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FLOQUET EXPONENT FOR A CLASS OF OSCILLATORY SCHRÖDINGER PROBLEMS

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An efficient »nonperturbative« method for solving the Hill equation of a certain class is discussed and applied to the problem of degaussing in $n \rightleftharpoons \overline{n}$ oscillation experiments.

1. Introduction

As an alternative to grand unified theories which require the proton decay, a class of left-right symmetric theories[1] leaves the proton stable and predicts the neutron-antineutron oscillations, $n \rightleftharpoons \overline{n}$. While the proton decay experiments are simple by themselves, the neutron oscillation experiments face an additional technical difficulty. In fact, the earth's static magnetic field B_0, interacting with the anomalous magnetic moments, pushes the neutron and antineutron states away from each other. The predicted transitions, caused by the fundamental baryon mixing force, are damped in this way by a factor of 1000 or so.

However, the oscillations are still there and the idea is to introduce an additional oscillatory magnetic field $B_1(t)$ with properly chosen amplitudes and frequencies and drive the oscillations as much as possible. Since anything that oscillates in nature may be driven and enhanced, the question arises to what extent this can be done in this particular case. The main task is then to determine the free parameters of the driving field $B(t)$ as to optimize the growth of the antineutron probability starting from $P_{\overline{n}}(0) = 0$ at $t = 0$.

2. *The underlying system of linear differential equations with periodic coefficients*

In the rest frame of the neutron beam where, besides the fundamental baryon mixing force, also a z-directional magnetic field is applied, the time evolution of the neutron (antineutron) wave function is governed by the self-adjoint system of linear differential equations with periodic coefficients[2],

$$
\begin{aligned}
\mathrm{d}n(t)/\mathrm{d}t &= -\mathrm{i}\omega_B(t)\,n(t) - \mathrm{i}\omega_m\,\bar{n}(t),\\
\mathrm{d}\bar{n}(t)/\mathrm{d}t &= -\mathrm{i}\omega_m\,n(t) + \mathrm{i}\omega_B(t)\,\bar{n}(t),
\end{aligned}
\tag{1}
$$

where we have used the units $c = \hbar = 1$, ω_m is the fundamental frequency characterizing the baryon mixing force, $\omega_B(t)$ is the time-dependent frequency of the external magnetic field, while n and $\bar{n}$ are the neutron and antineutron wave functions, respectively.

We will assume the applied field frequency of the form

$$
\omega_B(t) = \omega_B(1 - r\sin\omega t + s\cos\omega t),
\tag{2}
$$

where ω_B is given by the earth's static field, while the amplitudes r and s and the driving frequency ω are free parameters. The minus sign in front of r is for convenience. One may, of course, use higher polynomials of sines and cosines and different driving frequencies in different terms; however, the form (2) is simple enough, so that all steps of calculation may be carried through analytically in a fairly simple way.

With the notations

$$
X(t) \equiv \begin{pmatrix} n(t) \\ \bar{n}(t) \end{pmatrix}, \quad A(t) \equiv -\mathrm{i}\begin{pmatrix} \omega_B(t) & \omega_m \\ \omega_m & -\omega_B(t) \end{pmatrix}
\tag{3}
$$

the system (1) assumes its canonical form

$$
\mathrm{d}X(t)/\mathrm{d}t = -A(t)\,X(t).
\tag{4}
$$

3. *The corresponding Hill equation*

Since $A(t)$ is an antihermitian periodic matrix, the corresponding second-order differential equation is a Hill equation,

$$
\begin{aligned}
\mathrm{d}^2\bar{n}(x)/\mathrm{d}x^2 + Q(x)\,\bar{n}(x) &= 0,\\
Q(x) &= \lambda_m^2 + \lambda_B^2(x) - \mathrm{i}\mathrm{d}\lambda_B(x)/\mathrm{d}x,
\end{aligned}
\tag{5}
$$

$$
\lambda_m = \omega_m/\omega, \quad \lambda_B(x) = \omega_B(x)/\omega, \quad x = \omega t.
\tag{6}
$$

The neutron wave function $n(x)$ is the solution of the corresponding complex conjugate equation.

