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VACUUM FIELDS AND THE CURVATURE OF SPACE

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Weyl type solutions of the vacuum field due to spheroidal distribution of charged mass are investigated and expressions for the principal space curvatures are derived. Diagrams are drawn showing the variation of curvature with distance from the source. The problem of motion of test particles in the frame of an observer moving freely in a gravitational field is also discussed. It is shown how the space curvatures appearing in the equations of motion change when observations are made from different locally Galilean frames. It is further shown that a suitably rotating frame is more convenient for determining the motion of the particles than a locally Galilean frame.

1. *Introduction*

In 1917 Weyl[1] obtained a whole class of solutions of Einstein's equations for static, axially symmetric, electrogravitational fields in empty space by assuming the electrostatic potential Φ and the component g_{44} of the metric tensor to be connected by a relation of the form

$$g_{44} = 1 + 2B\Phi + \Phi^2, \tag{1.1}$$

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where $B = -m/q$ is the ratio of the mass and charge densities. Assuming this relation and writing the line element in the form

$$ds^2 = -e^{-w+\nu}(dz^2 + d\varrho^2) - \varrho^2 e^{-w} d\Phi^2 + e^w dt^2 \tag{1.2}$$

he was able to reduce the field equations to Laplace's equation,

$$\chi_{zz} + \chi_{\varrho\varrho} + \varrho^{-1} \chi_\varrho = 0, \tag{1.3}$$

for the function

$$\chi = \int^{\Phi} e^{-w} d\Phi. \tag{1.4}$$

Solutions subject to the restriction (1.1) have been studied in detail by many authors. In contrast with the large number of papers on Weyl type solutions there are only a few papers on non-Weyl solutions[2–8]. In the present paper we consider only Weyl type solutions for spheroidal distributions of charged mass and calculate the curvature of the space part of the vacuum metric associated with such distributions. Misra[9] and Zipoy[10] studied the vacuum field for uncharged mass possessing spheroidal symmetry.

By a conformal transformation of z, ϱ we write the axially symmetric line element in the form (see Eq. (5.2))

$$ds^2 = -d\sigma^2 + e^w dt^2, \text{ with } d\sigma^2 = e^{-w} d\bar{\sigma}^2 =$$

$$= e^{-w} [e^\lambda (dx_1^2 + dx_2^2) + e^\nu dx_3^2] \tag{1.5}$$

and express the field equations in terms of λ, ν, w, Φ and the components of the tensor $\bar{T}_{ik} = \bar{R}_{ik} - \frac{1}{2} \bar{g}_{ik} \bar{R}$ of the conformal space defined by the line element $d\bar{\sigma}^2 \cdot \bar{T}_{ik}$ turns out to be the Ricci tensor $\bar{\alpha}_{ik}$ of the conformal space. To simplify the calculations w and Φ are removed from the equations and replaced by χ. For spheroidal fields χ is easily determined by integration and ν by separation of variables.

The calculation of curvature is carried out in two steps, first for the conformal space $\bar{V}_3$ and then for the whole space. It is found that the lines of curvature in the conformal space coincide with the coordinate lines for all axially symmetric fields if χ depends on one variable. This is not true of the whole space V_3. This is evident from the field equations (2.1)—(2.4) and the equations (3.4), (4.1), (4.6). The field equations show that $\bar{T}_{ij} \equiv \bar{\alpha}_{ij}$ becomes a diagonal tensor when χ is a function of only one variable. The conclusion for $\bar{V}_3$ then follows from (4.1). Because of the presence of the nonzero covariant derivative w_{12} (see (4.9)) in the relation (4.6) for $i = 1, j = 2$ the component α_{12} does not vanish and the lines of curvature in V_3 do not coincide with the coordinate lines even when χ is a function of one variable.

Various aspects of the problem of geodesic deviation in a general gravitational field are discussed in Secs. 7, 8, 9. It is shown amongst other that, contrary to the general belief, the equations assume much simpler forms in an appropriate system of rotating coordinates than in locally Galilean coordinates. The simplification is automatically achieved if the lines of curvature are chosen as the coordinate lines. Since the principal directions of curvature are tangential to the lines of curvature at every point of a V_3, the tensor α_{ij} becomes diagonal for such a choice of coordinates greatly simplifying the equations of geodesic deviation. This property is possessed by the Schwarzschild coordinates.

