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ON THE SOLITARY WAVE SOLUTIONS TO NON-LOCAL K-P EQUATION

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We have analysed the formation and modulation of solitary waves for the new nonlinear equations — the non-local Kadomstev — Petviashvillee equation, with the help of multiple-scale Fourier transform technique. This equation is a generalisation of both the BO equation and KdV equation in three dimensions. We have obtained the equation satisfied by the modulated amplitude and hence the corresponding solitary wave solutions.

1. Introduction

In a recent communication Dobrokhotov1) has deduced a new nonlinear equation in three dimension from the equation of hydrodynamics. This equation reads

$$H(u) = P \int_{-\infty}^{\infty} \frac{u(x', y, t)}{x - x'} \cdot \mathrm{d}x'$$

$$u_t = 2(\mathrm{i}\delta)^{-1} H D^2 u + 2uDu + 4\delta^{-2} Du$$

with

$$Du = u_x - u_y$$

where H denotes the standard Hilbert transform integral.

As $\delta \to 0$, this equation reduces to the Kadomstev-Petviashwillee equation. So we can consider this equation as a generalisation of both the Kortweg-de Vries equation and Boussinesq equation in three dimensions. It is still not known that this equation possesses a Lax pair. But the existence of the Hilbert transform like operator makes a direct approach for solitary wave solutions impossible. So we have applied the technique of multiple scale Fourier transform[2)] to study the formation and modulation of stable wave forms since, for this equation, a direct approach is impossible.

2. *Formulation*

Let us start by writing the equation in the form

$$u_t = 2a\, D^2\, H(u) + 2u\, Du + 4n\, Du \tag{1}$$

where the constants a and n are defined as

$$a = (\mathrm{i}\delta)^{-1}$$

$$n = \delta^{-2}$$

and

$$H(u) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(x', t)}{x - x'} \cdot \mathrm{d}x'.$$

As usual the Fourier transform of $u(x, y, t)$ is defined by

$$u(x, y, t) = \int u(k, q, t)\, \mathrm{e}^{\mathrm{i}(kx + qy)}\, \mathrm{d}k \cdot \mathrm{d}q. \tag{2}$$

In the approach of multiple scale we define the scaled variables

$$x_n = \varepsilon^n \cdot x; \qquad y_n = \varepsilon^n \cdot y; \qquad t_n = \varepsilon^n \cdot t$$

and the extended function

$$u(x_0, y_0, t_0, x_1, y_1, t_1, \ldots)$$

whence the corresponding Fourier transformation is defined through

$$u(x_0, y_0, t_0, x_1, y_1, t_1, \ldots) = \int_{-\infty}^{\infty} u(k_0, q_0, t_0, k_1, q_1, t_1, \ldots)\, \mathrm{e}^{\mathrm{i}\sum\limits_{n=0}^{N} (k_n x_n + q_n y_n)} \quad \Pi\, \mathrm{d}k_n\, \mathrm{d}q_n \tag{3}$$

k_n, q_n denoting the corresponding wave numbers. Using Eq. (2) we first convert Eq. (1) into the form

$$f(k, q) = 4a(k^2 + q^2) - 4n(k - q)$$

$$\frac{\partial u(k, q, t)}{\partial t} + \mathrm{i} f(k, q)\, u(k, q, t) = \mathrm{i}(k - q) \iint u(k', q', t)\, u(k - k', q - q', t) \cdot \\ \cdot \, \mathrm{d}k' \mathrm{d}q'. \tag{4}$$

It is then easy to extend Eq. (4) when the extended function $u(x_i, y_i, t_i)$ is used.

We simply get

$$\left(\frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \dots\right) u(k_0, q_0, t_0, k_1, q_1, t_1, \dots) + \\ + \mathrm{i} f(k_0 + \varepsilon k_1 + \dots, q_0 + \varepsilon q_1 + \dots)\, u(k_0, q_0, t_0, k_1, q_1, t_1, \dots) = \\ = \mathrm{i}\{(k_0 + \varepsilon k_1 + \dots) - (q_0 + \varepsilon q_1 + \dots)\} \times \iint u(k_0, q_0, t_0, \dots)\, u(k_0 - k_0', q_0 - \\ - q_0', \dots)\, \Pi\, \mathrm{d}k_n' \mathrm{d}q_n'. \tag{5}$$

