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ERGODIC THEORY AND CONTINUITY OF THE BOWEN-RUELLE MEASURE FOR GEOMETRICAL LORENZ FLOWS+

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In this paper we first derive a theorem about the weak stability of the absolutely continuous invariant measure for some expanding mappings of an interval. Using a regular invariant foliation, we can derive for the geometrical Lorenz flows most of the result which are known up to now for expanding maps of the interval, including our stability result for Bowen-Ruelle measures.

1. Introduction

Up to now there are only very few classes of dissipative flows for which satisfactory ergodic properties have been proven. Apart from the well known and important case of axiom A attractors[3], the geometrical Lorenz flows[1,6] are the only examples of flows which have been studied in some details. One of the most successful method for the investigation of these models is to construct an invariant (strongly stable) foliation[6]. The typical return map turns out to be a skew product over a map of the interval. The attractor of this return map is merely the pinched inverse limit of the associated map of the interval[7,11], which can be identified with the natural extension for most measure theoretic purposes. Thus one can use the large amount of informations which are now available for maps of the interval.

+ This article is dedicated to the memory of Vladimir Jurko Glaser.

In Section 2 of this paper we shall prove a continuity theorem for the invariant measure of some mappings of the interval. A related result was obtained in Ref. 12 using different assumptions. In Section 3 we shall first briefly recall the dynamics of the geometrical Lorenz flows. A slight extension of the result in Ref. 22 about the stability of the skew product description of these flows will then allow us to complete the above mentioned program. That is to say, we shall be able to translate the ergodic properties of the one-dimensional mappings into corresponding properties of the geometrical Lorenz flows. Some of these properties were obtained by Bunimovich and Sinai using an entirely different method[4,5] motivated by Ref. 1.

Our analysis applies on an open set of flows, as follows from the extension of the result in Ref. 22 (a particular case was already described in Refs. 13 and 22). In particular, Theorem 4 of Section 2 can be interpreted as a kind of weak stability property of the invariant measure. All these ideas can be similarly applied to those axiom A attractors which are inverse limits of expanding maps on one dimensional branched manifolds[26,27].

2. *Continuity of the invariant measure*

In this chapter, we shall prove a continuity result (see theorem 4 below) for the invariant measure of some mappings of the interval. Related results, with different hypothesis, were proven in Ref. 12. We shall first recall the definition of a Doeblin-Fortet operator.

Definition: *Let $\mathscr{B}$ and $\mathscr{C}$ be two Banach spaces with the following properties*:

i) $\mathscr{B} \subset \mathscr{C}$, *and the injection from $\mathscr{B}$ into $\mathscr{C}$ is compact.*

ii) *If $\|\cdot\|_{\mathscr{C}}$ (resp. $\|\cdot\|_{\mathscr{B}}$) denote the norm of $\mathscr{C}$ (resp. $\mathscr{B}$), then $\|x\|_{\mathscr{C}} \leq \|x\|_{\mathscr{B}}$ for any element of $\mathscr{B}$.*

iii) *if $(x_n)_{n \in \mathbf{N}} \subset \mathscr{B}$, $x_n \to x$ in $\mathscr{C}$ and $\sup_n \|x_n\|_{\mathscr{B}} = A < +\infty$, then $x \in \mathscr{B}$ and $\|x\|_{\mathscr{B}} \leq \leq A$.*

iv) *$\mathscr{B}$ is dense in $\mathscr{C}$.*

Then a bounded linear operator T from $\mathscr{B}$ to $\mathscr{B}$ and from $\mathscr{C}$ to $\mathscr{C}$ with norm 1 in $\mathscr{C}$ is a Doeblin-Fortet operator[19] *if there is $\alpha \in [0, 1[$, and $L \in \mathbf{R}^+$ such that*

$$\forall x \in \mathscr{B} : \|T x\|_{\mathscr{B}} \leq \alpha \|x\|_{\mathscr{B}} + L \|x\|_{\mathscr{C}}.$$

The following spectral theorem for Doeblin-Fortet operators is due to Ionescu-Tulcéa and Marinaru (see Ref. 19 for a proof).

Theorem 1: *Let $\mathscr{B}$ and $\mathscr{C}$ be two Banach spaces as above, and T a Doeblin-Fortet operator (with respect to $\mathscr{B}$ and $\mathscr{C}$). Then*

1) *The spectrum of T in $\mathscr{C}$ contains only finitely many different eigenvalues (c_i, $1 \leq \leq i \leq r$) of modulus 1. They have finite degeneracy and the corresponding eigenspaces belong to $\mathscr{B}$ (we shall denote by P_i, $1 \leq i \leq r$, the corresponding spectral projections).*

2) *There are numbers K and H in $\mathbf{R}^+$, and $q \in [0, 1[$ such that, if $S = T - \sum_{i=1}^{r} c_i P_i$: $\|P_i\|_{\mathscr{C}} \leq 1$, $\|P_i\|_{\mathscr{B}} \leq K$ and, $\forall n \in \mathbf{N} : \|S^n\|_{\mathscr{C}} \leq K$, $\|S^n\|_{\mathscr{B}} \leq H q^n$.*

We shall now consider a one parameter family of Doeblin-Forter operators.