2. *The field equations*

The field equations in vacuum for the line element (1.5) for charged mass are

$$-4\overline{T}_{11} = \lambda_1 \nu_1 - \lambda_2 \nu_2 + 2\nu_{22} + \nu_2^2 = P_{11} - P_{22}, \tag{2.1}$$

$$-4(\overline{T}_{11} + \overline{T}_{22}) = 2(\nu_{11} + \nu_{22}) + \nu_1^2 + \nu_2^2 = 0, \tag{2.2}$$

$$-4\overline{T}_{12} = \lambda_1 \nu_2 + \lambda_2 \nu_1 - 2\nu_{12} - \nu_1 \nu_2 = 2P_{12}, \tag{2.3}$$

$$-4\overline{T}_{33} = e^{\nu-\lambda}\, 2(\lambda_{11} + \lambda_{22}) = -(P_{11} + P_{22})\, e^{\nu-\lambda}, \tag{2.4}$$

$$\Delta_2 w = 2e^{-w} \overline{g}^{ab} \Phi_a \Phi_b, \quad \Delta_2 \Phi = \overline{g}^{ab} w_a \Phi_b, \tag{2.5}$$

where

$$\Delta_2 w = \overline{g}^{ab} w_{,ab} = \overline{g}^{ab} (w_{ab} - \overline{\Gamma}^{l}_{ab} w_l), \quad P_{ij} = w_i w_j - 4e^{-w} \Phi_i \Phi_j.$$

If ν can be written as $f(x_1) + g(x_2)$, then Laplace's equation becomes simply separable. For closed coordinate systems the value of the separation constant A in Eq. (2.2) is 2. For non-closed ones $A = 0$.

3. *Some general relations between the field variables*

From (1.1), (1.4) we obtain the relations:

$$\chi = (2\beta)^{-1} \ln \{[(B + \beta)\Phi + 1]/[(B - \beta)\Phi + 1]\} \tag{3.1}$$

with $\beta^2 = B^2 - 1$,

$$\Phi = \frac{e^{\beta\chi} - e^{-\beta\chi}}{(B + \beta)\, e^{-\beta\chi} - (B - \beta)\, e^{\beta\chi}}, \tag{3.2}$$

$$e^{-w} = (2\beta)^{-2}\, [(B + \beta)\, e^{-\beta\chi} - (B - \beta)\, e^{\beta\chi}]^2, \tag{3.3}$$

$$P_{\chi\chi} \equiv w_{\chi}^2 - 4e^{-w}\Phi_{\chi}^2 = 4\beta^2, \quad P_{ij} = 4\beta^2\chi_i\chi_j, \tag{3.4}$$

$$\Delta_2\chi = 0. \tag{3.5}$$

Eq. (3.5) reduces to (1.3) in an axially symmetric field.

4. The space curvature

The principal curvatures of any 3-dimensional space V_3 are the roots of the characteristic equation[11]

$$|\alpha_{ij} - \gamma g_{ij}| = 0, \tag{4.1}$$

where α_{ij} is the Ricci tensor

$$\alpha^{ik} = \frac{1}{4}\varepsilon^{ipq}\varepsilon^{krs}R_{pqrs}, \tag{4.2}$$

and

$$\alpha_{ik} = T_{ik} \equiv R_{ik} - \frac{1}{2}g_{ik}R \tag{4.3}$$

in any V_3. If χ is a function of x_1 only, the field equation (2.3) gives

$$\bar{\alpha}_{12} = 0, \quad \text{and} \quad \bar{\alpha}_{13} = \bar{\alpha}_{23} = 0 \quad \text{identicaly.}$$

Thus, when χ is a function of a single variable, the principal directions of curvature coincide with the coordinate lines at every point of $\bar{V}_3$. In this case Eq. (4.1) has the following roots:

$$-\bar{\gamma}_1 = \bar{\gamma}_2 = \bar{\gamma}_3 = e^{-\lambda}(\beta\chi_1)^2. \tag{4.4}$$