By expanding each of the functions according to

$$f(k_0 + \varepsilon k_1 + \dots, q_0 + \varepsilon q_1 + \dots) = f(k_0, q_0) + \varepsilon \left\{k_1 \frac{\partial f}{\partial k_0} + q_1 \frac{\partial f}{\partial q_0}\right\} + \\ + \varepsilon^2 \left\{k_2 \frac{\partial f}{\partial k_0} + q_2 \frac{\partial f}{\partial q_0} + \frac{1}{2} k_1^2 \frac{\partial^2 f}{\partial k_0^2} + k_1 q_1 \frac{\partial^2 f}{\partial k_0 \partial q_0} + \frac{1}{2} q_1^2 \frac{\partial^2 f}{\partial q_0^2}\right\} + \dots \tag{6}$$

and collecting different powers of ε, we can obtain equations for the amplitude modulation in various order. But our main intention is to obtain an equation which governs the slow spatial — temporal variations of the amplitude of the wave train not in the Fourier space but in the physical space time. But by taking recourse to the Fourier transform we were able to convert the original integro-differential equation into a sequence of partial differential equations. Taking inverse Fourier transform of (6) we get

$$\sum_{n=0}^{N} \varepsilon^n \cdot L_n(k_0, q_0)\, u(k_0, q_0, t_0, k_1, q_1, t_1, \dots) = \\ = \mathrm{i}(k_0 - q_0) - \sum_{n=1}^{N} \varepsilon^n \left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial y_n}\right) \int u(k_0', q_0', t_0', k_1', q_1', \dots) \times \\ \times u(k_0 - k_0', q_0 - q_0', k_1 - k_1', \dots)\, \mathrm{d}k_0' \mathrm{d}q_0' \tag{7}$$

with

$$L_0 = \frac{\partial}{\partial t_0} + \mathrm{i} f(k_0, q_0)$$

$$L_1 = \frac{\partial}{\partial t_1} + f'(k_0, q_0) \frac{\partial}{\partial x_1} + f'(k_0, q_0) \frac{\partial}{\partial y_1} \tag{7a}$$

$$L_2 = \frac{\partial}{\partial t_2} + \left(\frac{\partial f}{\partial k_0} \cdot \frac{\partial}{\partial x_2} + \frac{\partial f}{\partial q_0} \cdot \frac{\partial}{\partial y_2}\right) - \frac{\mathrm{i}}{2}\left(\frac{\partial^2 f}{\partial k_0^2} \cdot \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2 f}{\partial q_0^2} \cdot \frac{\partial^2}{\partial y_0^2}\right) -$$

$$- \mathrm{i}\left(\frac{\partial^2 f}{\partial k_0 \cdot \partial q_0} \cdot \frac{\partial^2}{\partial x_1 \cdot \partial y_1}\right). \tag{7b}$$

For effective computation in asymptotic perturbation we expand the nonlinear field as

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \dots \tag{8}$$

Substituting (8) in (7) and equating like powers of ε we obtain, in various order of ε, the governing equations for the expansion coefficients u_i.

From the linear term in ε we get

$$L_0 (k_0, q_0)\, u_1 (k_0, q_0) = 0$$

or

$$\left[\frac{\partial}{\partial t_0} + \mathrm{i} f(k_0, q_0)\right] u_1 (k_0, q_0) = 0. \tag{9}$$

Hereafter for simplicity of expressions we will not write down the spatial and temporal variables in the argument of u_n. As a solution to Eq. (9) which is appropriate to nonlinear self-modulation problem we obtain³⁾

$$u_1 (k_0, q_0)' = \mathrm{e}^{-\mathrm{i}\omega t_0}\, b(x_1, y_1, t_1, \dots)\, \delta(k_0 - k)\, \delta(q_0 - q) + b^*(x_1, y_1, t_1, \dots) \cdot$$

$$\cdot\, \mathrm{e}^{\mathrm{i}\omega t_0}\, \delta(k_0 + k)\, \delta(q_0 + q) \tag{10}$$

which corresponds to a quasi-monochromatic wave with specified wave numbers k and q and frequency ω satisfying the linear dispersion relation

$$\omega = f(k, q) = 4a(k^2 + q^2) - 4(k - q).$$

Equating coefficient of ε^2 we get

$$L_0 u_2 + L_1 u_1 = \mathrm{i}(k_0 - q_0) \int u_1 (k_0', q_0', t)\, u_1 (k_0 - k_0', q_0' - q_0, t)\, \mathrm{d}k_0'\, \mathrm{d}q_0' \tag{11}$$

where Eq. (10) is to be used for u_1.