Theorem 2: Let $\mathscr{B}$ and $\mathscr{C}$ be two Banach spaces satisfying the conditions (i—iv) *of the definition. Let $\varepsilon_0 > 0$ and let T_ε, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ be a one parameter family of Doeblin-Fortet operators for the pair of Banach spaces $\mathscr{B}$ and $\mathscr{C}$. Assume that the constants L and α of the definition, can be chosen uniform in ε. Assume also that* 1 *is a non-degenerate eigenvalue of T_ε for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, and let e_ε denote the corresponding eigenvector of $\mathscr{C}$-norm* 1. *Finally, assume that there is a fixed element β of $\mathscr{C}^*$ such that*

$$\forall \varepsilon \in [-\varepsilon_0, \varepsilon_0], \qquad \langle \beta, e_\varepsilon \rangle = 1$$

and also

$$\|T_0 e_\varepsilon - e_\varepsilon\|_{\mathscr{C}} \to 0 \quad \text{if} \quad \varepsilon \to 0.$$

Then:

$$\|e_\varepsilon - e_0\|_{\mathscr{C}} \to 0 \quad \text{if} \quad \varepsilon \to 0.$$

Proof: Let $(\varepsilon_i)_{i\in\mathbf{N}}$ be a sequence in $[-\varepsilon_0, \varepsilon_0]$ such that $\varepsilon_i \to 0$. From Theorem 1 we deduce

$$\forall \varepsilon \in [-\varepsilon_0, \varepsilon_0] \qquad \|c_\varepsilon\|_{\mathscr{B}} \leq L(1 - \alpha)^{-1},$$

and from compact injection of $\mathscr{B}$ in $\mathscr{C}$, we deduce that some subsequence $(e_{\varepsilon_{i_j}})_{j\in\mathbf{N}}$ converges in $\mathscr{C}$ to a vector e. Therefore, from the continuity of T_0, and the condition

$$\|T_0 e_{\varepsilon_{i_j}} - e_{\varepsilon_{i_j}}\|_{\mathscr{C}} \to 0,$$

we derive $T_0 e = e$. This implies $e = e_0$ since $\langle \beta, e \rangle = 1$ by continuity of β, and since 1 is a non degenerate eigenvalue of T_0. This implies that the only accumulation point of the sequence $(e_{\varepsilon i})_{i\in\mathbf{N}}$ is e_0. Therefore $e_{\varepsilon_i} \to \varepsilon_0$, and $e_\varepsilon \to e_0$ if $\varepsilon \to 0$. Q. E. D.

We want to apply the above results to the case where T is the Perron-Frobenius operator associated to some map τ of the unit interval. To this end, we shall first prove some easy facts about functions of bounded variations.

Lemma 3: Let e be a function of bounded variations on $[0, 1]$, *and assume that $\int_0^1 |e(s)| \, ds \leq 1$. Let c denote the variation of e. Then*

i) $\|e\|_{L_\infty} \leq 1 + c$, *so that if $0 \leq a \leq a + \eta \leq 1$, we have*

$$\int_a^{a+\eta} |e(s)| \, ds \leq (1 + c) \, \eta.$$

ii) *If $r \in L_\infty([0, 1])$, and $\|r\|_{L_\infty} \leq \Theta \leq 1/2$,*

$$\int_\Theta^{1-\Theta} |e(s + r(s)) - e(s)| \, ds \leq 2\Theta (1 + 2c).$$

Proof. If s and s' belong to $[0, 1]$, we have $|e(s) - e(s')| \leq c$, hence

$$|e(s)| \geq ||e(s')| - c| \geq |e(s')| - c.$$

We deduce

$$1 \geq \int_0^1 |e(s)|\, ds \geq |e(s')| - c$$

which implies

$$\forall s' \in [0, 1], \qquad |e(s')| \leq 1 + c.$$

We now prove ii). Assume first $\Theta < 1/4$. With $N = E\left[\frac{1}{2\Theta}\right] - 2$, we have:

$$\int_\Theta^{1-\Theta} |e(s + r(s)) - e(s)|\, ds =$$

$$= \sum_{j=0}^{N} \int_{(1+2j)\Theta}^{(3+2j)\Theta} |e(s + r(s)) - e(s)|\, ds + \int_{(2N+3)\Theta}^{1-\Theta} |e(s + r(s)) - e(s)|\, ds.$$

From i) the last integral is bounded by $2\Theta(1 + c)$. We also have

$$\sum_{j=0}^{N} \int_{(1+2j)\Theta}^{(3+2j)\Theta} |e(s + r(s)) - e(s)|\, ds =$$

$$= \int_\Theta^{3\Theta} \sum_{j=0}^{N} |e(s + 2j\Theta + r(s + 2j\Theta)) - e(s + 2j\Theta)|\, ds.$$

For any fixed $s \in [\Theta, 3\Theta]$, let a_j and b_j $0 \leq j \leq N$ be defined by

$$a_j = s + 2j\Theta, \quad b_j = s + 2j\Theta + r(s + 2j\Theta) \text{ if } r(s + 2j\Theta) \geq 0$$

$$a_j = s + 2j + r(s + 2j\Theta), \quad b_j = s + 2j\Theta \text{ if } r(s + 2j\Theta) \leq 0.$$

We have

$$0 \leq a_1 \leq b_1 \leq a_2 \leq \dots \leq a_N \leq b_N < 1$$

and this implies

$$\forall s \in [\Theta, 3\Theta] : \sum_{j=0}^{N} |e(s + r(s + 2j\Theta) + 2j\Theta) - e(s + 2j\Theta)| \leq c.$$

