YU ISSN 0015—3206
FZKAAA 18 (1) 81 (1986)

# AN APPLICATION OF GEOMETRIC QUANTIZATION TO AFFINE SYSTEMS

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Received 25 September 1985

UDC 530.145

Original scientific paper

It is shown that the quantum models for collective motion may be derived by the geometric quantization of the corresponding classical phase spaces. This quantization procedure is applied to affine systems, especially to the homogeneously deformable rotator, based on $SL(3, R)$, which is an algebraic formulation of an appropriate model of even-even nuclei.

## *1. Introduction*

Geometric quantization is a well defined procedure and has achieved great progress toward a clearer understanding of the relationship between the classical and the quantum description of a physical system. The theory formulates a definition of quantization suitable for arbitrary symplectic manifolds — configuration or phase space manifolds — by identifying the intrinsic geometric objects involved in the ordinary canonical quantization, i. e. the geometric quantization is essentially a globalization of the Schrödinger-Dirac quantization scheme in which the additional structure needed for the quantization is explicitly expressed in geometric terms.

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In this paper we would like to show how the quantum mechanics of an affine (collective) model emerges as the result of the geometric quantization of the classical description of its dynamics.

It is assumed that the reader is familiar with the theory of geometric quantization. The foundations of the theory are given in Refs. 1—4; moreover, a fundamental review of the mathematical background is given in Ref. 5.

We first recall the classical description of the affine model (Section 2) where the classical collective models are defined to be the phase spaces on which, in particular, the Lie group $SL(3, R)$ acts transitively and canonically. From the classification theorem of Kostant and Souriau, these phase spaces are realized as orbits of the representation of $SL(3, R)$. Having obtained the classical affine models, we would like to quantize these symplectic spaces (Section 3).

However, the configuration space of our model (in contrast to the conventional collective model, where the quantum state space is given by wavefunctions on the classical configuration space) may be taken to be an orbit of $SL(3, R)$ in the space of $V$,

$$V = \{v \in M \mid v^t = v,\ v > 0\}, \tag{1.1}$$

where $M$ denotes the configuration space.

A point of $V$ can physically be interpreted as the quadrupole moment for a system of $N$ particles at the point $(x_1, \ldots, x_N) \in R^{3N}$, and hence $V$ is sometimes called quadrupole space.

The action of $SL(3, R)$, the group of linear transformations of $R^3$ with unit determinant, is inherited from the natural action of $SL(3, R)$ on $R^{3N}$,

$$(x_1, \ldots, x_N) \xrightarrow{g} (gx_1, \ldots, gx_N) \text{ for } g \in SL(3, R). \tag{1.2}$$

Thus, $SL(3, R)$ acts on $V$ according to

$$g : V \to V$$

$$v \to gvg^t \text{ for } g \in SL(3, R). \tag{1.3}$$

The restriction to the $SL(3, R)$ orbit has the consequence that the volume element in $R^3$ is preserved; this property is reasonable for nuclei, since it is known[6] that the nuclear forces tend strongly to conserve the nuclear volume, but not the nuclear shape.

In summary, the quantum affine model is given by the complex-valued functions on a five-dimensional orbit of $SL(3, R)$ in $V$. The inner product is defined by the $SL(3, R)$ invariant measure on the orbit.

Moreover, there is an algebraic characterization of the quantum affine model, since it is defined as an irreducible unitary representation of the Lie group $SL(3, R)$. These irreducible unitary representations are given by Mackey's inducing construction (cf. Refs. 7, 8).

The corresponding algebra is generated by the angular momenta and the incompressible vibrational momenta and can be identified with $sl(3, R)$, the traceless three by three real matrices.

## 2. Classical description of affine models

The general mechanical theory of the affinely-rigid body in the classical case has been discussed in a series of papers[9-11] by Slawianowski; the geometrical background of the theory is given there.

In a previous article[12] we discussed the classical mechanics of a breathing top and compared this model with the affinely-rigid body. Moreover, in a slightly different way we investigated in the classical theory of a homogeneously deformable rotator and described the quantization of such a system with the help of Mackey's theory of systems of imprimitivity[7].

The main concept of such so-called affine systems is the concept of collective motion, i. e. isolating a subsystem of a given system which already describes the motion of the system, when only the collective degrees of freedom are excited; the other degrees of freedom are to be considered as »frozen«.

The models of the affinely-rigid body and the homogeneously deformable rotator are based on the Lie groups $GL(3, R)$, the group of rotations and deformations, and $SL(3, R)$, the group of rotations and volume preserving deformations, respectively. These Lie groups as configuration spaces supply the rotational and vibrational degrees of freedom.

In the following, we shall confine ourselves to the $SL(3, R)$ model. Physical applications of this model may be found in the theory of elasticity and hydrodynamics[12].

We review now some of the geometrical properties and establish the notations. Our classical affine model under consideration is defined to be a phase space with a transitive canonical $SL(3, R)$ action. Thus, the incompressible motion is realized on the orbit surfaccs of $SL(3, R)$ in the quadrupole space $V$, as defined in section 1. The relevant concepts of computing such orbits in the general case of an arbitrary matrix Lie group may be found in Ref. 13.