If we impose the non-secularity condition for u_2 which is

$$\frac{\partial b}{\partial t_1}+\frac{\partial f}{\partial k_0}\cdot\frac{\partial b}{\partial x_1}+\frac{\partial f}{\partial q_0}\cdot\frac{\partial b}{\partial y_1}=0 \tag{12}$$

which leads to

$$\left[\frac{\partial}{\partial t_0}+\mathrm{i}\,f(k_0,q_0)\right]u(k_0,q_0)=2\mathrm{i}\,(k-q)\,b^2\,\mathrm{e}^{-2\mathrm{i}\omega t_0}\,\delta(k_0-2k)\,\delta(q_0-2q)+$$
$$+\,2\,|b^2|\;\delta k_0\cdot\delta q_0+2\mathrm{i}\,(k-q)\,b^{*2}\,\mathrm{e}^{2\mathrm{i}\omega t_0}\,\delta(k_0+2k)\,\delta(q_0+2q). \tag{13}$$

Solving Eq. (13) we get:

$$u_2\,(2k,2q)=\frac{(k-q)\,b^2}{4a\,(k^2+q^2)}\cdot\mathrm{e}^{-2\mathrm{i}\omega t_0} \tag{14}$$

$$u_2\,(k,q)=0$$

and $u_2\,(0,0)=\Phi_2$ (say) where Φ is an arbitrary function of space-time variables $(x_1, y_1, t_1, \ldots)$, so that we can write

$$u_2=\frac{(k-q)\,b^2}{4a\,(k^2-q^2)}\cdot\mathrm{e}^{-2\mathrm{i}\omega t_0}\,\delta(k_0-2k)\,\delta(q_0-2q)+\frac{(k-q)\,b^{*2}}{4a\,(k^2+q^2)}\cdot$$
$$\cdot\,\mathrm{e}^{2\mathrm{i}\omega t_0}\cdot\delta(k_0+2k)\,\delta(q_0+2q)+\Phi_2\cdot\delta k_{0,0}\cdot\delta q_{0,0}. \tag{15}$$

Next equating coefficients of ε^3 equation for u_3 is obtained in the following form

$$L_0u_3+L_1u_2+L_2u_1=2\mathrm{i}\,(k_0-q_0)\int u_1\,(k_0',q_0',t_0)\,u_2\,(k_0-k_0',q_0-q_0',t_0)\,\cdot$$
$$\cdot\,\mathrm{d}k_0'\cdot\mathrm{d}q_0'-\left(\frac{\partial}{\partial x_n}-\frac{\partial}{\partial y_n}\right)\int u_1\,(k_0',q_0',t_0)\,u\,(k_0-k_0',q_0-q_0',t_0)\,\mathrm{d}k_0'\cdot\mathrm{d}q_0'. \tag{16}$$

Solving Eq. (16) and taking care of the non secularity condition we obtain

$$\frac{\partial b}{\partial t_1}+\frac{\partial\omega}{\partial k}\cdot\frac{\partial b}{\partial x_1}+\frac{\partial\omega}{\partial q}\cdot\frac{\partial b}{\partial y_1}=0 \tag{17}$$

and

$$u_3=\frac{b^3\,(k-q)^2}{16a^2\,(k^2+q^2)^2}\cdot\mathrm{e}^{-3\mathrm{i}\omega t_0}\,\delta(k_0-3k)\,\delta(q_0-3q)+$$
$$+\,\mathrm{i}b\left(\frac{\partial b}{\partial x_1}-\frac{\partial b}{\partial y_1}\right)\frac{\mathrm{e}^{-2\mathrm{i}\omega t_0}}{8\,(k^2+q^2)}\cdot\delta(k_0-2k)\,\delta(q_0-2)q+\Phi_3\cdot\delta k_0\cdot\delta q_0 \tag{18}$$

along with

$$\frac{\partial \Phi_2}{\partial t_1} = -2\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}\right)|b|^2. \tag{19}$$

Lastly we deduce the secularity condition in the variables (x_2, y_2, t_2) as

$$\frac{1}{i}\left(\frac{\partial b}{\partial t_2} + \frac{\partial \omega}{\partial k}\cdot\frac{\partial b}{\partial x_2} + \frac{\partial \omega}{\partial q}\cdot\frac{\partial b}{\partial y_2}\right) - \frac{i}{2}\left(\frac{\partial^2 \omega}{\partial k^2}\cdot\frac{\partial^2 b}{\partial x_1^2} + \frac{\partial^2 \omega}{\partial q^2}\cdot\frac{\partial^2 b}{\partial y_1^2}\right) =$$

$$= \frac{(k-q)\, b\, |b|^2}{20\,(k^2+q^2)} + 2k(k-q)\cdot b\Phi_2. \tag{20}$$

Defining new variables

$$t' = t - ax - \beta y$$

and eliminating Φ_2 from (20) and (19) we get

$$\frac{\partial}{\partial t}\left[i\frac{\partial b}{\partial t} + A\frac{\partial^2 b}{\partial x^2} + B\frac{\partial^2 b}{\partial y^2} + c\,|b|^2\, b\right] = Db\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)|b|^2,$$

where

$$A = 4a;\quad B = 4a;\quad C = \frac{(k-q)^2}{2a\,(k^2+q^2)};\quad D = 4\,(k-q) \tag{21}$$

Eq. (21) is the required nonlinear modulational equation for the amplitude in 2 space and one time dimensions.

