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THEORY OF LANGMUIR SOLITONS

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The purpose of this paper is to give a theory of Langmuir solitons when the ion-nonlinearities (which become important when the Mach number of the soliton approaches unity) are included. When a charge-neutrality in the low-frequency plasma response is assumed, one obtains two models — with the low-frequency electron-density perturbation satisfying, respectively, a Korteweg-de Vries equation and a Boussinesq equation and the ponderomotive force acts as the driving term, while the Langmuir field obeys the nonlinear Schrödinger equation. The Boussinesq equation model is shown to admit only a single-humped soliton solution while the Korteweg-de Vries equation model is shown to admit only a double-humped soliton solution. A Lagrangian formalism is given for both models.

1. *Introduction*

Modulational instability of Langmuir wave arises through local depressions in density and corresponding increases in the energy density of the Langmuir field, and leads to Langmuir solitons (Zakharov[1)]). The density depressions are produced by the low-frequency ponderomotive force on electrons expelling them (and hence ions as well through ampipolar effects) from the region of strong fields. Mathematically, these solitons are localised stationary solutions of a nonlinear Schrödinger equation for the Langmuir field with an effective potential proportional to the low-frequency electron-density perturbation. The latter in turn obeys an equation for non-acoustic waves driven by the ponderomotive force. Specifically, one has for the high-frequency Langmuir field E and the low-frequency electron-density perturbation n,

$$\mathrm{i}\varepsilon E_t + \frac{3}{2} E_{xx} = \frac{n}{2} E \tag{1}$$

$$n_{tt} - n_{xx} = \frac{1}{4} (|E|^2)_{xx} \tag{2}$$

where we have nondimensionalised the various quantities using the ion-plasma period ω_{pi}^{-1}, the Debye length λ_{De}, n with respect to the unperturbed value N, and E using $4\sqrt{\pi KNT_e}$, T_e being the electron temperature and $\varepsilon = \sqrt{m_e/m_i}$.

The system of Eqs. (1) and (2) has the stationary soliton solution

$$E = 4\sqrt{\left(\frac{M^2}{6} - \varepsilon\sigma\right)(1 - M^2)}\ \mathrm{sech}\left[\sqrt{\frac{2}{3}\left(\frac{M^2}{6} - \varepsilon\sigma\right)}(x - Mt)\right] \times$$

$$\times\, \mathrm{e}^{\mathrm{i}\left(\varepsilon \frac{Mx}{3} - \varepsilon\sigma\right)}$$

$$n = -\frac{|E|^2}{4(1 - M^2)} \tag{3}$$

where σ is a frequency shift which is a free parameter. This solution moves faster but becomes smaller and narrower as M increases, and disappears when $M \approx 1$, i. e., when the soliton moves with a speed close to the ion-acoustic speed. Makhankov[2)] and Nishikawa et al.[3)] pointed out that the ion-nonlinearities become important for the latter case. Makhankov[2)] and Nishikawa et al.[3)] accounted for the ion-nonlinearities by assuming a charge-neutrality in the low-frequency plasma response and the low-frequency electron-density perturbation then obeyed, respectively, a Korteweg-de Vries equation and a Boussinesq equation with the pondermotive force as the driving term, while the Langmuir field continued to obey the nonlinear Schrödinger equation. Rao and Varma[4)] considered a fully-nonlinear system of equations without any assumption of charge neutrality in the low-frequency plasma-response, and gave a solution which shows a smooth transition from a single-hump soliton solution of Zakharov[1)] to a double-hump soliton solution of Nishikawa et al.[3)] as the Mach number of the soliton increases. In the present paper, we first make further studies of the Boussinesq equation model and the Korteweg-de Vries equation model. The former is shown to possess only a single-humped soliton solution which is valid both for $M \ll 1$ and $M \approx 1$. The latter is shown to possess only a double-humped soliton with somewhat different parameters than the ones given by Nishikawa et al.[3)]. This solution is valid for $M \approx 1$ and not for $M \ll 1$. Then, a Lagrangian formalism is given for both sets of model equations.

2. *The model equations of Makhankov*

Upon including only the nonlinearity arising from the Boltzmann distribution of the electrons, one obtains

$$n_{tt} - n_{xx} - n_{xxtt} + (n^2)_{tt} = \frac{1}{4}(|E|^2)_{xx}. \tag{4}$$

Observe that the equation given by Makhankov$^{2)}$ has a sign error for the fourth term on the left-hand side. It can be seen from the following that this sign error, unless corrected, will not lead to the existence of soliton-solutions of Eq. (4).