Therefore

$$\int_\Theta^{1-\Theta} |e(s + r(s)) - e(s)|\, ds \leq 2\Theta(1 + 2c).$$

If $\Theta \geq 1/4$, we observe that for each fixed s with $r(s) \geq 0$,

$$0 \leq s \leq s + r(s) \leq 1,$$

hence

$$|e(s + r(s)) - e(s)| \leq c,$$

and similarly for $r(s) \lesssim 0$. This implies

$$\int_{\Theta}^{1-\Theta} |e(s + r(s)) - e(s)| \, ds \leq (1 - 2\Theta) c \leq 2\Theta (1 + 2c) \text{ if } \Theta \geq 1/4.$$

Q. E. D.

Let τ_ε, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ ($\varepsilon_0 > 0$) be a one parameter family of piecewise C^1 maps of the interval $[-1, 1]$. We shall assume that τ_ε has the following properties $\mathscr{H}_1, \ldots, \mathscr{H}_4$ (see Ref. 29)

$\mathscr{H}_1$) $\inf\limits_{\varepsilon \in [-\varepsilon_0, \varepsilon_0]} |\tau'_\varepsilon| = K_1 > 1.$

$\mathscr{H}_2$) $1/\tau'_\varepsilon$ has bounded variations on $[-1, 1]$, and

$$\sup_{\varepsilon \in [-\varepsilon_0, \varepsilon_0]} (\bigvee_{-1}^{1} 1/\tau'_\varepsilon) = K_2 < +\infty.$$

$\mathscr{H}_3$) $\varepsilon \to \tau'_\varepsilon$ is continuous in $L^1([-1, 1], dx)$.

$\mathscr{H}_4$) There is a finite sequence of points $a_1(\varepsilon) < a_2(\varepsilon) < \ldots < a_n(\varepsilon)$ in $[-1, 1]$, such that if x is not one of these points, τ'_ε is continuous and finite at x. Let $b_i(\varepsilon) = \tau_\varepsilon(a_i(\varepsilon))$, we shall also impose that $\varepsilon \to a_i(\varepsilon)$ and $\varepsilon \to b_i(\varepsilon)$ are continuous for $1 \leq i \leq n$, and that $\inf\limits_{\varepsilon \in [-\varepsilon_0, \varepsilon_0]} (\inf\limits_{i \neq j} |b_i(\varepsilon) - b_j(\varepsilon)|) > 0$.

According to Ref. 29 if conditions $\mathscr{H}_1$, and $\mathscr{H}_2$ are satisfied, τ_ε admits an absolutely continuous invariant measure (a. c. i. m) with density e_ε of bounded variations. Any such density satisfies the equation $T_\varepsilon e_\varepsilon = e_\varepsilon$ where T_ε is the Perron-Frobenius operator for the map τ_ε and is given by

$$(T_\varepsilon \varphi)(x) = \sum_{y, \tau_\varepsilon(y) = x} \varphi(y)/|\tau'_\varepsilon(y)| \text{ for } \varphi \in L^1([-1, 1], dx)$$

(see Ref. 10 for example).

Moreover, a function e is $L^1([-1, 1], dx)$ is the density of an a. c. i. m if and only if e is a fixed point of T_ε. It was shown in Ref. 29 that T_ε is a Doeblin-Fortet operator for the couple $(\mathscr{B}, \mathscr{C})$ where $\mathscr{B}$ is the space of functions of bounded variations on $[-1, 1]$, and $\mathscr{C} = L^1([-1, 1], dx)$. We shall moreover assume that for any ε in $[-\varepsilon_0, \varepsilon_0]$, T_ε has a unique fixed point e_ε which is a probability density (see below for an explicit case).

Theorem 4. *Let* $\varepsilon_0 > 0$, *and for* $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, *let* $\varepsilon \to \tau_\varepsilon$ *be a one parameter family of piecewise* C^1 *maps of the interval* $[-1, 1]$ *satisfying the hypothesis* $\mathcal{H}_1, \ldots, \mathcal{H}_4$. *Assume also that* τ_ε *has a unique a. c. i. m. which is a probability measure. Then,* $\varepsilon \to e_\varepsilon$ *is continuous in* $\varepsilon = 0$.

Proof of Theorem 4: From $T_\varepsilon e_\varepsilon = e_\varepsilon$, we have

$$\|T_0 e_\varepsilon - e_\varepsilon\|_{L^1} = \|(T_0 - T_\varepsilon) e_\varepsilon\|_{L^1},$$

and we shall denote $(T_\varepsilon - T_0) e_\varepsilon$ by g_ε. Moreover, it follows from $\mathcal{H}_1$ and $\mathcal{H}_2$ that we have (see Ref. 29)

$$\sup_{\varepsilon \in [-\varepsilon_0, \varepsilon_0]} \|T_\varepsilon\|_{\mathcal{B}} \le K_3 \quad \text{and} \quad \sup_{\varepsilon \in [-\varepsilon_0, \varepsilon_0]} \|e_\varepsilon\|_{\mathcal{B}} \le K_3$$

where K_3 is some finite number and $\mathcal{B}$ denotes the set of functions of bounded variations. Therefore $\forall \varepsilon \in [-\varepsilon_0, \varepsilon_0]$ $\|g_\varepsilon\|_{\mathcal{B}} \le 2K_3^2$. Let $(\tilde{b}_i)_{1 \le i \le n}$ denote the ordered set corresponding to $\{b_i \,|\, 1 \le i \le n\}$. We now construct two sequences c_i and c'_i, $0 \le i \le n$ by setting

$$c_i(\varepsilon) = \sup(\tilde{b}_i(\varepsilon), \tilde{b}_i(0)) \text{ if } 1 \le i \le n,\ c_0(\varepsilon) = 0,$$

$$c'_i(\varepsilon) = \inf(\tilde{b}_{i+1}(\varepsilon), \tilde{b}_{i+1}(0)) \text{ if } 0 \le i \le n-1,\ c'_n(\varepsilon) = 1.$$

