YU ISSN 0015—3206
FZKAAA 20 (3) 337 (1988)

DOMAIN-LIKE STRUCTURE OF Q CONFIGURATION IN GENERAL RELATIVITY

CARL WOLF

Department of Physics, North Adams State College, North Adams, MA 01247 U. S. A.

Received 24 June 1988

UDC 530.12

Original scientific paper

Successive layers of conserved Q charges are studied in general relativity. The external metric can be written as a combination of these charges to serve as a probe for such configurations. A simple two layer model is used to illustrate the general approach.

1. Introduction

Global symmetries in nature reflect the presence of conserved quantum numbers which are not always easy to probe because they do not lead to the corresponding gauge fields that can be probed in particle experiments[1]. A classic example of this is the recently hotly debated nature of the fifth force which could be a manifestation of the gauging of baryonic charge which was normally thought to be a global charge[2,3]. Lepton number is another example of such a global charge whose spontaneous breakdown could very well lead to the correct prediction of Majorana neutrino masses along with the existence of neutral scallar particles mediating generation changes called »majorans« as described by the Gelmini-Roncadelli model[4,5]. In response to these issues, Coleman has invented the generic name »Q Ball« to symbolize the state of matter stabilized by a globally conserved quantum number that acts much like an angular momentum which gives rise to orbital stability by preventing the collapse of the orbit of a circulating particle[6]. Applied to configurations of fermions and scalar fields the notion of an L ball has led to stable objects that decay only slowly by fermion emission at the surface[7]. In a previous note we have discussed the general relativistic stability of Q matter in general relativity with the result that for a crude model with Higgs-like potential the Q ball will be stable providing it does not get too large[8]. We have also discussed this problem within the context of the bimetric theory of gravitation and shown that Q balls are also stable in this alternate theory of gravity having prior geometry[9]. This seems to be a relevant issue since such a theory prevents black hole formation and it seems of relevance of whether or not it admits bound objects with a globally conserved charge. These investigations suggest that we ask another relevant question, can Q matter lead to a domain-like structure with alternating maximum and minimum of the relevant potential of the scalar field. The point of this investigation is to discuss such a configuration, wherein the potential experiences max and min where the boundary conditions (B. C.) on the scalar field for each region are sufficient to insure the conservation of the relevant Q charge within each domain.

2. *Domain-like structure of Q matter*

To begin we write down the Lagrangian for a SO_2 scalar doublet with Higgs-like potential, $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$

$$\mathscr{L} = \left[\frac{\partial^\mu \varphi^T \partial_\mu \varphi}{2} - \frac{A_2}{4}\left(\varphi^T \varphi - \frac{A_1}{A_2}\right)^2\right]\sqrt{-g} \tag{2.1}$$

which is invariant under $\delta\varphi_1 = -\varepsilon\,\varphi_1$, $\delta\varphi_2 = \varepsilon\,\varphi_1$.

If we impose the B. C. for two constructive layers as indicated in Fig. 1, we have

$$\begin{aligned} &\varphi(r) = 0, \quad r = 0 \\ &\frac{\mathrm{d}\varphi(r)}{\mathrm{d}r} = 0, \quad r = R_1, \\ &\varphi(r) = 0, \quad r = R_2. \end{aligned} \tag{2.2}$$

We have a conserved charge within each domain given by

$$Q = \int\int\int \left(\frac{\partial \mathscr{L}}{\partial \varphi_{1,4}}\,\delta\varphi_1 + \frac{\partial \mathscr{L}}{\delta \varphi_{2,4}}\,\delta\varphi_2\right) \mathrm{d}r\,\mathrm{d}\Theta\,\mathrm{d}\overline{\varphi} \tag{2.3}$$

since the boundary values of the spatial integral vanish by the B. C., when we integrate

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathscr{L}}{\partial \varphi_{1,4}}\,\delta\varphi_1\right) + \frac{\partial}{\partial x^u}\left(\frac{\partial \mathscr{L}}{\partial \varphi_{2,4}}\,\delta\varphi_2\right)$$

$$\text{over } \mathrm{d}r\,\mathrm{d}\Theta\,\mathrm{d}\varphi \tag{2.4}$$

for $\delta\varphi_1 = -\varepsilon\varphi_2$, $\delta\varphi_2 = \varepsilon\varphi_1$.