When χ is a function of two variables, $\bar{\alpha}_{12}$ is given by (2.3) and the principal curvatures are

$$-\bar{\gamma}_1 = \bar{\gamma}_2 = \bar{\gamma}_3 = \beta^2(\chi_1^2 + \chi_2^2)e^{-\lambda}. \tag{4.5}$$

The relation between the Riemannian invariants of the 3-space V_3 with the line element $d\sigma^2$ and those of $\bar{V}_3$ is

$$T_{ij} = \bar{T}_{ij} - \frac{1}{2}w_{,ij} - \frac{1}{4}w_{,i}\,w_{,j} + \frac{1}{2}\bar{g}_{ij}\Delta_2 w. \tag{4.6}$$

When χ is a function of a single variable we get from (3.3)

$$w_1 = 2H\chi_1, \quad w_{11} = 2e^w\chi_1^2 + 2H\chi_{11}, \tag{4.7}$$

where

$$H = \sqrt{(\beta^2 + e^w)} = \Phi + B.$$

For the type of the line element under consideration

$$w_{,11} = -H\lambda_1 \chi_1 + 2\,e^w \chi_1^2 + 2\,H\chi_{11}, \tag{4.8}$$

$$w_{,22} = H\lambda_1 \chi_1, \quad w_{,33} = e^{\nu-\lambda} H \nu_1, \chi_1, \quad w_{,12} = -H\lambda_2 \chi_1. \tag{4.9}$$

Hence,

$$\Delta_2 w = e^{-\lambda}[H\nu_1 \chi_1 + 2\,e^w \chi_1^2 + 2\,H\chi_{11}]. \tag{4.10}$$

5. *Spheroidal fields*

We write the line element in prolate coordinates in the form

$$ds^2 = -e^{-w}[e^\lambda(dx_1^2 + dx_2^2) +$$

$$+ a^2 \sinh^2 x_1 \cos^2 x_2\, d\Phi^2] + e^w dt^2 \tag{5.1}$$

by applying to (1.2) the conformal transformation

$$z + i\varrho = a \sin(ix_1 - x_2), \tag{5.2}$$

and obtain

$$e^\lambda = e^\nu a^2 (\sinh^2 x_1 + \cos^2 x_2).$$

For equipotential surfaces having prolate spheroidal symmetry we have

$$\chi_1 = -\frac{k}{\sinh x_1}, \quad \chi = -k \ln\left(\tanh\frac{x_1}{2}\right), \tag{5.3}$$

where k is positive for positive charge and negative for negative charge.

For the line element (5.1),

$$\nu_1 = 2\coth x_1, \qquad \nu_2 = -2\tan x_2, \qquad A = 2, \tag{5.4}$$

and by (2.3), (5.3) we get

$$w_1^2 - 4\,e^{-w}\Phi_1^2 = \frac{4k^2\beta^2}{\sinh^2 x_1}. \tag{5.5}$$

Substituting (5.4), (5.5) in (2.1) and (2.3) and solving for λ_1, λ_2 we obtain

$$\lambda_1 = 2 \coth x_1 \left[k^2 \beta^2 - \frac{(k^2 \beta^2 - 1) \sinh^2 x_1}{\sinh^2 x_1 + \cos^2 x_2}\right], \tag{5.6}$$

$$\lambda_2 = \frac{2 (k^2 \beta^2 - 1) \sin x_2 \cos x_2}{(\sinh^2 x_1 + \cos^2 x_2)}. \tag{5.7}$$

Integrating we get

$$e^{\lambda} = a^2 \left[\frac{\sinh^2 x_1}{\sinh^2 x_1 + \cos^2 x_2}\right]^{k^2\beta^2} (\sinh^2 x_1 + \cos^2 x_2). \tag{5.8}$$

The Gaussian curvature of $x_1 =$ constant surfaces is given by

$$K = \frac{e^w}{a^2} \frac{(\sinh^2 x_1 + \cos^2 x_2)^{k^2\beta^2-1}}{\sinh^{2k^2\beta^2} x_1} \left[1 - \frac{(k^2 \beta^2 - 1) \sin^2 x_2}{(\sinh^2 x_1 + \cos^2 x_2)}\right]. \tag{5.9}$$

This shows that for

$$k \beta = \pm 1 \tag{5.10}$$

the family of surfaces $x_1 =$ constant are independent of x_2, that is those become »spheres«. Here the positive sign is to be taken for positive charge distribution. Expression (5.9) shows that in this coordinate system no spherically symmetric solution will be possible for negative β^2.