According to the Floquet-Lyapunov theorem[3] the general solution of Eq. (5) has the form

$$\bar{n}(x) = f(x)\, e^{\alpha x} + g(x)\, e^{-\alpha x}, \tag{7}$$

where f and g are bounded periodic functions whose periodicity is the same as that of Q. The generally complex characteristic exponent a determines the stability of the solutions. Its determination is very difficult. The standard method of infinite determinants does not work here since Q is neither a real nor a symmetric function.

Let us first show that in our particular case the characteristic exponent is purely imaginary. As one sees from the definition (3), the matrix A is antihermitian, $A^\dagger = -A$, and this ensures the conservation of the total probability, $P_n + P_{\bar{n}} = 1$:

$$\frac{d}{dt}(|n(t)|^2 + |\bar{n}(t)|^2) = \frac{d}{dt}|X(t)|^2 = \frac{d}{dt}(X(t), X(t)) =$$

$$= (\dot{X}(t), X(t)) + (X(t), \dot{X}(t)) = ((A + A^+)X, X) = 0.$$

For the Floquet-Lyapunov solution with periodicity τ,

$$X(t) = f(t)\, e^{at},$$

this implies that

$$(X(t), X(t)) = (X(t+\tau), X(t+\tau)) = \exp[(a + a^*)\tau]\,(X(t), X(t)),$$

and therefore

$$a + a^* = 0, \quad a = i\gamma, \quad \gamma \text{ real}. \tag{8}$$

The solution is double oscillatory. Trivial periodicity is that of the matrix $A(t)$, $2\pi/\omega$, where ω is the frequency of the driving magnetic field in the definition (2). As it will be seen later, it has to be of the order of the static field frequency ω_B. For the earth's magnetic field of $0.5 \cdot 10^{-4}$ T, we have $\omega_B \approx 10^4\ s^{-1}$ and therefore $(2\pi/\omega) \approx 10^{-4}$ s. In the zero-field limit, the exact solution of the system (1) for the antineutron wave function is

$$n(t) = -i \sin \omega_m t, \tag{9}$$

and we, therefore, expect that the characteristic exponent should be of the same order of magnitude, $\gamma \approx \omega_m$. According to the estimate used in Ref. 2, $\omega_m \approx \approx 10^{-4}\ s^{-1}$ and, therefore, $(2\pi/\gamma) \approx 10^4$ s. The nontrivial periodicity due to γ is very slow in comparison with that which is due to ω, $(2\pi/\gamma) \gg (2\pi/\omega)$.

4. A »nonperturbative« approximate solution

In the following we present a fairly general method to find the characteristic exponent and an approximate solution of the problem for those cases when the function $\omega_B(x)$ is given by Eq. (4) or a similar polynomial of sines and cosines. Numerical experience with oscillatory Schrödinger problems tells us that the solutions always appear in the form when some sort of rapid oscillations of small magnitude are superimposed on slow oscillations of large magnitude. For commulative experiments, the latter type of variation is physically more interesting and, as it will be seen later, our method somehow »smears« the rapid variations and takes into account only the average general oscillation of large magnitude.

Let us now concentrate on the formal (canonical) solution of the system of Eqs. (1). In order to get rid of diagonal terms, we perform a unitary transformation

$$n(x) = N(x) \exp\left[-i \int_0^x dx' \, \lambda_B(x')\right],$$
$$\bar{n}(x) = \bar{N}(x) \exp\left[i \int_0^x dx' \, \lambda_B(x')\right], \tag{10}$$

and obtain the simpler system

$$dN(x)/dx = -i\lambda F_+(x) \bar{N}(x),$$
$$d\bar{N}(x)/dx = -i\lambda F_-(x) N(x), \tag{11}$$