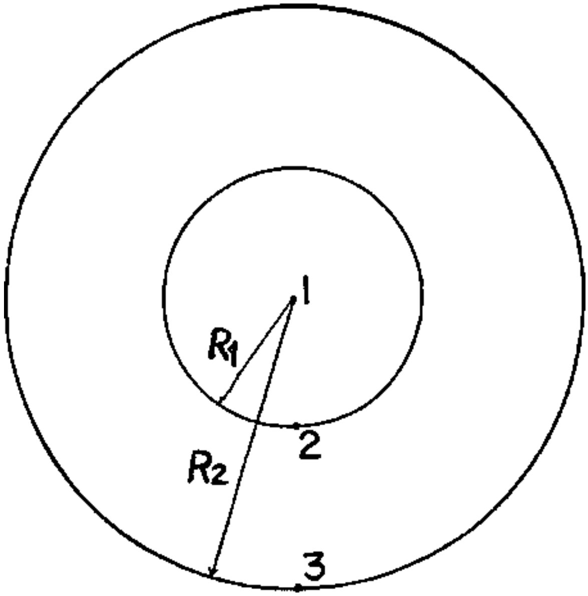

Fig. 1. Value of spatial part of scalar field; $\varphi(r) = 0$ at $r = 0$; $\varphi(r)_{,r} = 0$ at $r_i = R_1$; $\varphi(r) = 0$ for $r \geq R_2$.

This gives

$$Q_l = \frac{4\pi\omega_l}{c} \int_{R_l}^{R_{l+1}} |\varphi_l(r)|^2 \, e^{\frac{(\lambda-\nu)}{4}} r^2 \, \mathrm{d}r \tag{2.5}$$

for $\delta\varphi_1 = -\varepsilon\varphi_2$, $\delta\varphi_2 = \varepsilon\varphi_1$, where

$$\varphi_l(r, t) = \begin{pmatrix} \varphi_l(r) \cos\omega_l t \\ \varphi_l(r) \sin\omega_l t \end{pmatrix}. \tag{2.5'}$$

(Note in Eq. (2.3), Eq. (2.4) that φ_1, φ_2 denote components of a scalar doublet, while in Eq. (2.5) φ_l refers to *ith* domain) to represent the solution for the SO_2 doublet with separate ω_l, $\varphi_l(r)$, R_l, R_{l+1} for each domain.

The equation for the scalar field in each domain is found from the Lagrangian equation (2.1)

$$\mathscr{L} = \left[\frac{e^{-\nu}\varphi_i^2(r)\omega_i^2}{2c^2} - \frac{e^{-\lambda}(\varphi_{l,r})^2}{2} - \frac{A_2}{4}\left((\varphi_l(r))^2 - \frac{A_1}{A_2}\right)^2\right] r_i^2 \, e^{\frac{\lambda+\nu}{2}} \sin\Theta,$$

$$\left(\frac{e^{-\nu}\varphi_l(r)\omega_i^2}{c^2}\right) r^2 e^{\frac{\lambda+\nu}{2}} - A_2\varphi_l(r)\left((\varphi_l(r))^2 - \frac{A_1}{A_2}\right) r^2 e^{\frac{\lambda+\nu}{2}}$$

$$+ \frac{\mathrm{d}}{\mathrm{d}r}\left((\varphi_l, r)\, r^2 \, e^{\frac{(\nu-\lambda)}{2}}\right) = 0. \tag{2.6}$$

Note the time dependence cancels out of equation for scalar field after using Eq. (2.5′).