The field with oblate spheroidal symmetry is obtained from the prolate type by the usual substitutions:

$$a = ic, \quad \cosh x_1 = -i \sinh y_1, \quad \cos x_2 = \sin y_2 \quad \text{and} \quad k \to -ik. \tag{5.11}$$

In the case of oblate fields spherically symmetric solutions exist only if β is imaginary and the condition $k\mu = 1$, $\mu = i\beta$, is fulfilled. Solutions for charged mass reduce to those for pure mass in the limit

$$B \to \infty, \quad B\chi \to w/2. \tag{5.12}$$

6. *The space curvature for spheroidal fields*

. Using the results of Secs. 4 and 5 we obtain the following expressions for the principal curvatures of the space part of the line element for prolate spheroidal fields due to charged and uncharged mass

$$\gamma_3 = e^{w-\lambda} (H^2 \chi_1^2 - \frac{1}{2} H \nu_1 \chi_1), \tag{6.1}$$

$$\gamma_{\pm} = \frac{1}{2} e^{w-\lambda} \left(\frac{1}{2} H \nu_1 \chi_1 - \beta^2 \chi^2 \right) \pm \frac{1}{2} e^{w-\lambda} [\{H (\lambda_1 + \frac{1}{2} \nu_1) \chi_1 -$$

$$- (3 \beta^2 + 2 e^{w}) \chi_1^2\}^2 + (H \lambda_2 \chi_1)^2]^{1/2}, \tag{6.2}$$

$$\gamma_3^0 = e^{w-\lambda} \left(\frac{1}{4} w_1^2 + \frac{1}{2} w_{11} \right), \tag{6.3}$$

$$\gamma_{\pm}^0 = \frac{1}{2} e^{w-\lambda} \left[\frac{1}{2} \nu_1 w_1 - \frac{1}{4} w_1^2 + \frac{1}{2} w_{11} \pm \right.$$

$$\left. \pm \left\{ \left(- \frac{3}{4} w_1^2 + \frac{1}{2} \lambda_1 w_1 - \frac{1}{2} w_{11} \right)^2 + \frac{1}{4} (\lambda_2 w_1)^2 \right\}^{1/2} \right]. \tag{6.4}$$

Fig. 1 shows the nature of variation of principal space curvatures of field due to pure mass having prolate spheroidal symmetry as the source is approached along the axis of symmetry and along an equatorial radius, respectively. As expected,

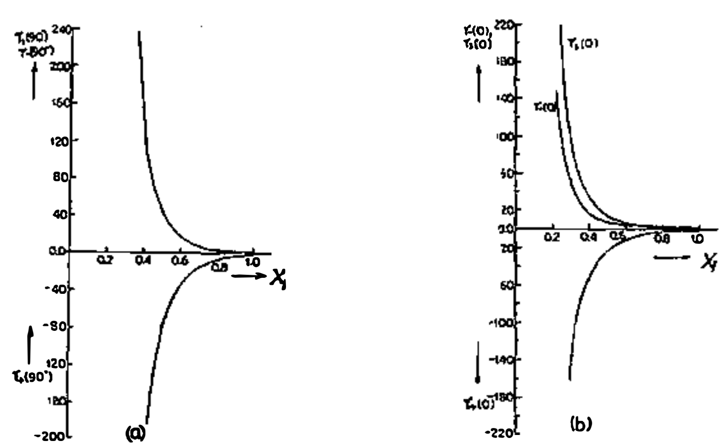

Fig. 1. Variation of principal space curvatures of field due to pure mass as the source is approached along the axis of symmetry (a) and along an equatorial radius (b).

the singularity at the origin is stronger in the former case and γ_3, γ_- are identical. Fig. 2 shows the variation of the same parameters of a field of similar symmetry due to a source with the electronic mass-charge ratio. Singularity of the γ's in this case arises due to the singularity of e^{w} at the point where $\chi = \frac{\pi}{2}$.

The expressions (6.3), (6.4) have been derived independently from the field equations of Sec. 2 for $\Phi = 0$. They also follow from (6.1), (6.2) in the limit (5.12).