(i) *Exact solutions*

Look for stationary solutions of the form,

$$\begin{aligned} E(x,t) &= \mathscr{E}(\xi)\, e^{i\left(\frac{\varepsilon M x}{3} - \sigma t\right)} \\ n(x,t) &= n(\xi) \end{aligned} \tag{5}$$

where

$$\xi = x - Mt.$$

Then Eqs. (1) and (4) give

$$\mathscr{E}'' - \frac{2}{3}\left(\frac{\varepsilon^2 M^2}{3} - \varepsilon\sigma\right)\mathscr{E} - \frac{n}{3}\mathscr{E} = 0 \tag{6}$$

$$(M^2 - 1)\, n - M^2 n'' + M^2 n^2 = \frac{1}{4}\mathscr{E}^2 \tag{7}$$

where primes denote differentiation with respect to ξ.

Look for a single-humped soliton solution of the form

$$\begin{aligned} \mathscr{E} &= \gamma \operatorname{sech} \beta\xi \\ n &= a \operatorname{sech}^2 \beta\xi \end{aligned} \tag{8}$$

so that Eqs. (6) and (7) give,

$$\left[-\frac{2}{3}\left(\frac{\varepsilon^2 M^2}{6} - \varepsilon\sigma\right)\gamma - \frac{1}{3}a\gamma - \gamma\beta^2]\right] + \left[2\gamma\beta^2 + \frac{1}{3}a\gamma\right]\tanh^2 \beta\xi = 0 \tag{9}$$

$$\left[a(M^2 - 1) + 2a\beta^2 M^2 + M^2 a^2 - \frac{\gamma^2}{4}\right] + \left[-a(M^2 - 1) + \frac{\gamma^2}{4} - {}\right.$$

$$\left. - 8a\beta^2 M^2 - 2M^2 a^2\right]\tanh^2 \beta\xi + [6a\beta^2 M^2 + M^2 a^2]\tanh^4 \beta\xi = 0 \tag{10}$$

from which one obtains

$$a = -4\left(\frac{\varepsilon^2 M^2}{6} - \varepsilon\sigma\right) \tag{11}$$

$$\beta^2 = -\frac{a}{6}, \quad \frac{\gamma^2}{4} = a\left[(M^2 - 1) + \frac{2M^2 a}{3}\right]. \tag{12}$$

By an appropriate choice of σ, it is possible to ensure that the single-humped soliton solution given by (8) is well-behaved for both $M \ll 1$ and $M \approx 1$.

In this model, as shown by (8) and (12), observe that the density perturbation is linearly proportional to the high-frequency electric field amplitude as $M \Rightarrow 1$, in contrast to the second power of the latter in Zakharov's solution (3). Therefore, in this model, a substantial density cavity is produced by a relatively weak high-frequency electric field.

Note that the low-frequency shift σ is a free parameter for the single-humped soliton solutions (as for Zakharov's solution (3)).

Let us now try a double-humped soliton solution

$$\begin{aligned} \mathcal{E} &= \gamma \operatorname{sech} \beta\xi \cdot \tanh \beta\xi \\ n &= a \operatorname{sech}^2 \beta\xi \end{aligned} \tag{13}$$

so that Eqs. (6) and (7) give

$$\left[-5\gamma\beta^2 + \frac{2}{3}\left(\frac{\varepsilon^2 M^2}{6} - \varepsilon\sigma\right)\gamma + \frac{1}{3}a\gamma\right]\tanh \beta\xi + \left(6\gamma\beta^2 + \frac{1}{3}a\gamma\right)\tanh^3 \beta\xi = 0 \tag{14}$$

$$\begin{aligned} &[a(M^2 - 1) + 2a\beta^2 M^2 + M^2 a^2] + \\ &+ \left[-a(M^2 - 1) - \frac{\gamma^2}{4} - 8a\beta^2 M^2 - 2M^2 a^2\right]\tanh^2 \beta\xi + \\ &+ \left[\frac{\gamma^2}{4} + 6a\beta^2 M^2 + M^2 a^2\right]\tanh^4 \beta\xi = 0 \end{aligned} \tag{15}$$

from which one obtains

$$a = -\frac{9(M^2 - 1)}{8M^2}, \quad \beta^2 = \frac{M^2 - 1}{16M^2}, \quad \frac{\gamma^2}{4} = -\frac{27(M^2 - 1)^2}{32M^2}. \tag{16}$$

The ill-behaved nature of (16) shows that Eqs. (1) and (4) do not possess double-humped soliton solutions.