It follows from $\mathcal{H}_4$ that if ε_0 is small enough we have

$$c_i(\varepsilon) < c'_i(\varepsilon) \le c_{i+1}(\varepsilon) \qquad 0 \le i \le n-1$$

and also,

$$\sup_i |c_{i+1}(\varepsilon) - c'_i(\varepsilon)| \to 0 \text{ if } \varepsilon \to 0.$$

We have

$$\|g_\varepsilon\|_{L^1} = \sum_{i=0}^{n} \int_{c_i(\varepsilon)}^{c'_i(\varepsilon)} |g_\varepsilon(s)|\, ds + \sum_{i=0}^{n-1} \int_{c'_i(\varepsilon)}^{c_{i+1}(\varepsilon)} |g_\varepsilon(s)|\, ds.$$

Using Lemma 3i) we deduce

$$\sum_{i=0}^{n-1} \int_{c'_i(\varepsilon)}^{c_{i+1}(\varepsilon)} |g_\varepsilon(s)|\, ds \le n(1 + 2K_3^2) \sup |c_{i+1}(\varepsilon) - c'_i(\varepsilon)| \to 0$$

$$\text{if } \varepsilon \to 0.$$

We also have

$$g_\varepsilon(x) = \sum_{\substack{y \\ \tau_\varepsilon(y) = x}} e_\varepsilon(y) |\tau'_\varepsilon(y)|^{-1} - \sum_{\substack{z \\ \tau_0(z) = x}} e_\varepsilon(z) |\tau'_0(z)|^{-1}.$$

From our choice of $c_l(\varepsilon)$ and $c_l'(\varepsilon)$, given x in $[c_l(\varepsilon), c_l'(\varepsilon)]$, there is a one to one correspondence between the set of y's such that $\tau_\varepsilon(y) = x$, and the set of z's such that $\tau_0(z) = x$. We shall denote by $\mathscr{P}$ this bijection (which depends on x and ε). We can write

$$g_\varepsilon(x) = s_1(x, \varepsilon) + s_2(x, \varepsilon) + s_3(x, \varepsilon),$$

where

$$s_1(x, \varepsilon) = \sum_{y, \tau_\varepsilon(y) = x} e_\varepsilon(y) \left[|\tau_\varepsilon'(y)|^{-1} - |\tau_0'(y)|^{-1}\right]$$

$$s_2(x, \varepsilon) = \sum_{z, \tau_0(z) = x} e_\varepsilon(\mathscr{P}(z)) \left[|\tau_0'(\mathscr{P}(z))|^{-1} - |\tau_0'(z)|^{-1}\right]$$

$$s_3(x, \varepsilon) = \sum_{z, \tau_0(z) = x} |\tau_0'(z)|^{-1} \left[e_\varepsilon(\mathscr{P}(z)) - e_\varepsilon(z)\right].$$

Since τ_ε' and τ_0' have the same sign on $\tau_\varepsilon^{-1}([c_l(\varepsilon), c_l'(\varepsilon)])$, we have

$$\int_{c_l(\varepsilon)}^{c_l'(\varepsilon)} |s_1(x, \varepsilon)| \, \mathrm{d}x \leq \int_{c_l(\varepsilon)}^{c_l'(\varepsilon)} \sum_{\substack{y \\ \tau_\varepsilon(y) = x}} e_\varepsilon(y) \left|(\tau_\varepsilon'(y))^{-1} - (\tau_0'(y))^{-1}\right| \mathrm{d}x$$

$$\leq \int_{\tau_\varepsilon^{-1}([c_l(\varepsilon), c_l'(\varepsilon)])} (1 + K_3) \left|(\tau_\varepsilon'(y))^{-1} - (\tau_0'(y))^{-1}\right| |\tau_\varepsilon'(y)| \, \mathrm{d}y.$$

From Lemma 3 and $\mathscr{H}_1$ this quantity is bounded by $K_1^{-1}(1 + K_3) \|\tau_\varepsilon' - \tau_0'\|_{L^1}$ which tends to zero if ε tends to zero.

For the integral of s_2, we have

$$\int_{c_l(\varepsilon)}^{c_l'(\varepsilon)} |s_2(x, \varepsilon)| \, \mathrm{d}x \leq \int_{\tau_0^{-1}([c_l(\varepsilon), c_l'(\varepsilon)])} (1 + K_3) |\tau_0'(z)| \left|\tau_0'(\mathscr{P}(z))^{-1} - \tau_0'(z)^{-1}\right| \mathrm{d}z.$$

Assume that $\mathscr{P}(z) - z \to 0$ if $\varepsilon \to 0$, then since $(\tau_0')^{-1}$ is continuous on each connected component of $\tau_0^{-1}(]\bar{b}_l(0), \bar{b}_{l+1}(0)[)$ and uniformly bounded (by K_1^{-1}), it follows from Lebesgue's dominated convergence theorem that the above integral tends to zero if ε tends to zero.