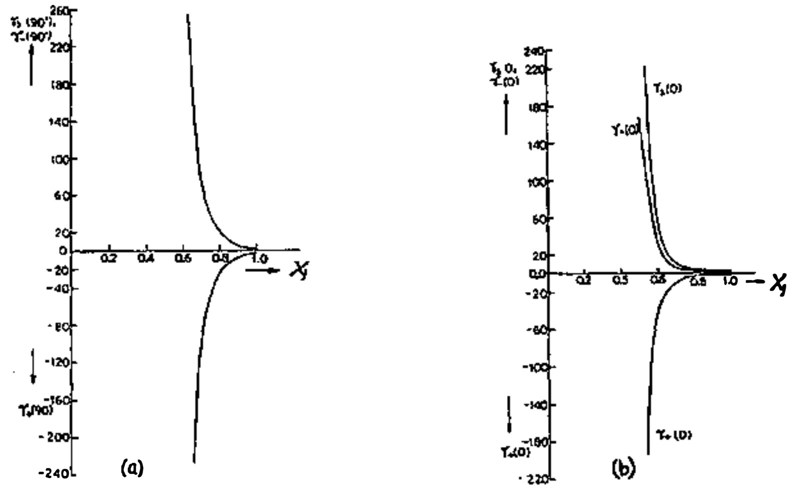

Fig. 2. Variation of principal space curvatures of field due to a source with electronic mass-charge ratio as the source is approached along the axis of symmetry (a) and along an equatorial radius (b).

7. Geodesic deviation

The nature of the gravitational field experienced by an observer moving in an arbitrary manner in a region where no real gravitational field exists has been discussed at length in two previous papers[13,14]. We shall now consider the opposite case and describe the experiences of an observer moving freely in a real gravitational field. This is best done by making use of the equations of geodesic deviation in which space — time curvature plays a prominent role[15-20].

The invariant equations of geodesic deviation[11,15,16] in an n-dimensional Riemannian space are

$$\xi^{(\lambda)}_{|\tau\tau} = - B^{\lambda}_{\mu\cdot\varrho\sigma}\, b^{\mu}\, b^{\varrho}\, \xi^{\sigma}, \tag{7.1}$$

where ξ^{μ} is the vector leading from a point on the geodesic B of an observer to the corresponding point on a neighbouring geodesic, b^{μ} is the tangent vector to B, $\xi^{(\tau)}_{|\tau\tau}$ is the λ component of the second covariant derivative of ξ^{μ} along B and τ is the proper time of the observer. When the expressions for $\xi^{(\lambda)}_{|\tau\tau}$ and the Riemann tensor $R^{\lambda}_{\mu\cdot\varrho\sigma}$ are used, the equation in general coordinates assumes the form

$$\frac{\mathrm{d}^2\, \xi^{\lambda}}{\mathrm{d}\tau^2} + 2\, \Gamma^{\lambda}_{\nu\varrho}\, b^{\nu}\, \frac{\mathrm{d}\xi^{\varrho}}{\mathrm{d}\tau} + \left(\frac{\partial}{\partial \chi^{\sigma}} \Gamma^{\lambda}_{\mu\varrho}\right) b^{\mu}\, b^{\varrho}\, \xi^{\sigma} = 0. \tag{7.2}$$

This form seems to be new and is more convenient for determining ξ than the equation in locally Cartesian coordinates.

In locally Galilean coordinates in the (3 + 1)-dimensional space of relativity Eq. (7.1) takes the form

$$\frac{d^2 y_i}{dt^2} = R_{4i\ 4k}\, y_k. \tag{7.3}$$

Using the field equations for empty space and the definition of the Ricci tensor we obtain

$$R_{4i\ 4j} = \alpha_{ij}, \tag{7.4}$$

$$\alpha_{11} + \alpha_{22} + \alpha_{33} = 0. \tag{7.5}$$

Eq. (7.3) represents harmonic motion with frequencies ω given by

$$|\alpha_{ij} + \omega^2\, \delta_{ij}| = 0. \tag{7.6}$$

This is a cubic equation in ω^2 and the sum of its roots vanishes by (7.5). Comparing (7.6) with (4.1) we see that $\omega_i^2 = -\gamma_i$. The results are well known.