(ii) *A Lagrangian formalism*

In view of the nondimensionalisation of the various quantities described below Eqs. (1) and (2) before, and the stationary propagation of the waves, we may replace the t-derivatives in the third and fourth terms on the l. h. s. in Eq. (4) by the x-derivatives, redefine E suitably, and write Eq. (4) in the form

$$n_{tt} - n_{xx} - n_{xxxx} + (n^2)_{xx} = \frac{1}{2}(|E|^2)_{xx}. \tag{17}$$

Introducing

$$n = \psi_{xx} \tag{18}$$

and proceeding along the lines of the treatment for Zakharov's equations given by Gibbons et al.5), we find that the Lagrangian for the system of Eqs. (1) and (17) is given by

$$L = \frac{i\varepsilon}{2}(EE_t^* - E^*E_t) + \frac{3}{2}|E_x|^2 - \frac{1}{2}\psi_{xx}|E|^2 + \frac{1}{2}\psi_{tt}\psi_{xx} -$$

$$- \frac{1}{2}\psi_{xx}^2 + \frac{1}{2}\psi_{xxx}^2 + \frac{1}{3}\psi_{xx}^3 . \tag{19}$$

Using (19), the »momenta« conjugate to the variables E, E^* and ψ_t are given by

$$\pi_E = \frac{\partial L}{\partial E_t} = - \frac{i\varepsilon}{2} E^*$$

$$\pi_{E^*} = \frac{\partial L}{\partial E_t^*} = \frac{i\varepsilon}{2} E$$

$$\pi_{\psi_t} = \frac{\partial L}{\partial \psi_{tt}} = \frac{1}{2}\psi_{xx} = \frac{n}{2}. \tag{20}$$

Using (19) and (20), the Hamiltonian for the system of Eqs. (1) and (17) is given by

$$H = \pi_E E_t + \pi_{E^*} E_t^* + \pi_{\psi_t}\psi_{tt} - L =$$

$$= \frac{1}{2}\left(- \frac{3}{2}|E_x|^2 + \frac{1}{2}n|E|^2 + \frac{1}{2}n^2 - \frac{1}{2}n_x^2 - \frac{1}{3}n^3\right). \tag{21}$$

Using (21), Hamilton's »equation of motion« is

$$\psi_{tt} = \frac{\delta H}{\delta \pi_{\psi_t}} = -2\frac{\delta H}{\delta n} = \frac{1}{2}|E|^2 + n + n_{xx} - n^2 \tag{22a}$$

from which one obtains

$$n_{tt} - n_{xx} - n_{xxxx} + (n^2)_{xx} = \frac{1}{2}(|E|^2)_{xx}. \tag{22b}$$

Next,

$$(\pi_{E^*})_t = \mathrm{i}\varepsilon \frac{E_t}{2} = -\frac{\delta H}{\delta E} = \frac{1}{2}\left(-\frac{3}{2}E_{xx} + \frac{n}{2}E\right) \tag{23a}$$

from which one obtains

$$\mathrm{i}\varepsilon E_t + \frac{3}{2}E_{xx} = \frac{n}{2}E. \tag{23b}$$

The »constants of motion« are given by

$$N = \int_{-\infty}^{\infty} |E|^2 \, \mathrm{d}x$$

$$P = \int_{-\infty}^{\infty} [\mathrm{i}(EE_x^* - E^*E_x) + n^2] \, \mathrm{d}x$$

$$H = \frac{1}{2}\int_{-\infty}^{\infty}\left[-\frac{3}{2}|E_x|^2 + \frac{1}{2}n|E|^2 + \frac{1}{2}n^2 - \frac{1}{2}n_x^2 - \frac{1}{3}n^3\right]\mathrm{d}x. \tag{24}$$

N is called the plasmon number. The fact that the energy of an electrostatic field in vacuum is proportional to $\int |E\mathrm{e}^{-\mathrm{i}\omega t}|^2 \, \mathrm{d}x$ suggests that N may be the zeroth-order energy of the system (Gibbons et al.[5]), whilst H is the first-order correction resulting from nonzero wavenumbers of the electric field and from ion-density perturbations. P is the (linear) momentum.

Using the exact solution given by (5), (8), (11) and (12), one may verify that unlike the case with the system of Eqs. (1) and (2), (for which a Lagrangian formalism was developed by Gibbons et al.[5]), the »constants of motion« for the solution (5), (8) and (11) of the system ofEqs. (1) and (4) do not vanish as $M \Rightarrow 0$.

