consequence of insufficiently correct elimination of «nonphysical» states in both representations.

Concluding following can be said:

- a) Holstein-Primakoff representation gives incorrect energy spectra of bound states.
- b) According to this representation the states with two excitations at one lattice point contribute to the energies of bound states which can also be estimated as a fault of this representation.
- c) Since the problem of bound states is soluble exactly and it can be solved in many different ways, we conclude that Holstein-Primakoff representation of spin-operators is not convenient for description of two-particle states. It is clear from the fact that the eigenvalues of  $(1 - B^+ B)^{1/2}$  are imaginary on the states with two bosons. However, the use of this representation in the low temperature thermodynamics of spin systems gives the correct results.

#### References

- 1) F. J. Dyson, Phys. Rev. **102** (1956) 1217 and 1230;
- 2) M. Wortis, Phys. Rev. **132** (1963) 85;
- 3) D. I. Lalović, B. S. Tošić, J. B. Vujaklija and R. B. Žakula, Nuovo Cimento **68B** (1970) 75;
- 4) T. Holstein and H. Primakoff, Phys. Rev. **58** (1940) 1098;
- 5) V. M. Agranovič and B. S. Tošić, Zh. eksp. teor. Fiz. **53** (1967) 149;
- 6) S. Maleev, Zh. eksp. teor. Fiz. **33** (1957) 1010;
- 7) T. Marumori, M. Yamamura and G. Tokunaga, Prog. Theor. Phys. (Kyoto) **31** (1964) 1009.

HOLSTEIN-PRIMAKOFF REPREZENTACIJA SPINSKIH OPERATORA  
I VEZANA STANJA

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Testirana je Holstein-Primakoffova reprezentacija na problemu vezanih stanja spinskih valova. Pokazano je da se rezultat dobijen korištenjem ove reprezentacije bitno razlikuje od odgovarajućih rezultata dobivenih u Pauli reprezentaciji spinskih operatora kao i u drugim bozonskim reprezentacijama spinskih operatora.