The Gaussian curvature of a geodesic surface defined by the directions u, v having components u^i, v^j is given by

$$K = \frac{R_{hijk}\, u^h\, v^i\, u^j\, v^k}{(g_{hj}\, g_{ik} - g_{hk}\, g_{ij})\, u^h\, v^i\, u^j\, v^k}. \tag{7.7}$$

Therefore, for the geodesic surfaces containing the time axis we have

$$\begin{gathered} K(14) = -K(23) = -\alpha_{11}, \qquad K(24) = -K(13) = -\alpha_{22} \\ K(34) = -K(12) = -\alpha_{33}, \end{gathered} \tag{7.8}$$

where $K(i4)$ is the Gaussian curvature of the $(i4)$ surface. Hence

$$K(14) + K(24) + K(34) = 0. \tag{7.9}$$

To bring out the advantages of using Eq. (7.2) instead of (7.3) for solving problems of geodesic deviation we consider the case of an observer moving in a circular orbit in the earth's field. If a is the radius of the orbit and ω the angular velocity of orbital motion, then

$$r = a, \quad \Theta = \pi/2, \quad \Phi = \omega t, \quad \omega^2 = m\, a^{-3}, \quad v^2 = m\, a^{-1}. \tag{7.10}$$

Bringing the observer to rest by a Lorentz transformation in the Φ-direction we find that

$$b^3 = \omega\mu^{-1}, \qquad b^4 = \mu^{-1}, \tag{7.11}$$

where $\mu^2 = 1 - 3ma^{-1}$, while b^1 and b^2 both vanish. It is convenient to assume the orthogonality of b^μ and ξ^μ expressed by the relation

$$g_{\mu\nu}\, b^\mu\, \xi^\nu = 0. \tag{7.12}$$

With the vector b fully specified the relation reduces to

$$\xi^4 = G^{-1}\, a^2\, \omega\, \xi^3, \tag{7.13}$$

with $G = 1 - 2ma^{-1}$. We now proceed to find the form of Eq. (7.2) for the special kind of motion considered here. The equation for $\lambda = 2$ is seen to separate out and to have the form

$$\frac{d^2 \xi^2}{d\tau^2} + \omega^2\, (1 + 3v^2)\, \xi^2 = 0. \tag{7.14}$$

The remaining three equations are coupled. Of these the equations for $\lambda = 3, 4$ become identical in virtue of the orthogonality relation (7.13). The common form to which they reduce is

$$\frac{d^2 \xi^3}{d\tau^2} + 2\omega\, (a\,\mu)^{-1}\, \frac{d\xi^1}{d\tau} = 0. \tag{7.15}$$

The equation for $\lambda = 1$ is

$$\frac{d^2 \xi^1}{d\tau^2} = 2a\,\omega\,\mu\, \frac{d\,\xi^3}{d\tau} + 3G\,\omega^2\,\mu^{-2}\,\xi^1. \tag{7.16}$$

The first order terms in (7.15), (7.16) are the components of the Coriolis force generated by the rotation and the term involving ξ^1 in (7.16) represents the sum of the centrifugal and gravitational forces. These are equations of motion with constant coefficients solvable by elementary methods. They follow directly from (7.2) when Schwarzschild coordinates are used. On the other hand, the equations of motion obtained from (7.3) after parallel transport of a tetrad have variable coefficients. The complication arises from the time-dependence of the Riemann tensor in tetrad coordinates.

It would be interesting to investigate the relativistic motion of particles in the field of an ellipsoidal mass distribution. The powerful method described in the preceding paragraph is especially suited for handling such problems. The general motion in such a field will be difficult to determine. Only special motions for which the geodesic equations can be solved approximately or exactly are expected to be mathematically tractable. The simplest case is that of motion in the equatorial plane of the ellipsoid. This problem and that of motion in a parabolic orbit in the Schwarzschild field will be considered in another paper.