The first step is to define a set of orbit representatives in $V$, the second one to compute the isotropy subgroup which gives a useful form for orbits. Finally, the action of the Casimir invariants on the orbits is determined which yields the physical interpretation to different orbits.

For rotational and volume preserving vibrational motion, it is obvious that the constraint surface is an orbit of the group $SL(3, R)$. In order to construct the quantization (see Section 3), a set of orbit representatives must be chosen, i. e. it is necessary to enumerate the orbits of $SL(3, R)$ by choosing precisely one point $v$ from each orbit surface.

An orbit $\Theta_v$ of $SL(3, R)$ is the surface $gvg^t$ as $g$ ranges over $SL(3, R)$. Since $v$ is real, symmetric, and positive-definite, each orbit contains a diagonal matrix $\lambda I$ with $\lambda > 0$[14], and the diagonal matrix contained in an orbit is unique. Hence, a choice of orbit representatives is given by

$$\{\lambda I, \lambda > 0\}. \tag{2.1}$$

The orbit containing the point $\lambda I$ is

$$\Theta_\lambda = \{\lambda g g^t \mid g \in SL(3, R)\}. \tag{2.2}$$

The most convenient description of the orbit is given by its identification with the coset space of $SL(3, R)$, $SL(3, R)$/isotropy subgroup. Recall that $SL(3, R)$ acts transitively on $V$, i. e. as $g$ ranges over $SL(3, R)$, an initial point, for cxample $v = I$, the identity matrix, is transformed into every other point of $V$, $I \to gg^t$, i. e. the entire space $V$ is a single orbit of $SL(3, R)$. The isotropy subgroup at the identity matrix in $V$ is the rotation group.

Hence, the orbit $\Theta_\lambda$ is identified with $SL(3, R)/SO(3, R)$ and the dimension of the orbit is $\dim \Theta_\lambda = 5$.

In Ref. 15, it was observed that almost every element of $SL(3, R)$ can uniquely be written as the product of an element of $SO(3, R)$ and element of $Z$,

$$SL(3, R) = SO(3, R) \cdot Z \tag{2.3}$$

where $Z$ is the group of upper triangular matrices with unit determinant and positive diagonal entries.

Since this decomposition is almost everywhere unique, we can identify $SL(3, R)/SO(3, R)$ with the group $Z$.

The physically important invariant of $SL(3, R)$ assumes constant values on the orbit:

$$\det v = \lambda^3, \tag{2.4}$$

where $\lambda$ is a measure for the volume of the affine system.

## 3. SL(3, R) quantization

In Ref. 16 we have discussed the quantum models for different types of affine systems using the ordinary Schrödinger-Dirac quantization of the corresponding classical phase spaces. Because of the physical importance of such systems (in particular the homogeneously deformable rotator as an appropriate model for even-even nuclei), we apply in this paper the Kostant-Souriau theory to a special affine model based on $SL(3, R)$. Notice that this quantization scheme of a symplectic manifold is defined provided the phase space meets the generalized Bohr-Sommerfeld quantization condition.

Hence, the first step in the quantization structure is to determine which orbits meet the Bohr-Sommerfeld conditions and find the unitary characters $\chi$. Consider an orbit $\Theta_y \approx SL(3, R)/SO(3, R)$ of $SL(3, R)$, containing the point $y \in sl(3), R$, the corresponding Lie algebra of $SL(3, R)$. Let $so(3, R)$ be the corresponding Lie algebra of $SO(3, R)$, the isotropy subgroup of $SL(3, R)$. The quantization condition is given on the maximal compact subalgebra $so(3, R)$ such that the Lie algebra homomorphism

$$so(3, R) \to iR \tag{3.1}$$
$$Z \to 2\pi i < F, Z >, \; Z \in so(3, R)$$

is the derived representation of a unitary character $\chi$ of $SO(3, R)$:

$$\chi : SO(3, R) \to U(1). \tag{3.2}$$

Hence, if $\chi$ exists, then it is given by

$$\chi(\exp(\Theta Z)) = \exp(2\pi i \Theta < F, Z >) = 1 \text{ with } Z \in so(3, R), \; \Theta \in R\}. \tag{3.3}$$

Thus, the orbit $\Theta_y$ is quantizable.

There is a natural representation $\pi$ of $SL(3, R)$, known as prequantization, on the space

$$Q = \{\psi : SL(3, R) \to C \mid \psi(gh) = \chi^{-1}(h) \psi(g), \; g \in SL(3, R), \; h \in SO(3, R)\}$$

with

$$(\pi(g') \psi)(g) = \psi(g'^{-1} g), \quad g', g \in SL(3, R) \text{ and } \psi \in Q. \tag{3.4}$$

This prequantization does not yield an irreducible representation of $SL(3, R)$.