We now show that $\mathscr{P}(z) - z \to 0$ if $\varepsilon \to 0$. We have by definition of $\mathscr{P}$

$$\tau_0(z) - \tau_\varepsilon(z) = \tau_0(z) - \tau_0(\mathscr{P}(z)) = \int_{\mathscr{P}(z)}^{z} \tau_0'(s) \, \mathrm{d}s.$$

From $|\tau_0'| > 1$ and the continuity of τ_0' on $[z, \mathscr{P}(z)]$ (resp. $[\mathscr{P}(z), z]$), we derive

$$|\tau_0(z) - \tau_\varepsilon(z)| \geq |\mathscr{P}(z) - z|.$$

We shall now estimate $|\tau_0(z) - \tau_\varepsilon(z)|$. Let i be defined by $z \in]a_i(0), a_{i+1}(0)[$. Assume $a_i(0) \geq a_i(\varepsilon)$ (the other case is similar). We have

$$\tau_\varepsilon(z) - \tau_0(z) = \int_{a_i(\varepsilon)}^{z} [\tau'_\varepsilon(s) - \tau'_0(s)]\,\mathrm{d}s + \int_{a_i(\varepsilon)}^{a_0(\varepsilon)} \tau'_0(s)\,\mathrm{d}s + \tau_\varepsilon(a_i(\varepsilon)) - \tau_0(a_0(\varepsilon)).$$

Therefore

$$|\tau_\varepsilon(z) - \tau_0(z)| \leq \|\tau'_\varepsilon - \tau'_0\|_{L^1} + |b_i(\varepsilon) - b_i(0)| + \int_{a_i(\varepsilon)}^{a_0(\varepsilon)} |\tau'_0(s)|\,\mathrm{d}s,$$

and if $\varepsilon \to 0$, the first two terms tend to zero by hypothesis, and the third one also tends to zero since τ'_0 is in L^1. We now estimate s_3. We have

$$\int_{c_i(\varepsilon)}^{c'_i(\varepsilon)} |s_3(x, \varepsilon)|\,\mathrm{d}x \leq \int_{\tau_0^{-1}([c_i(\varepsilon), c'_i(\varepsilon)])} |e_\varepsilon(\mathscr{P}(z)) - e_\varepsilon(z)|\,\mathrm{d}z \leq 2n(1 + 2K_3) \sup |\mathscr{P}(z) - z|$$

by Lemma 3ii), and the result follows from the above estimation. Q. E. D.

3. *Applications to the geometric Lorenz flows*

3.1. *One parameter families of geometric flows*

In this section, we will apply Theorem 4 of the previous section to the geometric Lorenz flows, introduced in Ref. 6 to modelize the Lorenz equations[17]. We shall first recall how these flows are constructed and present the main bifurcations which can occur in one parameter families of such flows. One can start with a vector field in $\mathbf{R}^3$ which is linear in the box[(*)]:

$$P: \{(x, y, z) \mid -1 \leq x \leq 1;\ -\frac{1}{2} \leq y \leq \frac{1}{2};\ 0 \leq z \leq 1\},$$

and given by

$$\lambda_1 x \frac{\partial}{\partial x} - \lambda_3 y \frac{\partial}{\partial y} - \lambda_2 z \frac{\partial}{\partial z};\quad \lambda_1, \lambda_2, \lambda_3 > 0;\quad \lambda_3 > \lambda_2.$$

The origin O is then a critical point of saddle type. Its unstable manifold has two branches $W_0^{u\pm}$ which coincide with the x-axis before leaving P, and a two-dimensional stable manifold W_0^s whose local part in P is given by $x = 0$. This linear part of the flow induces a mapping from the rectangles $R^+ : 0 < x \leq 1$ and $R^- : -1 \leq$

(*) For another presentation, involving branched manifolds, see Refs. 7 and 28.

$\leq x < 0$ on the upper face of P, respectively to the faces $F^+ : x = 1$ and $F^- : x = -1$, which is defined by associating to a point M in $R^\pm$, the first intersection of the orbit starting from M with $F^\pm$. This mapping is given by:

$$T_0(x, y, 1) = \begin{cases} (1, y\,|x|^{\lambda_3/\lambda_1}, \ |x|^{\lambda_2/\lambda_1}) & \text{if } M \in R^+, \\ (-1, y\,|x|^{\lambda_3/\lambda_1}, \ |x|^{\lambda_2/\lambda_1}) & \text{if } M \in R^-, \end{cases}$$

and transforms the rectangles $R^\pm$ in cusped triangles $T^\pm$ in $F^\pm$, as shown in Fig. 1.

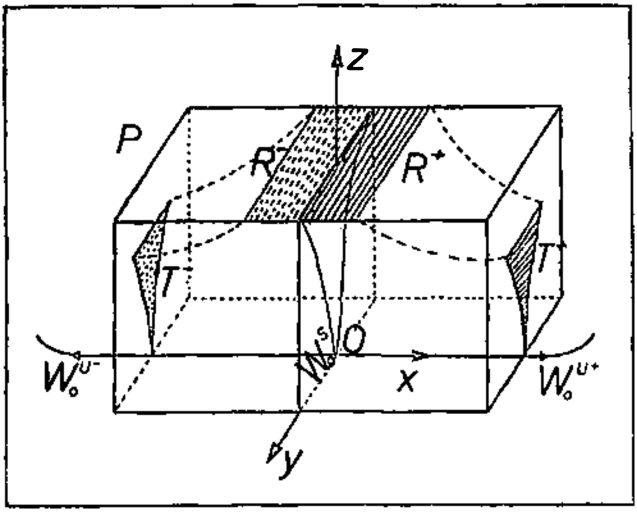

Fig. 1.

