The orbit containing the point λI is

$$\Theta_\lambda = \{\lambda g g^t \mid g \in SL(3, R)\}. \tag{2.2}$$

The most convenient description of the orbit is given by its identification with the coset space of $SL(3, R)$, $SL(3, R)$/isotropy subgroup. Recall that $SL(3, R)$ acts transitively on V, i. e. as g ranges over $SL(3, R)$, an initial point, for cxample $v = I$, the identity matrix, is transformed into every other point of $V, I \to gg^t$, i. e. the entire space V is a single orbit of $SL(3, R)$. The isotropy subgroup at the identity matrix in V is the rotation group.

Hence, the orbit Θ_λ is identified with $SL(3, R)/SO(3, R)$ and the dimension of the orbit is $\dim \Theta_\lambda = 5$.

In Ref. 15, it was observed that almost every element of $SL(3, R)$ can uniquely be written as the product of an element of $SO(3, R)$ and element of Z,

$$SL(3, R) = SO(3, R) \cdot Z \tag{2.3}$$

where Z is the group of upper triangular matrices with unit determinant and positive diagonal entries.

Since this decomposition is almost everywhere unique, we can identify $SL(3, R)/SO(3, R)$ with the group Z.

The physically important invariant of $SL(3, R)$ assumes constant values on the orbit:

$$\det v = \lambda^3, \tag{2.4}$$

where λ is a measure for the volume of the affine system.

3. $SL(3, R)$ quantization

In Ref. 16 we have discussed the quantum models for different types of affine systems using the ordinary Schrödinger-Dirac quantization of the corresponding classical phase spaces. Because of the physical importance of such systems (in particular the homogeneously deformable rotator as an appropriate model for even-even nuclei), we apply in this paper the Kostant-Souriau theory to a special affine model based on $SL(3, R)$. Notice that this quantization scheme of a symplectic manifold is defined provided the phase space meets the generalized Bohr-Sommerfeld quantization condition.

Hence, the first step in the quantization structure is to determine which orbits meet the Bohr-Sommerfeld conditions and find the unitary characters χ. Consider an orbit $\Theta_y \approx SL(3, R)/SO(3, R)$ of $SL(3, R)$, containing the point $y \in sl(3), R$, the corresponding Lie algebra of $SL(3, R)$. Let $so(3, R)$ be the corresponding Lie algebra of $SO(3, R)$, the isotropy subgroup of $SL(3, R)$. The quantization condition is given on the maximal compact subalgebra $so(3, R)$ such that the Lie algebra homomorphism

$$so\,(3, R) \to \mathrm{i}R \qquad (3.1)$$
$$Z \to 2\pi \mathrm{i} < F, Z >, \; Z \in so\,(3, R)$$

is the derived representation of a unitary character χ of $SO\,(3, R)$:

$$\chi : SO\,(3, R) \to U\,(1). \qquad (3.2)$$

Hence, if χ exists, then it is given by

$$\chi\,(\exp\,(\Theta Z)) = \exp\,(2\pi \mathrm{i}\,\Theta < F, Z >) = 1 \text{ with } Z \in so\,(3, R), \; \Theta \in R\}. \qquad (3.3)$$

Thus, the orbit Θ_y is quantizable.

There is a natural representation π of $SL\,(3, R)$, known as prequantization, on the space

$$Q = \{\psi : SL\,(3, R) \to C \mid \psi\,(gh) = \chi^{-1}\,(h)\,\psi\,(g), \; g \in SL\,(3, R), \; h \in SO\,(3, \mathrm{R})\}$$

with (3.4)

$$(\pi\,(g')\,\psi)\,(g) = \psi\,(g'^{-1}\,g), \quad g', g \in SL\,(3, R) \text{ and } \psi \in Q.$$

This prequantization does not yield an irreducible representation of $SL\,(3, R)$.

Notice that the functions in Q are essentially defined on the coset space $SL\,(3, R)/SO\,(3, R)$. Thus, if we choose canonical coordinates for the phase space Θ_y for example $(q_1 \ldots, q_n, p_1 \ldots, p_n)$, then the elements of Q are given by $\psi\,(q_1, \ldots, p_n)$. Hence, although the unitary character has allowed us to define a representation on the phase space, we haven't got a quantization in which the wave functions ψ depend only on n variables.

Therefore, in order to quantize, it is necessary to restrict the wavefunctions. This can be done by introducing a polarization. Thus, the next step is to select a polarization.

