YU ISSN 0015-3206
FZKAAA 19 (3) 319 (1987)

LETTER TO THE EDITOR

# ON THE CONSERVATION LAWS AND LIE SYMMETRY FOR THE POISSON EQUATION

SRIKANTA SASMAL and ASESH ROY CHOWDHURY

*High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta—700 032, India*

Received 9 May 1986

Revised manuscript received 20 June 1987

UDC 530.182

Original scientific paper

We have obtained the Lie point symmetries and the reduced ordinary differential equation for the Poisson equation. Some of the conservation laws are also deduced by the help of the technique of differential forms. These conservation laws are useful for studying the interaction and evolution of soliton like waves in the system.

Analysis of Lie point symmetries for various non-linear equations gives important clues regarding many properties of the equation. Reduction to Painleve class in certain cases also gives an indication regarding the complete integrability, of late many such equations have been analysed in this outlook. Here we report of an analysis of the nonlinear Poisson equation through Lie symmetry. We also deduce some of the conservation laws associated with the equation through the use of the differential forms, which are useful in discussing the interaction of solitary waves.

The equation under consideration reads

$$\Phi_{tt} + 2\Phi_x \Phi_{xt} - (1 - \Phi_x^2)\,\Phi_{xx} = 0. \tag{1}$$

Let us consider the transformation

$$\left.\begin{aligned} \Phi^* &= \Phi + \varepsilon\,\eta \\ x^* &= x + \varepsilon\,\zeta \\ t^* &= t + \varepsilon\,\tau. \end{aligned}\right\} \tag{2}$$

and demand invariance of (1) under (2) which yields

$$\begin{aligned} \zeta &= x\lambda + \mu \\ \tau &= t\lambda + \nu \\ \eta &= \Phi\lambda + \lambda t + h. \end{aligned} \tag{3}$$

The Lagrange equations

$$\frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}x}{\zeta} = \frac{\mathrm{d}\Phi}{\eta}. \tag{4}$$

upon integration yields

$$c_1 = \frac{\lambda t + \nu}{\lambda x + \mu}$$

and

$$\Phi = \frac{\lambda t + \nu}{\lambda} \log(\lambda t + \nu) - \frac{h - \nu}{\lambda} + Q\left(\frac{\lambda t + \nu}{\lambda x + \mu}\right)(\lambda t + \nu) \tag{5}$$

where $Q$ is an arbitrary function.

Substituting (5) in Eq. (1) we observe that $Q$ thought of as function of

$$\xi = \frac{\lambda t + \nu}{\lambda x + \mu}$$

satisfies the following ordinary first order differential equation:

$$x \frac{\mathrm{d}x}{\mathrm{d}\xi} f_1(\xi) + x_\xi f_2(\xi) + 1 = 0 \tag{6}$$

where

$$\left.\begin{aligned} f_1(\xi) &= (2 + \xi^3)\,\xi \\ f_2(\xi) &= (1 - \xi^2) \\ x &= Q\ \xi^2. \end{aligned}\right\} \tag{7}$$

Hence the generator of Lie point symmetry can be written as

$$X = (\lambda x + \mu)\frac{\partial}{\partial x} + (\lambda t + \nu)\frac{\partial}{\partial t} + (\Phi\lambda + \lambda t + h)\frac{\partial}{\partial \Phi}. \tag{8}$$

From the form of Eq. (6) it is not immediately clear that it is of Painleve type. However, it seems that by some special transformation, it will be possible to solve it. We next proceed to obtain some non-trivial conservation laws which are useful in absence of a Lax pair for study of the interaction of solitary waves.

The equation under consideration can be written in the following form

$$q_t + 2pp_t - (1 - p^2) p_x = 0 \tag{9}$$

if we define the auxiliary variables $(q, p)$ through

$$p = \Phi_x, \quad q = \Phi_t \tag{10}$$

for the purpose of putting it in the language of differential form. It is easily seen that

$$\left.\begin{aligned} a_1 &= \mathrm{d}\Phi \wedge \mathrm{d}t - p\,\mathrm{d}x \wedge \mathrm{d}t \\ \beta_1 &= \mathrm{d}\Phi \wedge \mathrm{d}x - q\,\mathrm{d}t \wedge \mathrm{d}x. \\ \nu_1 &= \mathrm{d}q \wedge \mathrm{d}x + 2p\,\mathrm{d}p \wedge \mathrm{d}x + (1 - p^2)\,\mathrm{d}p \wedge \mathrm{d}t \end{aligned}\right\} \tag{11}$$

yields the equation set under proper sectioning. The search for the conservation law now proceeds through the construction of forms

$$\begin{aligned} \omega &= F\,\mathrm{d}x + G\,\mathrm{d}t. \\ F &= F(p, q, \Phi, x, t) \\ G &= G(p, q, \Phi, x, t), \end{aligned} \tag{12}$$

such that the exterior derivative $\mathrm{d}\omega$ belongs to the closed ideal generated by $\alpha_1$, $\beta_1$, $\nu_1$ and by $\omega$ itself. Written mathematically, this means

$$\mathrm{d}\omega = (\tilde{f}a_1 + \tilde{g}\beta_1 + \tilde{h}\nu_1) + (a\,\mathrm{d}x + b\,\mathrm{d}t) \wedge \omega. \tag{13}$$