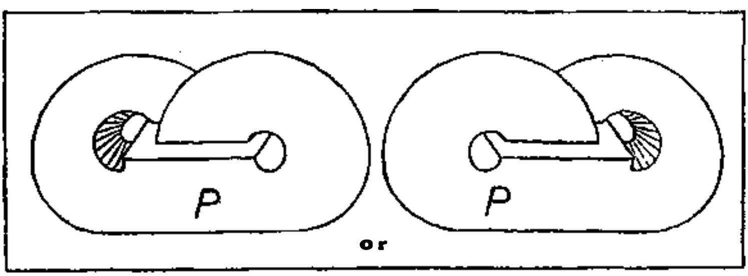

Fig. 2.

We now suppose that the flow outside P transports $T^\pm$ back on the upper face of P, as represented in Fig. 2, yielding a map:

$$T_1 : T^+ \cup T^- \rightarrow \overline{R^+ \cup R^-}.$$

Then $T = T_1 \circ T_0$ is a Poincaré map from $R^+ \cup R^-$ to $\overline{R^+ \cup R^-}$.

We next impose the very strong hypothesis that the set of lines $x = cte$ in $R^+ \cup R^-$ is preserved by T so that

$$T(x, y) = (f(x), g(x, y)),$$

where $\frac{\partial g}{\partial y} \to 0$ as $x \to 0\pm$ because of the strong contraction in the y direction.

Assuming the flow to be C^1:

$$f(x) = \begin{cases} -a_0^+ + a_1^+ |x|^\alpha + \text{h. o. t.} & \text{if } x > 0 \\ +a_0^- - a_1^- |x|^\alpha + \text{h. o. t} & \text{if } x < 0 \end{cases}$$

where $a_1^+, a_1^- > 0$; $\alpha = \frac{\lambda_2}{\lambda_1}$.

The dynamics of the flow can, for a large extent, be understood by studying f, since the ω-limit set of the flow in the cell represented in Fig. 2 is merely a suspension over the natural extension[24] of f.

For the sake of simplicity, we shall only consider here flows which are invariant under the change of variables

$$(x, y, z) \to (-x, -y, z),$$

so that $f(x) = -f(-x)$. Then $a_i^+ = -a_i^-$ and we shall suppress the superscripts $\pm$.

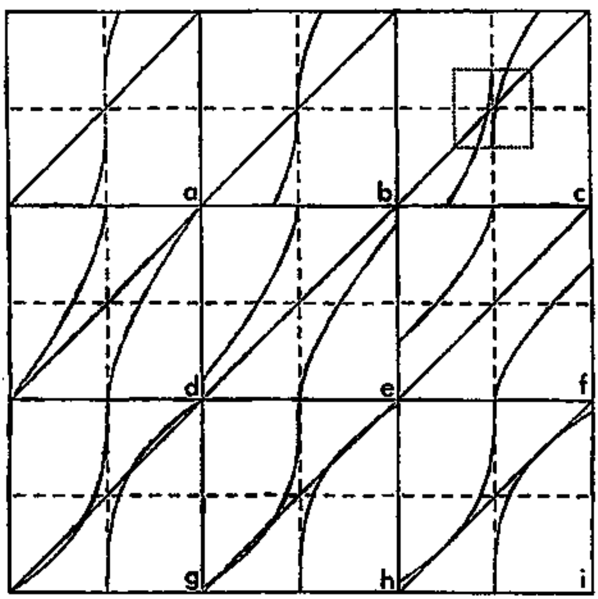

Fig. 3.

Furthermore, we shall suppose that $\lambda_1 > \lambda_2$, i. e. $\alpha < 1$ if $a_0 > 0$, which implies $|f'| \to \infty$ as $x \to 0\pm$.

We now describe the main bifurcations which can occur in one parameter families of such flows, in terms of the changes of the map f.

a) *The first homoclinic bifurcation.* This occurs when a_0^+ crosses O from above, as represented in Figs. 3 a, b and c. For $a_0 > 0$, but small enough, $f([-a_0, a_0])$ covers $[-a_0, a_0]$ twice and the topological entropy is Ln 2. Furthermore, almost every orbits eventually escape from this interval if f is supposed to be C^2 (see Ref. 2 for a counterexample in the C^1 case).

We shall denote by f_{a_0} the map f viewed as a function of the parameter a_0. Some monotonicity hypotheses are to be made to isolate the most relevant bifurcations. We shall suppose that:

— f'_{a_0} is monotonous on $\mathbf{R}^+$ and $\mathbf{R}^-$

— the interval $[-a_0, a_0]$ becomes invariant for a_0 large enough say $a_0 \geq a_{0,A}$ where $f(a_{0,A}) = a_{0,A}$.