3. *Solution of the equation*

We seek a solution of (21) in the form

$$b = g\,(x, y)\, e^{-i\Omega t} = \xi\,(x, y)\, e^{i\eta(x,y) - i\Omega t}. \tag{22}$$

Furthermore we assume that the amplitude ξ and phase η have the simple dependence

$$\eta = \eta\,(x - \lambda_1 y)$$

and

$$\xi = \xi\,(x - \lambda_2 y)$$

so that Eq. (21) yields

$$\eta_x = \frac{D(1+\lambda_1)}{2\Omega(A+B\lambda^2)} \cdot \xi^2 \tag{23}$$

$$\xi_{xx} = A_1\xi + B_1\xi^3 + C_1\xi^5 \tag{24}$$

where

$$A_1 = -\frac{\Omega}{A+B\lambda_1^2}; \quad B_1 = -\frac{C}{A+B\lambda_1^2}$$

$$C_1 = \frac{\mu}{A+B\lambda_1^2}; \quad \mu = \frac{D^2(1+\lambda_1^2)}{4\Omega^2(A+B\lambda_1^2)}.$$

Integrating (24) according to the usual procedure we get

$$x - \lambda_2 y = \int_0^{\xi} \frac{\mathrm{d}\varrho}{\sqrt{V(\varrho)}} \tag{25}$$

where:

$$V(\varrho) = C_3\varrho + 4A_1\varrho^2 + 2B_1\varrho^3 + \frac{4}{3}C_1\varrho^4.$$

Eq. (25) indicates that the integral involved is of elliptic type. To integrate we are to analyse the roots of the polynomial $V(\varrho) = 0$. Let the roots be $\varrho_1, \varrho_2, \varrho_3, \varrho_4$. Then from the symmetric function of the roots, we can ascertain that at least one of the roots may vanish. If we make a substitution$^{4)}$

$$k'^2 = \frac{(\varrho_2-\varrho_3)(\varrho_1-\varrho_4)}{(\varrho_1-\varrho_3)(\varrho_2-\varrho_4)} \tag{26}$$

then the solution is

$$\sqrt{\frac{4C_1}{3}}(x-\lambda_2 y) = \frac{2}{\sqrt{(\varrho_1-\varrho_3)(\varrho_2-\varrho_4)}} S_n^{-1} \sqrt{\frac{(\varrho_2-\varrho_4)(\varrho-\varrho_3)}{(\varrho_2-\varrho_3)(\varrho-\varrho_4)}}.$$

If we let

$$\frac{1}{2}\left\{\frac{4C_1}{3}(\varrho_1-\varrho_3)(\varrho_2-\varrho_4)\right\}^{1/2}(x-\lambda_2 y) = \delta(x,y) \tag{26a}$$

then,

$$S_n^2(\delta,k) = \frac{(\varrho_2-\varrho_4)(\varrho-\varrho_3)}{(\varrho_2-\varrho_3)(\varrho-\varrho_4)} \tag{27}$$

whose solution yields

$$\varrho = \frac{\varrho_4 (\varrho_2 - \varrho_3) S_n^2 (\delta, k) - (\varrho_2 - \varrho_4) \varrho_3}{(\varrho_2 - \varrho_3) S_n^2 (\delta, k) - (\varrho_2 - \varrho_4)}. \tag{28}$$

Eq. (29) gives the wave solution of the original nonlinear BO equation. In the limit $\varrho_1 \to \varrho_2$ and $Sn^{-1} \to \tanh^{-1}$ we get:

$$\varrho = \frac{\varrho_4 (\varrho_2 - \varrho_3) \tanh^2 \delta - (\varrho_2 - \varrho_4) \varrho_1}{(\varrho_2 - \varrho_3) \tanh^2 \delta - (\varrho_2 - \varrho_4)}. \tag{29}$$