8. Relativity of space curvature

The principal curvatures of space appearing in the solutions of Eq. (7.3) are likely to change for a different choice of the LG frame. That is, generally, found to be the case. Let S and S' be two LG frames and let S' have an instantaneous velocity v relative to S along the common x-axis. Then the components of the Riemann tensor appearing in the equations of motion in the two frames are easily seen to be connected by the relations:

$$R'_{4141} = R_{4141}, \tag{8.1}$$

$$R'_{4242} = \gamma^2 [v^2 R_{1212} + R_{4242} + 2v R_{1242}], \tag{8.2}$$

$$R'_{4142} = \gamma (R_{4142} + v R_{3432}), \tag{8.3}$$

$$R'_{4243} = \gamma^2 [(1 + v^2) R_{4243} + v R_{1243} + v R_{4213}]. \tag{8.4}$$

The corresponding relations for motion of S' in the y- and z-directions are obtained by interchanging 1, 2 and 1, 3 respectively in the above relations.

The relations do not simplify further unless one assumes that Riemann symbols of the type R_{ijk4} with 4 occuring only once amongst the indices vanish. This condition is automatically fulfilled for a LG frame momentarily at rest in a static field. Otherwise, we assume this condition to hold and write the relations in the simplified form:

$$\begin{gathered} \alpha'_{11} = \alpha_{11}, \quad \alpha'_{22} = \gamma^2 (\alpha_{22} - v^2 \alpha_{33}), \quad \alpha'_{33} = \gamma^2 (\alpha_{33} - v^2 \alpha_{22}), \\ \alpha'_{12} = \gamma \alpha_{12}, \quad \alpha'_{23} = \gamma^2 (1 + v^2) \alpha_{23}, \quad \alpha'_{31} = \gamma \alpha_{31}. \end{gathered} \tag{8.5}$$

From Eqs. (7.5), (8.5) we see that the curvatures remain unchanged for motion along the symmetry axis of a static field.

We shall now compute the change of curvature by an alternative method in the case left out by Pirani[21,22], that of an inertial observer going round the earth in an equatorial circle and making observation on a test particle starting from the same point with a small relative velocity in the Θ-direction. As the observer completes his journey following a great circle, the particle also makes one oscillation in the Θ-direction. The periodic time τ_0 of oscillation of a test particle will therefore be the same as that of the observer himself when measured with his own clock. To find τ_0 we make use of the following result of general relativity for circular orbits

$$\left(\frac{2\pi}{\tau_0}\right)^2 = \omega_0^2 = \frac{m}{r^3} (1 + 3v^2). \tag{8.6}$$

9. *Geodesic deviation in Schwarzschild field*

Resuming the discussion of Sec. 7 on geodesic deviation we consider again the case of an observer moving in a circular orbit in the earth's field. We first show that the Riemann tensor in this case assumes a particularly simple form in a Cartesian frame moving with the observer and spinning about the axis of the earth with the same angular velocity ω. The directions of the rotating axes at any instant together with the direction of the world line of the observer form an orthonormal tetrad $\overset{0}{\lambda}{}^{\mu}_{(a)}$ carrying two indices, the coordinate index μ and the tetrad index (a). We determine the components of the tetrad and collect them in Table (9.2) below. Using the notation and the results of Sec. 7 Eqs. (7.10), (7.11) we can write

$$\overset{0}{\lambda}{}^{\mu}_{(4)} = b^{\mu} = (0, 0, \omega\, \mu^{-1}, \mu^{-1}). \tag{9.1}$$

Because of the rotation $\overset{0}{\lambda}{}^{\mu}_{(1)}$ points always in the r-direction and $\overset{0}{\lambda}{}^{\mu}_{(2)}$ always in the Θ-direction. Their nonzero components are, $\overset{0}{\lambda}{}^{1}_{(1)} = G^{1/2}$, $\overset{0}{\lambda}{}^{2}_{(2)} = a^{-1}$. With $\overset{0}{\lambda}{}^{\mu}_{(3)}$ determined by the orthonormality we, thus, obtain the following table for the components of the tetrad:

	1	2	3	4	
(1)	$G^{1/2}$	0	0	0	
(2)	0	a^{-1}	0	0	
(3)	0	0	$G^{1/2}(a\mu)^{-1}$	$G^{-1/2}\, a\, \omega\, \mu^{-1}$	(9.2)
(4)	0	0	$\omega\, \mu^{-1}$	μ^{-1}	