where we have defined

$$F_\pm(x) = \exp\left[\pm 2i \int_0^x dx' \, \lambda_B(x')\right]. \tag{12}$$

Now we write

$$Y(x) = \begin{pmatrix} N(x) \\ \bar{N}(x) \end{pmatrix} \tag{13}$$

and introduce the Pauli matrices σ_1 and σ_3, so that we may rewrite Eq. (4) in the form

$$dY(x)/dx = B(x) \, Y(x), \tag{14}$$

where

$$B(x) = -i\lambda_m \exp\left[2i\sigma_3 \int_0^x dx' \, \lambda_B(x')\right] \sigma_1. \tag{15}$$

For a given initial value $Y(0)$, the purely formal solution may be found, e. g., in any textbook on field theory,

$$Y(x) = T \exp \left[\int_0^x \mathrm{d}x' \, B(x') \right] Y(0), \tag{16}$$

where T is the time-ordering operator.

In our particular case of spontaneous $n \rightleftarrows \bar{n}$ transitions we assume that at the initial (dimensionless) time $x = 0$ the beam contains only neutrons, so that $n(0) = N(0) = 1$ and $\bar{n}(0) = \bar{N}(0) = 0$. Hence

$$Y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{17}$$

Let us now write the solution (16) in a down-to-the-earth form

$$\begin{aligned} \bar{N}(x) = (-\mathrm{i}\lambda_m) \int_0^x \mathrm{d}x_1 F_-(x_1) + (-\mathrm{i}\lambda_m)^3 \int_0^x \mathrm{d}x_1 F_-(x_1) \times \\ \times \int_0^{x_1} \mathrm{d}x_2 F_+(x_2) \int_0^{x_2} \mathrm{d}x_3 F_-(x_3) + \ldots \equiv \sum_{k=0}^{\infty} N_{2k+1}(x), \end{aligned} \tag{18}$$

$$N(x) = 1 + (-\mathrm{i}\lambda_m) \int_0^x \mathrm{d}x_1 F_+(x_1) \int_0^{x_1} \mathrm{d}x_2 F_-(x_2) + \ldots \equiv \sum_{k=0}^{\infty} N_{2k}(x). \tag{19}$$

We proceed by evaluating $F_\pm(x)$ according to the definitions (12), (6) and (2) and obtain

$$F_\pm(x) = \mathrm{e}^{\mp 2\mathrm{i}\alpha} \exp\left[\pm 2\mathrm{i}ax/r \pm 2\mathrm{i}a \cos x \pm 2\mathrm{i}\beta \sin x\right], \tag{20}$$

where we have introduced the abbreviations

$$a \equiv \omega_B r/\omega \quad \text{and} \quad \beta = \omega_B s/\omega. \tag{21}$$

The trigonometric functions in the exponent are resolved by expanding them in terms of Bessel functions[4]:

$$\begin{aligned} \mathrm{e}^{2\mathrm{i}\alpha \cos x} &= \sum_{n=-\infty}^{\infty} \mathrm{i}^n J_n(2a)\, \mathrm{e}^{\mathrm{i}nx}, \\ \mathrm{e}^{2\mathrm{i}\beta \sin x} &= \sum_{m=-\infty}^{\infty} J_m(2\beta)\, \mathrm{e}^{\mathrm{i}mx}. \end{aligned} \tag{22}$$