Equating coefficients of the basic two forms $\mathrm{d}x \wedge \mathrm{d}t$, $\mathrm{d}x \wedge \mathrm{d}\Phi$, etc. on both sides of Eq. (13) we get equations for $F$ and $G$ (which for the present we assume not to depend on $x$ and $t$ explicitly). Solving these we obtain

$$F = ap^2 + \beta p + \gamma + aq, \qquad G = a\left(p - \frac{1}{3}p^3\right). \tag{14}$$

Other forms of $F$ and $G$ can be generated by applying the same consideration to the differentiated version of Eq. (1). Let us differentiate Eq. (1) with respect to $t$, so that we obtain:

$$\Phi_{ttt} + 2\Phi_{xt}^2 + 2\Phi_x\Phi_{xtt} - (1 - \Phi_x^2)\,\Phi_{xxt} + 2\Phi_x\Phi_{xx}\Phi_{xt} = 0. \tag{15}$$

We define extra variables

$$Z = p_t \text{ or } q_x, \quad \text{and} \quad S = p_x, \; u = q_t,$$

so that the basic set of forms are

$$\alpha_1 = \mathrm{d}\Phi \wedge \mathrm{d}t - p\,\mathrm{d}x \wedge \mathrm{d}t$$

$$\beta_1 = \mathrm{d}\Phi \wedge \mathrm{d}x - q\,\mathrm{d}t \wedge \mathrm{d}x$$

$$\gamma_1 = \mathrm{d}q \wedge \mathrm{d}x + 2p\,\mathrm{d}p \wedge \mathrm{d}x + (1 - p^2)\,\mathrm{d}p \wedge \mathrm{d}t$$

$$\alpha_2 = \mathrm{d}p \wedge \mathrm{d}x - Z\,\mathrm{d}t \wedge \mathrm{d}x$$

$$\beta_2 = \mathrm{d}p \wedge \mathrm{d}t - S\,\mathrm{d}x \wedge \mathrm{d}t$$

$$\gamma_2 = \mathrm{d}u \wedge \mathrm{d}x + (2Z^2 + 2pSZ)\,\mathrm{d}t \wedge \mathrm{d}x + 2p\,\mathrm{d}Z \wedge \mathrm{d}x + (1 - p^2)\,\mathrm{d}Z \wedge \mathrm{d}t.$$

Proceeding as before we get

$$F = \alpha u + \beta p^2 + \beta q + 2\alpha Z p$$

$$G = \beta\left(p - \frac{p^3}{3}\right) - \alpha\,p^2 Z$$

which is linearly independent of the previous one.

The same considerations also applies if we differentiate Eq. (1) with respect to $x$. The result is:

$$F = \alpha Z + \beta p^2 + \beta q + 2\alpha p S$$

$$G = \beta\left(p - \frac{p^3}{3}\right) - \alpha p^2 S.$$

In the above paragraphs we have discussed the symmetry structure and conservation laws associated with the nonlinear Poisson equation. Since until now no Lax pair is known for this system it seems that such considerations are the only means to study its properties.

## References


1) G. W. Bluman and J. D. Cole, *Similarity Methods in Partial Differential Equation*, (Springer--Verlag), Berlin, New-York 1978;
2) E. L. Ince, *Ordinary Differential Equation*, (Dover), New-York, 1936;
3) H. Flander, *Differential Forms*, (Academic Press), New-York, London, 1963.

# ZAKONI SAČUVANJA I LIEVA SIMETRIJA ZA POISSONOVU JEDNADŽBU

SRIKANTA SASMAL i ASESH ROY CHOWDHURY

*High Energy Division, Department of Physics, Jadavpur University, Calcutta — 700 032, India*

UDK 530.182

Originalni znanstveni rad

Dobili smo Lievu simetriju i reduciranu običnu diferencijalnu jednadžbu za Poissonovu jednadžbu. Neki zakoni sačuvanja izvedeni su pomoću tehnike diferencijalnih formi. Ovi zakoni sačuvanja korisni su za studij međudjelovanja i evolucije solitonskih valova u sistemu.