The components of the Riemann tensor in the coordinates of the tetrad are given in terms of the components in the Schwarzschild coordinates by

$$\overset{0}{R}_{ijkl} = \overset{S}{R}_{\alpha\beta\gamma\delta}\, \overset{0}{\lambda}{}^{\alpha}_{(i)}\, \overset{0}{\lambda}{}^{\beta}_{(j)}\, \overset{0}{\lambda}{}^{\gamma}_{(k)}\, \overset{0}{\lambda}{}^{\delta}_{(l)}. \tag{9.3}$$

The values of $\overset{0}{R}_{4i4j} \equiv \overset{0}{\alpha}_{ij}$ obtained from this formula are

$$\overset{0}{\alpha}_{11} = \gamma^2 A\,(v^2 + 2), \quad \overset{0}{\alpha}_{22} = -\gamma^2 A\,(2v^2 + 1), \quad \overset{0}{\alpha}_{33} = -A,$$

$$\overset{0}{\alpha}_{ij} = 0 \text{ for } i \neq j, \text{ with } A = m/r^3. \tag{9.4}$$

These are found to agree with the values of α'_{ij} computed from (8.5) when 1 and 3 are interchanged. Once the components of the Riemann tensor are known it is a straightforward matter to set up and solve the equations of geodesic deviation in any system of coordinates. However, we shall not adopt this procedure here but shall obtain the solutions more easily by analyzing the motion of test particles relative to the observer.

The simplest solution is obtained when a test particle moves in the same orbit as the observer. It is

$$\overset{0}{X}{}^1 = 0, \qquad \overset{0}{X}{}^3 = A. \tag{9.5}$$

Another simple solution for low velocities is

$$\overset{0}{X}{}^3 = B, \quad \overset{0}{X}{}^3 = a\,\delta\,\Phi = a\,\delta\,(\omega\,t) = -\frac{3}{2}\,B\,\Phi, \tag{9.6}$$

corresponding to motion in a circular orbit infinitely near the observer's. The relativistic form of the solution is

$$\overset{0}{X}{}^1 = B, \quad \overset{0}{X}{}^3 = -\frac{3}{2}\left(1 - \frac{v^2}{2}\right) B\,\Phi, \tag{9.7}$$

obtained by inserting the factor $G^{-1/2}\,\mu$ in (9.6_2). The third solution is periodic and corresponds to motion in a neighbouring elliptic orbit whose equation in general relativity is

$$\frac{1}{r} = \frac{m}{D^2}\left\{1 + \varepsilon\cos\left(1 - \frac{3m^2}{D^2}\right)\Phi\right\}. \tag{9.8}$$

For values of the eccentricity $\varepsilon \ll 1$ the argument of cosine in this equation can be written approximately as $\mu^2\,\Phi = \mu\,\omega\,\tau$. The radial oscillations are therefore given by $\overset{0}{X}{}^1 = C\cos\mu^2\,\Phi$. In the Φ-direction, the only force acting on the test particle is the Coriolis force and, consequently, its equation of motion is

$$\frac{\mathrm{d}^2\overset{0}{X}{}^3}{\mathrm{d}\tau^2} = -2\omega\,\frac{\mathrm{d}\overset{0}{X}{}^1}{\mathrm{d}\tau}. \tag{9.9}$$

The periodic solution is, therefore,

$$\overset{0}{X}{}^1 = \mathrm{C}\cos\mu^2\,\Phi, \qquad \overset{0}{X}{}^3 = -\,2\,C\,\mu^{-1}\sin\mu^2\,\Phi. \tag{9.10}$$

Eqs. (9.5), (9.7)—(9.10) remain unchanged in a purely relativistic treatment of the problem.

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VAKUUMSKA POLJA I ZAKRIVLJENOST PROSTORA

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Istraživana su rješenja Weylovog tipa za vakuumsko polje koje potječe od sferoidalne raspodjele naboja. Izvedene su relacije za glavna zakrivljenja prostora. Također je diskutiran problem gibanja probnih čestica u sustavu opažača koji se slobodno giba u gravitacionom polju. Pokazano je kako se mijenjaju zakrivljenosti prostora koje se pojavljuju u jednadžbama gibanja kada se opažanja vrše iz raznih, lokalno Galilejanskih sustava.