— the kneading invariant of f_{a_0}, defined when $a_0 \geq a_{0,A}$, varies monotonically with a_0 (for the kneading theory of such maps, see e. g. Ref. 7 or 21).

b) The appearance of an attractor. This occurs when $a_0 = a_{0,A}$; then $f_{a_0}(a_0)$ crosses a_0 from above. For $a_0 > a_{0,A}$, it is convenient to perform an a_0-dependent normalization of f_{a_0} so that $f_{a_0}(0\pm) = \mp 1$, in order to work in the fixed interval $[-1, 1]$. There are then mainly two different situations to be considered, according to whether $f'_{a_0,A}(1) > 1$ (Fig. 3d) or $f'_{a_0,A}(1) < 1$ (Fig. 3g).

b-1) The case $f'_{a_0,A}(1) > 1$. Then the further evolution of f_{a_0} is as represented in Fig. 3e: for $a_0 > a_{0,A}$, but not too big, $|f'|$ is everywhere larger than one : nearby orbits diverge exponentially fast, and if f is C^2, there is a unique absolutely continuous probabilistic invariant measure with »good« ergodic properties, as will be detailed below. Indeed, f has an a. c. i. p. m. as long as $|f^{(n)\prime}| > 1$ for some iterate f^n (Fig. 3f).

b-2) The case $f'_{a_0,A}(1) < 1$. Then for $a_0 > a_{0,A}$, but not too big, f has two pairs of fixed points (for $x < 0$ and $x > 0$). Each pair consist of a stable and an unstable fixed point (see Fig. 3h), and can eventually disappear in a saddle-node bifurcation (Fig. 3i) when increasing further the parameter. This last bifurcation yields maps like in Fig. 3f which are not uniformly expanding, but it may occur that some iterate is, up to (but not including) the bifurcation point. Then, as in case *b*-1, one has good ergodic properties if f is C^2 and, using Theorem 4 and Ref. 18, one has an intermittency transition[20)] at the saddle-node.

3.2. *Ergodic properties and stability of the geometric Lorenz flow*

We shall first recall the known results about the ergodic theory of the geometric Lorenz flows (g. L. f. for short) as defined previously. We shall then discuss more general cases which are obtained by perturbation of the above flows (for topological results see Refs. 7, 21 and 28).

Most ergodic properties of the g. L. f. follow from the corresponding results for the associated one dimensional map f. Indeed the following theorem is an easy consequence of the skew product structure of the map T together with the known results about the map f. It summarizes the main ergodic properties of T.

Theorem 5: *Consider a geometric Lorenz flow defined as above, and let T be the Poincaré map, assuming as above that T is given by*

$$T(x, y) = (f(x), g(x, y)).$$

We shall assume that $I = [-1, 1]$ is an invariant interval for f and that f is increasing and differentiable on $[-1, 0[$ and $]0, 1]$ with:

i) $f(-x) = -f(x)$
ii) $f(0^-) = 1$, $f(-1) < 0$
iii) $1/f'$ is of bounded variations

iv) $\exists n \geq 1$ such that $|f^{n\prime}| \geq A > 1$, wherever $f^n = f \circ \dots \circ f$ (n times) is differentiable.

Then T has the following properties

a) *There is a unique invariant measure μ such that almost every conditional measure along a local unstable manifold is absolutely continuous with respect to the Lebesgue measure.*

b) *This measure is ergodic and has the Bowen-Ruelle property, i. e. there is an open neighborhood U of Ω such that for any function g continuous in U and for Lebesgue almost every point M in U, we have*

$$n^{-1} \sum_{n=0}^{n-1} g(T^j(M)) \underset{n \to +\infty}{\to} \int_{\Omega} g \, d\mu.$$

c) *If $n = 1$ and $A > \sqrt{2}$, the dynamical system (Ω, T, μ) is Bernoulli.*

d) *Correlations of the measure μ decay exponentially fast, i. e. if g_1 and g_2 are two Hölder continuous functions on Ω, there is a number $\lambda \in [0, 1[$ and a constant C such that*

$$\left|\int_{\Omega} g_1 \circ T^n g_2 \, d\mu - \int_{\Omega} g_1 \, d\mu \int_{\Omega} g_2 \, d\mu\right| \leq C \lambda^n.$$

The central limit theorem is also true, i. e. let g be a real valued Hölder continuous function on Ω, and define σ^2 by

$$\sigma^2 = \sum_{n=0}^{+\infty} \int_{\Omega} g \circ T^n g \, d\mu.$$

Assume that $\sigma \neq 0$ $(\sigma \geq 0)$ and $\int_{\Omega} g \, d\mu = 0$. Then if $a < b$ are two real numbers, we have

$$\mu(\{M \in \Omega \mid a \leq \frac{1}{\sqrt{n \sigma}} \sum_{j=0}^{n-1} g(T^j M) \leq b\}) \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} dx.$$

e) The Pesin formula holds, i. e. if $h(\mu)$ is the entropy of μ and λ^u the unstable Lyapunov exponent, we have

$$h(\mu) = \lambda^u.$$

If λ^s is the stable Lyapunov exponent ($\lambda^s < 0$), the Hausdorff dimension d of Ω is given by

$$d = h(\mu)\left(\frac{1}{\lambda^u} - \frac{1}{\lambda^s}\right).$$

Proof: It follows from a result of Wong[29] that f has a finite number of ergodic a. c. i. p. m.. Moreover, the support of each a. c. i. p. m. contains an interval[16], and some iterate of this interval must contain the point of discontinuity. Therefore there is only one a. c. i. p. m. and this measure is ergodic. We shall denote it by ϱ. We shall first show that the entropy of ϱ (with respect to f) is strictly positive. Let I denote the smallest interval containing the attractor of f. We shall denote by $\mathscr{B}$ the partition of I in the two atoms $[-1, 0]$ and $]0, 1]$. It is easy to verify that $\mathscr{B}$ is a generating partition[25]. Let $\mathscr{B}_n = \bigvee\limits_0^n f^{-i}(\mathscr{B})$, we shall give a lower bound for the number S_n defined by

$$S_n = -n^{-1} \sum_{B \in \mathscr{B}_n} \varrho(B) \operatorname{Log} \varrho(B).$$