For the negative sign in the exponent we use the symmetry properties[4)] $J_n(-z) = J_{-n}(z) = (-1)^n J_n(z)$ and obtain

$$F_{\pm}(x) = e^{\mp 2i\alpha} \sum_{n,m} i^n (\pm 1)^{n+m} J_n(2a) J_m(2\beta) \exp[ix(n+m\pm\nu)], \tag{23}$$

where

$$\nu = 2a/r = 2\omega_B/\omega. \tag{24}$$

From now on we will assume that ν is an integer. This will greatly simplify our calculation since we may now renumerate the series (23) and use the symmetry properties of Bessel functions. In this way we obtain $F_{\pm}(x)$ in a more compact form

$$F_{-}(x) = F_{+}^{*}(x) = e^{2i\alpha} \sum_{\ell=-\infty}^{\infty} S(a, \beta; \nu+\ell)\, e^{i\ell x}, \tag{25}$$

where the coefficients of this Fourier expansion are defined by

$$S(a, \beta; n) = (-1)^n \sum_{k=-\infty}^{\infty} i^k J_k(2a) J_{n-k}(2\beta). \tag{26}$$

The general term in the expansion (19) now takes the form

$$N_{2k}(x) = (-i\lambda_m)^{2k} \sum_{\ell_1,\ell_2,\ldots,\ell_{2k}} \int_0^x dx_1\, e^{-i\ell_1 x_1}\, S^*(a, \beta; \nu+\ell_1)$$
$$\times \int_0^{x_1} dx_2\, e^{i\ell_2 x_2}\, S(a, \beta; \nu+\ell_2) \ldots$$
$$\ldots \int_0^{x_{2k-2}} dx_{2k-1}\, e^{i\ell_{2k-1} x_{2k-1}}\, S^*(a, \beta; \nu+\ell_{2k-1}) \times$$
$$\times \int_0^{x_{2k-1}} dx_{2k}\, e^{i\ell_{2k} x_{2k}}\, S(a, \beta; \nu+\ell_{2k}).$$

At first sight, the complexity of this expression is not encouraging. However, it is not quite so. We note, namely, that when the integers ℓ_n run from $-\infty$ to $+\infty$, the product of exponential functions oscillates wildly and averages to a value roughly equal to zero, except for the case of a »resonant configuration« when all exponent vanish. This is the giant part of the integrals and we pick it out. As a result of equating the exponents to zero, we get δ symbols and may perform the summation over the indices $\ell_1, \ell_2, \ldots, \ell_{2k}$. If we write

$$N_{2k}(x) = N_{2k}^{R}(x) + N_{2k}^{NR}(x),$$

this resonant contribution is easily found to be

$$N^R_{2k}(x) = \frac{(-\mathrm{i}\,\lambda_m x)^{2k}}{(2k)!}|S(\alpha, \beta; \nu)|^{2k}. \tag{27}$$

This result makes the series (19) for $N(x)$ absolutely convergent and may be summed up to give

$$N^R(x) = \cos\{\lambda_m x\,|S(\alpha, \beta; \nu)|\}. \tag{28}$$

Using the same procedure, with slightly more algebra with phase factors, we also obtain the antineutron wave function,

$$\overline{N}^R(x) = \mathrm{e}^{\mathrm{i}(2\alpha-\pi/2)}\frac{S(\alpha, \beta; \nu)}{|S(\alpha, \beta; \nu)|}\sin[\lambda_m x\,|S(\alpha, \beta; \nu)|]. \tag{29}$$

What remains is to see how large the nonresonant part is.

5. *Discussion*

How good are the solutions (28) and (29) and which properties of the exact solutions do they preserve? First, we see that they are manifestly unitary,

$$|N^R(x)|^2 + |\overline{N}^R(x)|^2 = 1.$$

In addition, in the limit of zero magnetic field, characterized by $\omega_B = r = s = 0$ in the relation (2), or consequently, $\alpha = \beta = \nu = 0$, the solution $\overline{N}^R(x)$ tends to the exact solution (9). This is easily seen by recalling that $\lambda_m x = \omega_m t$ and that all Bessel functions with nonzero integer indices vanish at the origin, so that, according to the definition (26), we have $S(0, 0; 0) = J_0^2(\lambda) = 1$.