In analogy to Ref. 13, the Poisson bracket is given by

$$\{\lambda\,(X_1), \lambda\,(X_2)\}\,(y) = 0, \; X_1, X_2 \in so\,(3, R), \qquad (3.5)$$

where $y \in so\,(3, R)$ is a point contained in an orbit as above.

Since the Poisson bracket is non-degenerate on the orbit, it is clear that there exists a polarization which has the following form:

$$p = so\,(3, R). \qquad (3.6)$$

Then, in general, the quantum state space Q^p is defined to be

$$Q^p = \{\psi \in Q \mid L_X\,\psi = 2\,\pi \mathrm{i} < y, X > \text{ for all } X \in p\}. \qquad (3.7)$$

In the special case of $SL\,(3, R)$, the Lie derivative $L_X\,\psi = 0$.

The quantum Hilbert space is given by

$$H^p = L^2 (SL(3, R)/SO(3, R), \mu) = \\ = \{\psi : SL(3, R)/SO(3, R) \to C \mid \int_{SL(3,R)/SO(3,R)} \mathrm{d}\mu (gr) \mid \psi (gr) \mid^2 < \infty\} \tag{3.8a}$$

where $g \in SL(3, R)$, $r \in SO(3, R)$ and μ the invariant measure on $SL(3, R)/SO(3, R)$.

On this Hilbert space, the action of $SL(3, R)$ is determined to be

$$\pi : SL(3, R) \to U(L^2 (SL(3, R)/SO(3, R)) \\ (\pi (g') \psi)(gr) = \psi (g'^{-1} gr) \tag{3.9a}$$

where $g' \in SL(3, R)$, $gr \in SL(3, R)/SO(3, R)$ and $\psi \in L^2 (SL(3, R)/SO(3, R))$.

Such wave functions on the coset space $SL(3, R)/SO(3, R)$ may be regarded as functions on $SL(3, R)$ which are right invariant under $SO(3, R)$, i. e.

$$H^n = L^2 (SL(3, R)/SO(3, R, \nu) \\ = \{\psi : SL(3, R) \to C \mid \\ (1)\ \psi (gr) = \psi (g) \text{ for all } g \in SL(3, R),\ r \in SO(3, R) \\ (2) \int_{SL(3,R)} \mathrm{d}\nu (g) \psi \mid (g) \mid^2 < \infty\} \tag{3.8b}$$

where ν is the invariant measure on $SL(3, R)$.

The inner product on H^n is given by

$$\langle \psi_1 \psi_2 \rangle = \int_{SL(3,R)} \mathrm{d}\nu (g) (\psi_1 (g), \psi_2 (g)), \tag{3.10}$$

and the action $SL(3, R)$ on the Hilbert space H^n has the form:

$$(\pi^\lambda (g') \psi)(g) = \psi (g'^{-1} g) \tag{3.9b}$$

for $g', g \in SL(3, R)$ and $\psi \in H^n$.

Observe that there is an obvious isomorphism of H^p with the carrier space H^n for the irreducible representations of $SL(3, R)$ given by the inducing construction. It is easily found that π^p is unitarily equivalent to π^λ. Moreover, the physical admissible models are just those irreducible unitary representations that occur in the decomposition of the representation of $SL(3, R)$[7].

As we have seen in this paper, the geometric quantization is a more relevant construction in nuclear theory, in particular in the formulation of affine (collective) models, than the Schrödinger-Dirac quantization. In fact, the latter one takes full account of the $3N$ degrees of freedom, although not all of these degrees of freedom are necessary for the description of collective rotational and vibrational motion.

Hence, geometric quantization plays a fundamental rule not only in the usual quantum mechanics but also in the formulation of approximate theories of physical phenomena like the nuclear structure physics. Moreover, this concept can be applied to a theory quite different from the affine models, namely the Hartree--Fock theory, which can be understood most naturally in terms of geometric quantization ideas[17].

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PRIMJENA GEOMETRIJSKE KVANTIZACIJE NA AFINE SISTEME

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Pokazano je da je kvantnomehaničke modele kolektivnih gibanja moguće izvesti geometrijskom kvantizacijom odgovarajućih klasičnih faznih prostora. Ta procedura kvantizacije primjenjena je na afine sisteme, specijalno na rotor koji se može homogeno deformirati. Taj rotor baziran je na $SL(3, R)$, te je u stvari algebarska formulacija odgovarajućeg modela parno-parne jezgre.