3. *The model equations of Nishikawa et al.*

Upon including both the convective nonlinearity in the equations governing the ion motions and the nonlinearity arising from the Boltzmann distribution of the electrons, one obtains

$$n_t + nn_x + \frac{1}{2}n_{xxx} = -\frac{1}{2}(|E|^2)_x. \tag{25}$$

(i) *Exact solutions*

Look for stationary solutions of the form,

$$E(x, t) = \mathscr{E}(\xi)\, \mathrm{e}^{-\mathrm{i}\left(\frac{\varepsilon M x}{3} + \sigma t\right)} \tag{26}$$

$$n(x, t) = n(\xi) \tag{27}$$

where,

$$\xi = x + Mt.$$

Then, Eqs. (1) and (25) give

$$\mathscr{E}'' - \frac{2}{3}\left(\frac{\varepsilon^2 M^2}{3} - \varepsilon\sigma\right)\mathscr{E} - \frac{n}{3}\mathscr{E} = 0 \tag{28}$$

$$Mn' + nn' + \frac{1}{2} n''' = -\mathscr{E}\mathscr{E}' \tag{29}$$

where primes denote differentiation with respect to ξ.

Look for a double-humped soliton solution of the form

$$\begin{aligned} \mathscr{E} &= \gamma \operatorname{sech} \beta\gamma \quad \tanh \beta\xi \\ n &= a \operatorname{sech}^2 \beta\xi \end{aligned} \tag{30}$$

so that Eqs. (28) and (29) give

$$\left[-5\gamma\beta^2 + \frac{2}{3}\left(\frac{\varepsilon^2 M^2}{6} - \varepsilon\sigma\right)\gamma + \frac{1}{3}a\gamma\right] \tanh \beta\xi + \\ + \left(6\gamma\beta^2 + \frac{1}{3}a\gamma\right) \tanh^3 \beta\xi = 0 \tag{31}$$

$$(-2a\beta M - 2a^2\beta + 8a\beta^3 + \gamma^2\beta) \tanh \beta\xi + (2a\beta M + 4a^2\beta - \\ - 20a\beta^3 - 3\gamma^2\beta) \tanh^3 \beta\xi + (-2a^2\beta + 12a\beta^3 + 2\gamma^2\beta) \tanh^5 \beta\xi = 0 \tag{32}$$

from which one obtains

$$a = -\frac{9}{20} M, \quad \gamma^2 = \frac{27}{100} M^2, \quad \beta^2 = \frac{1}{40} M. \tag{33}$$

This solution is qualitatively but not quantitatively the same as the one given by Nishikawa et al.[3)] because of different values of the parameters. Further, a major difference is the possibility of a supersonic propagation indicated by the present solution whereas the one given by Nishikawa et al.[3)] shows only a subsonic propagation. A few more observations of the solution given by (26), (27), (30) and (33) are in order:

(a) this solution is well-behaved for $M \approx 1$, but vanishes for $M \Rightarrow 0$;

(b) unlike the single-humped soliton solutions given in Sections 1 and 2, this solution has only one free parameter, namely, M. However, this solution is of a special type satisfying

$$\xi = 0, \; E = 0 \tag{34}$$

so that the loss of the other free parameter, namely, $\varepsilon\sigma$, may appear to be a consequence of this special feature.

(c) like the solution (8) given by Makhankov's model, the density perturbation is linearly proportional to the high-frequency electric field amplitude as $M \Rightarrow 1$, in contrast to the second power of the latter in Zakharov's solution (3). Therefore, in this model a substantial density cavity is produced by a relatively weak high-frequency electric field.

Let us now try a single-humped soliton solution of the form

$$\mathscr{E} = \gamma \operatorname{sech} \beta\xi, \quad n = a \operatorname{sech}^2 \beta\xi \tag{35}$$

so that Eqs. (28) and (29) give

$$\left[-\frac{2}{3}\left(\frac{\varepsilon^2 M^2}{6} - \varepsilon\sigma\right)\gamma - \frac{1}{3}\cdot a\gamma - \gamma\beta^2\right] + \left(2\gamma\beta^2 + \frac{1}{3}a\gamma\right)\tanh^2 \beta\xi = 0 \tag{36}$$

$$\begin{aligned}[2a\beta M - 2a^2\beta + 16a\beta^3 - \gamma^2\beta]\tanh \beta\xi + [-2a\beta M + 4a^2\beta - 40a\beta^3 + \\ + \gamma^2\beta]\tanh^3 \beta\xi + (-2a^2\beta + 24a\beta^2)\tanh^5 \beta\xi = 0\end{aligned} \tag{37}$$

which clearly shows that a single-humped soliton solution is not compatible with the system (28) and (29).