When discussing the corresponding Hill equation (5), we have seen that the exact solution is double oscillatory: rapid ω oscillations of small magnitude (whose trivial periodicity is equal to that of the function Q) are superimposed upon much slower oscillations due to the characteristic exponent γ. Our resonant solutions (28) and (29) somehow smear the faster oscillations and follow the »averaged« functional dependence which is physically more relevant. Whatever physical phenomena are described by the system of equations of the type (1), the slower oscillations of larger amplitude contain physically significant information measured by experiments. In our particular case of $n \rightleftarrows \overline{n}$ oscillations, it is the general sinusoidal growth of the antineutron wave function (hence, the antineutron probability) and not the small local fluctuation which matters for the experiments.

The value of the characteristic exponent which governs the growth of the antineutron probability starting at $P_{\overline{n}}(0) = 0$ is given by

$$\gamma = \lambda_m |S(\alpha, \beta; \nu)|. \tag{30}$$

For the particular choice (2) of the function $\omega_B(t)$, γ may be evaluated analytically by using a theorem[5] for Bessel functions:

$$\sum_{k=-\infty}^{\infty} e^{ik\Phi_2} J_{n-k}(r_1) J_k(r_2) = e^{in\Phi} J_n(r),$$

where

$$r = (r_1^2 + r_2^2 + 2r_1 r_2 \cos \Phi_2)^{1/2},$$

$$e^{i\Phi} = (r_1 + r_2 e^{i\Phi_2})/r.$$

Choosing $\varphi_2 = \pi/2$ and $r = 2(a^2 + \beta^2)^{1/2}$, we may apply this theorem and perform the summation in the definition (26):

$$S(a, \beta; n) = \frac{(-\beta - ia)^n}{(a^2 + \beta^2)^{n/2}} J_n [2(a^2 + \beta^2)^{1/2}]. \tag{31}$$

One may now raise the question of choosing the free parameters (a, β and ω) in such a way as to achieve the optimal growth of the antineutron probability $P_{\bar{n}}(x)$. This was studied numerically in Ref. 2 and analytically for the simplified case $\beta = 0$ in Ref. 6. In the limit $\beta = 0$, our solution (29) leads to the same result, namely,

$$\overline{N}^R_{\beta=0}(x) = \exp[i(2a - \pi\nu/2 - \pi/2)] \sin[\lambda_m x J_\nu(2a)]. \tag{32}$$

From the purely experimental point of view, the antineutron probability corresponding to the solution (29) contains one more free parameter and therefore more flexibility in degaussing the earth's magnetic field B_0 by applying the sine, or the cosine, or both types of the driving magnetic field $B(t)$. On the other hand, the utility of our computational method depends on how large $\overline{N}^{NR}(x)$ is.

The direct way to see the fine structure lost by our nonperturbative solution is to go to the limit of small times and to evaluate $\overline{N}(x)$ to the first order:

$$\overline{N}_{(1)}(x) = (-i\lambda_m) \int_0^x dx' F_-(x') = (-i\lambda_m) e^{2i\alpha} [xS(a, \beta; \nu) +$$

$$+ \sum_{\ell \neq 0} S(a, \beta; \nu + \ell) \int_0^x e^{i\ell x'} dx'] = \overline{N}^R_{(1)}(x) + \overline{N}^{NR}_{(1)}(x). \tag{33}$$

$\overline{N}^R_{(1)}(x)$ is just the first term of the expansion of the resonant solution (29), while the second term describes the fine wiggling around the »resonant« curve. Having in mind the expression (31) for S and recalling that the Bessel functions decrease rapidly when their indices increase[7], this fine structure may be easily taken into account with desired accuracy. Of course, the oscillation due to the Lyapunov exponent is now completely lost.

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FLOQUETOV EKSPONENT ODREĐENE KLASE OSCILATORNIH SCHRÖDINGEROVIH PROBLEMA

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Predložena je djelotvorna »neperturbativna« metoda za rješavanje Hillove jednadžbe određene klase i primijenjena na zadaću uklanjanja magnetskog polja u $n \rightleftarrows \bar{n}$ oscilacijskim eksperimentima.