Notice that the functions in $Q$ are essentially defined on the coset space $SL(3, R)/SO(3, R)$. Thus, if we choose canonical coordinates for the phase space $\Theta_y$ for example $(q_1 \ldots, q_n, p_1 \ldots, p_n)$, then the elements of $Q$ are given by $\psi(q_1, \ldots, p_n)$. Hence, although the unitary character has allowed us to define a representation on the phase space, we haven't got a quantization in which the wave functions $\psi$ depend only on $n$ variables.

Therefore, in order to quantize, it is necessary to restrict the wavefunctions. This can be done by introducing a polarization. Thus, the next step is to select a polarization.

In analogy to Ref. 13, the Poisson bracket is given by

$$\{\lambda(X_1), \lambda(X_2)\}(y) = 0, \; X_1, X_2 \in so(3, R), \tag{3.5}$$

where $y \in so(3, R)$ is a point contained in an orbit as above.

Since the Poisson bracket is non-degenerate on the orbit, it is clear that there exists a polarization which has the following form:

$$p = so(3, R). \tag{3.6}$$

Then, in general, the quantum state space $Q^p$ is defined to be

$$Q^p = \{\psi \in Q \mid L_X \psi = 2\pi i < y, X > \text{ for all } X \in p\}. \tag{3.7}$$

In the special case of $SL(3, R)$, the Lie derivative $L_X \psi = 0$.

The quantum Hilbert space is given by

$$H^p = L^2 (SL (3, R)/SO (3, R), \mu) = \\ = \{\psi : SL (3, R)/SO (3, R) \to C \mid \int_{SL (3,R)/SO (3,R)} d\mu (gr) \mid \psi (gr) \mid^2 < \infty\} \tag{3.8a}$$

where $g \in SL (3, R)$, $r \in SO (3, R)$ and $\mu$ the invariant measure on $SL (3, R)/SO (3, R)$.

On this Hilbert space, the action of $SL (3, R)$ is determined to be

$$\pi : SL (3, R) \to U (L^2 (SL (3, R)/SO (3, R)) \\ (\pi (g') \psi) (gr) = \psi (g'^{-1} gr) \tag{3.9a}$$

where $g' \in SL (3, R)$, $gr \in SL (3, R)/SO (3, R)$ and $\psi \in L^2 (SL (3, R)/SO (3, R))$.

Such wave functions on the coset space $SL (3, R)/SO (3, R)$ may be regarded as functions on $SL (3, R)$ which are right invariant under $SO (3, R)$, i. e.

$$H^n = L^2 (SL (3, R)/SO (3, R, \nu) \\ = \{\psi : SL (3, R) \to C \mid \\ (1)\ \psi (gr) = \psi (g) \text{ for all } g \in SL (3, R),\ r \in SO (3, R) \\ (2) \int_{SL(3,R)} d\nu (g) \psi \mid (g) \mid^2 < \infty\} \tag{3.8b}$$

where $\nu$ is the invariant measure on $SL (3, R)$.

The inner product on $H^n$ is given by

$$\langle \psi_1 \psi_2 \rangle = \int_{SL(3,R)} d\nu (g) (\psi_1 (g), \psi_2 (g)), \tag{3.10}$$

and the action $SL (3, R)$ on the Hilbert space $H^n$ has the form:

$$(\pi^\lambda (g') \psi) (g) = \psi (g'^{-1} g) \tag{3.9b}$$

for $g', g \in SL (3, R)$ and $\psi \in H^n$.

Observe that there is an obvious isomorphism of $H^p$ with the carrier space $H^n$ for the irreducible representations of $SL (3, R)$ given by the inducing construction. It is easily found that $\pi^p$ is unitarily equivalent to $\pi^\lambda$. Moreover, the physical admissible models are just those irreducible unitary representations that occur in the decomposition of the representation of $SL (3, R)$[7].

As we have seen in this paper, the geometric quantization is a more relevant construction in nuclear theory, in particular in the formulation of affine (collective) models, than the Schrödinger-Dirac quantization. In fact, the latter one takes full account of the $3N$ degrees of freedom, although not all of these degrees of freedom are necessary for the description of collective rotational and vibrational motion.

Hence, geometric quantization plays a fundamental rule not only in the usual quantum mechanics but also in the formulation of approximate theories of physical phenomena like the nuclear structure physics. Moreover, this concept can be applied to a theory quite different from the affine models, namely the Hartree-Fock theory, which can be understood most naturally in terms of geometric quantization ideas[17].

## Acknowledgment

I would like to thank Professor H. D. Doebner for many helpful discussions about the theory of affine systems and the IAEA and UNESCO for the hospitality at the International Centre for Theoretical Physics, Trieste, during the last period of this work.

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# PRIMJENA GEOMETRIJSKE KVANTIZACIJE NA AFINE SISTEME

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Originalni znanstveni rad

Pokazano je da je kvantnomehaničke modele kolektivnih gibanja moguće izvesti geometrijskom kvantizacijom odgovarajućih klasičnih faznih prostora. Ta procedura kvantizacije primjenjena je na afine sisteme, specijalno na rotor koji se može homogeno deformirati. Taj rotor baziran je na $SL(3, R)$, te je u stvari algebarska formulacija odgovarajućeg modela parno-parne jezgre.