(ii) *A Lagrangian formalism*

Introducing

$$n = \psi_x$$

the Lagrangian for the system of Eqs. (1) and (25) is given by

$$L = \frac{i\varepsilon}{2}(EE_t^* - E^*E_t) + \frac{3}{2}|E_x|^2 + \frac{1}{2}\psi_x|E|^2 + \frac{1}{2}\psi_t\psi_x + \frac{1}{6}\psi_x^3 - \frac{1}{4}\psi_{xx}^2 . \tag{38}$$

Using (38), the »momenta« conjugate to the variables E, E^* and ψ are given by

$$\pi_E = \frac{\partial L}{\partial E_t} = -\mathrm{i}\varepsilon \frac{E^*}{2}$$

$$\pi_{E^*} = \frac{\partial L}{\partial E^*_t} = \mathrm{i}\varepsilon \frac{E}{2} .$$

$$\pi_\psi = \frac{\partial L}{\partial \psi_t} = \frac{1}{2}\psi_x = \frac{n}{2}. \qquad (39)$$

Using (38) and (39), the Hamiltonian for the system of Eqs. (1) and (25) is given by

$$H = \pi_E E_t + \pi_{E^*} E^*_t + \pi_\psi \psi_t - L = \frac{1}{2}\left(-\frac{3}{2}|E_x|^2 - \frac{1}{2} n |E|^2 - \frac{1}{6} n^3 + \frac{1}{4} n_x^2\right). \qquad (40)$$

Using (40), Hamilton's »equation of motion« is

$$\psi_t = \frac{\delta H}{\delta \pi_\psi} = 2\frac{\delta H}{\delta n} = -\frac{1}{2}|E|^2 - \frac{1}{2} n^2 - \frac{1}{2} n_{xx} \qquad (41a)$$

from which one obtains

$$n_t + n n_x + \frac{1}{2} n_{xxx} = -\frac{1}{2}(|E|^2)_x. \qquad (41b)$$

Next,

$$(\pi_{E^*})_t = \mathrm{i}\varepsilon \frac{E_t}{2} = -\frac{\delta H}{\delta E} = \frac{1}{2}\left(-\frac{3}{2} E_{xx} + \frac{n}{2} E\right) \qquad (42a)$$

from which one obtains

$$\mathrm{i}\varepsilon E_t + \frac{3}{2} E_{xx} = \frac{n}{2} E. \qquad (42b)$$

The »constants of motion« are given by

$$N = \int_{-\infty}^{\infty} |E|^2 \, \mathrm{d}x$$

$$P = \int_{-\infty}^{\infty} \frac{1}{2} (E E^*_x - E^* E_x) + n^2] \, \mathrm{d}x$$

$$H = \int_{-\infty}^{\infty} \frac{1}{2}\left[-\frac{3}{2} E_x^2 - \frac{1}{2} n |E|^2 - \frac{1}{6} n^3 + \frac{1}{4} n_x^2\right] \mathrm{d}x. \qquad (43)$$

Using the exact solution given by (26), (27), (30) and (33), it is possible to verify that unlike the case with the single-humped soliton solutions, the »constants of motion« for the double-humped soliton solution vanish as $M \Rightarrow 0$. This further shows that the double-humped soliton solution, though well-behaved as $M \Rightarrow 1$, is not valid as $M \Rightarrow 0$.

Acknowledgments

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References

1) V. E. Zakharov, Sov. Phys. JETP **35** (1972) 908;
2) V. G. Makhankov, Phys. Lett. **50A** (1974) 42;
3) K. Nishikawa, H. Hojo, K. Mima and H. Ikezi, Phys. Rev. Lett. **33** (1974) 149;
4) N. N. Rao and R. K. Varma, J. Plasma Phys. **27** (1982) 195;
5) J. Gibbons, S. G. Thornhill, M. J. Wardrop and D. ter Haar, J. Plasma Phys. **17** (1977) 153.

TEORIJA LANGMUIROVIH SOLITONA

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Razmatra se ponašanje Langmuirovih solitona u plazmi uzimajući u obzir nelinearne efekte koji potiču od iona. Primjenjuje se Lagrangeov formalizam a posebno se ispituje ponašanje solitonskih rješenja kada su odgovarajući Machovi brojevi bliski jedinici.