We first observe that if $B \in \mathscr{B}_n$, then $f^n(B) \in \mathscr{B}$, and f_B^n is differentiable. Therefore B is an interval of length smaller than DA^{-n} where D is the length of I. Let h denote the density of the measure ϱ with respect to the Lebesgue measure. h is of bounded variations[29], and we shall denote by K its total variation. It follows from Lemma 3 that h is bounded by $1 + K$. Therefore, if $B \in \mathscr{B}_n$, we have $\varrho(B) \leq$ $\leq D(1 + K) A^{-n}$, and we get

$$S_n \geq -n^{-1} \sum_{B \in \mathscr{B}_n} \varrho(B)(-n \log A + \log(D(1 + K)))$$

$$\geq \operatorname{Log} A - n^{-1}(\operatorname{Log} D(1 + k)).$$

Therefore the entropy of ϱ is larger than $\operatorname{Log} A$ and hence is positive.

Let μ denote the natural extension of ϱ[24]. μ is ergodic and since its entropy is positive, it follows from Ref. 14 that the conditional measure of μ along almost every local unstable manifold is absolutely continuous with respect to the Lebesgue measure. Let $\tilde{\mu}$ be another T invariant probability measure with the same property. It is obvious that $\tilde{\mu}$ induces on the interval an f invariant measure $\tilde{\varrho}$ which is absolutely continuous with respect to the Lebesgue measure. Therefore $\tilde{\varrho} = \varrho$, and from the construction of the natural extension we get $\mu = \tilde{\mu}$.

b) follows at once from the absolute continuity of the strong stable foliation, the absolute continuity of the conditional measures and the Bowen-Ruelle property for the measure ϱ.

c) follows from the result of Ref. 22. Notice that in the symmetrical case, using the theory of kneading sequences for unimodal mappings, one can prove the same result assuming only that the Lyapunov exponent is larger than Log $\sqrt{2}$.

d) is a consequence of results in Refs. 10 and 30.

e) Pesin's formula follows from Ref. 14. The formula for the Hausdorff dimension follows from Ref. 15.

Remark: There are by now many versions of the results about expanding maps of the interval. We have given here some recent references without trying to be exhaustive.

As a matter of fact, it turns out that the above result extends to open sets of C^k flows in $\mathbf{R}^3$. In particular, one has the following theorem.

Theorem 6: *Let* $k \geq 2$, *and* $e^{k\lambda_3} e^{\lambda_2} e^{\lambda_1} < 1$ *in the definition of the* g. L. f.. *Then there is an open set* $\mathcal{N}_k$ *of all* C^k *flows which are* C^k *near the geometric model flow, such that every flow in* $\mathcal{N}_k$ *has a* C^k *strong stable foliation in the neighborhood of the attractor.*

This result is a straightforward extension of the corresponding C^1 result proven in Ref. 23. In order to prove the above C^k version, it suffices to use the following theorem instead of theorem 6.3 of Ref. 8.

Theorem 7: *Let* $\Phi : M \to M$ *be a* C^k *Anosov diffeomorphism of a compact smooth manifold* M *with Anosov splitting* $TM = E^s + E^u$. *Define* a, b *and* c *by* $a = \| T\Phi^{-1} | E^u \| < 1$, $b = \| T\Phi | E^s \| < 1$ *and* $c = \| T\Phi | E^u \| > 1$. *Then if* $abc^{\ell} < 1$ *with* $\ell \in \,]0, k]$, *the stable foliation is* C^{ℓ}.

The proof of this last statement is almost a repetition of the proof of Theorem 6.3 in Ref. 8, using the C^{ℓ} version[9] of their invariant section theorem (Theorem 6.1 in Ref. 8).

We observe that our hypothesis are stronger than those of Refs. 4 and 5 (we need a stronger contraction in the strong stable direction). However, the reduction to one dimensional maps, as done here, allows to use Theorem 4 of Section 2 to get some weak stability results for the Bowen-Ruelle measure of Poincaré maps of Lorenz-like flows. For example, the entropy is continuous and the Bowen-Ruelle measure is weakly continuous. In particular, one can analyze rigorously one parameter families of Lorenz-like flows exhibiting the intermittency transition[18,20].

We end with the remark that a $C^{1+\varepsilon}$ version of Wong's theorem[29,31] would lead to results of the same generality as those of Bunimovich and Sinai, as well as to a $C^{1+\varepsilon}$ version of the above weak stability result.

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ERGODSKA TEORIJA I KONTINUIRANOST BOWEN-RUELLOVE MJERE ZA GEOMETRIJSKE LORENZOVE TOKOVE

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U radu je prvi puta dokazan teorem o slaboj stabilnosti apsolutno kontinuirane invarijantne mjere za neko ekspandirajuće preslikavanje na interval. Koristeći regularnu invarijantnu folijaciju, dobivena je za geometrijski Lorenzov tok većina sada poznatih rezultata za ekspandirajuća preslikavanja intervala. To uključuje rezultate stabilnosti za Bowen-Ruellove mjere.