Now from Eq. (23) we get

$$\eta_x = \mu_1 \frac{\varrho_4 (\varrho_2 - \varrho_3) \tanh^2 \delta - (\varrho_2 - \varrho_4) \varrho_2}{(\varrho_2 - \varrho_3) \tanh^2 \delta - (\varrho_2 - \varrho_4)}$$

with

$$\mu_1 = \frac{D (1 + \lambda_1)}{2\Omega (A + B\lambda^2)}$$

so that

$$\eta_x = \frac{\omega_1 \tanh^2 \delta (x, y) - \omega_2}{\omega_3 \tanh^2 \delta (x, y) - \omega_4}$$

$$\omega_1 = \mu_1 \varrho_4 (\varrho_2 - \varrho_3) \tag{30}$$

$$\omega_2 = \mu_1 \varrho_1 (\varrho_2 - \varrho_4)$$

$$\omega_3 = \varrho_2 - \varrho_3$$

$$\omega_4 = \varrho_2 - \varrho_4.$$

On integrating (30) we obtain

$$\eta = \omega_6 x + \frac{\omega_5}{2} \ln \frac{1 + \tanh \delta (x, y)}{1 - \tanh \delta (x, y)} - \frac{\omega_5}{2\omega_7} \ln \frac{\omega_7 \mp \tanh \delta (x, y)}{\omega_7 - \tanh \delta (x, y)}$$

where

$$\omega_5 = \left(\frac{\omega_1 \omega_4}{\omega_3} - \omega_2 \right) \frac{1}{\omega_3 - \omega_4}$$

$$\omega_6 = \frac{\omega_1}{\omega_3}$$

$$\omega_7 = \sqrt{\frac{\omega_4}{\omega_3}}.$$

To get the actual structure of the solitary wave form, we make a possible simplified choice of the constants involved. We let $\varrho_3 = 0$ so that the roots ϱ_1, ϱ_2, ϱ_4 are determined from the following set of equations

$$2\varrho_2 + \varrho_4 = -\frac{3C}{2\mu}$$

$$\varrho_2^2 = -\frac{3C_3}{4C_1}$$

$$\varrho_2^2 + 2\varrho_2\varrho_4 = \frac{3A_1}{C_1}.$$

Then δ, defined by (26a) is of the form

$$\delta = \frac{1}{2}\left\{\frac{\mu\varrho_2}{6a(1+\lambda_1^2)}\left(3\varrho_2 - \frac{3C}{2\mu}\right)\right\}^{1/2}(x - \lambda_2 y)$$

assigning some admissible numerical values to these parameters we have plotted both the x and y section of the three dimensional static soliton. Such diagrams are displayed in Figs. 1 and 2.

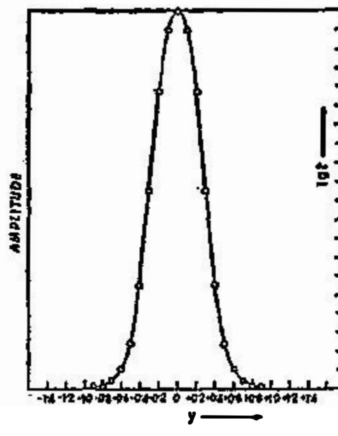

Fig. 1. Plot of amplitude vs. y.

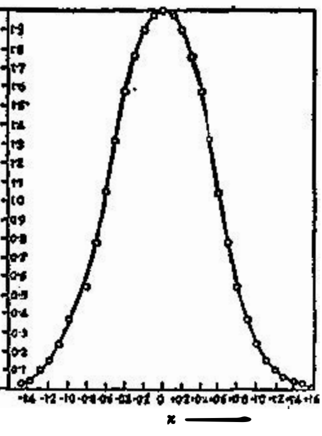

Fig. 2. Plot of amplitude vs. x.

In our above computations we have shown that the method of multiple scale Fourier transform can yield both periodic and solitary wave like solutions of non-local KP equation. To ascertain the structure of these we have plotted the amplitude of the $x = 0$ and $y = 0$ section of the solitary wave in Figs. 1 and 2 which clearly depict the structure of it in the form of a inverted bell shape. Finally, we may add that in absence of any inverse problem the reductive perturbation like approach, which we have used here is the only technique for solving such nonlinear non-local equation.

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SOLITONSKA RJEŠENJA NELOKALNE K-P JEDNADŽBE

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Pomoću višestruko baždarske Fourierove tehnike analizirali smo formiranje i modulaciju solitonskih valova kod nelinearne i nelokalne jednadžbe Kadomstev-Petviashvilleea. Ta jednadžba je generalizacija BO i KdV jednadžbe na tri dimenzije. Mi smo dobili jednadžbu koju zadovoljava amplitudno modulirani solitonski val.