We solve Eq. (2.6) subject to the relevant boundary conditions and in the approximation $e^{\nu} \cong e^{\lambda} \cong 1$, we then evaluate the frequency from Eq. (2.5) in terms of the relevant charge Q_t.

For the energy momentum tensor we have

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial \mathscr{L}}{\partial g^{\mu\nu}} = \partial_\mu \varphi^T \partial_\nu \varphi - \frac{g_{\mu\nu}}{2} \partial^\alpha \varphi^T \partial_\alpha \varphi + g_{\mu\nu} \frac{A_2}{4} \left(\varphi^T \varphi - \frac{A_1}{A_2}\right)^2$$

or

$$T^\mu_\nu = \partial^\mu \varphi^T \partial_\nu \varphi - \frac{\delta^\mu_\nu}{2} (\partial^\alpha \varphi^T \partial_\alpha \varphi) + \delta^\mu_\nu \frac{A_2}{4} \left(\varphi^T \varphi - \frac{A_1}{A_2}\right)^2 \tag{2.7}$$

with

$$T^4_4 = \frac{e^{-\nu}\omega^2}{2c^2} (\varphi(r))^2 + \frac{1}{2} e^{-\lambda} (\varphi(r)_{,r})^2 + \frac{A_2}{4}\left((\varphi(r))^2 - \frac{A_1}{A_2}\right)^2 \tag{2.8}$$

after using Eq. (2.5′).

For the $^4_4$ component of the Einstein equation we have for the metric

$$(ds)^2 = e^{\nu}(dx^4)^2 - e^{\lambda}(dr)^2 - r^2 (d\Theta)^2 - r^2 \sin^2\Theta (d\overline{\varphi})^2$$

$$\frac{d}{dr}(r\, e^{-\lambda}) = 1 - \frac{8\pi G}{c^4} T^4_4 r^2. \tag{2.9}$$

For region I $(0 < r < R_1)$ we have upon integration

$$(R_1 e^{-\lambda})_{R1} = R_1 - \frac{8\pi G}{c^4} \int_0^{R_1} T^4_4 r^2 \, dr$$

$$(e^{-\lambda})_{R1} = 1 - \frac{8\pi G}{R_1 c^4} \int_0^{R_1} T^4_4 r^2 \, dr \tag{2.10}$$

where $\varphi(r = 0) = 0$, $\varphi(r)_{,r} = 0$ at $r = R_1$.

For region II $(R_1 < r < R_2)$ we have

$$(R_2 e^{-\lambda})_{R2} = (R_1 e^{-\lambda})_{R1} + (R_2 - R_1) - \frac{8\pi G}{c^4} \int_{R_1}^{R_2} r^2 T^4_4 \, dr \tag{2.11}$$

where

$$\varphi(r)_{,r} = 0 \quad \text{at} \quad r = R_1$$

$$\varphi(r) = 0 \quad \text{at} \quad r = R_2. \tag{2.12}$$

In general by successive integration we can evaluate $e^{-\lambda}$ at the final boundary and determine the mass from

$$e^{-\lambda} = 1 - \frac{2GM}{R_F c^2} - \frac{8\pi G V_{EFF} R_F^2}{3c^4} \tag{2.13}$$

where

$$V_{EFF} = \frac{A_2}{4}\left((\varphi(r = R_F))^2 - \frac{A_1}{A_2}\right)^2.$$

Here Eq. (2.13) represents the metric exterior to the Q configuration and $V_{effective}$ represents the potential outside the last layer and is equal to the potential at the last boundary.

To illustrate the above idea we construct an approximate solution as follows: in the approximation $e^{\nu} \cong e^{-\lambda} \cong 1$, Eq. (2.6) becomes in a given domain

$$r^2 \varphi(r)_{,rr} + 2r\varphi(r)_{,r} + \frac{r^2}{c^2}\omega^2 \varphi(r) - A_2 \varphi(r)\left[(\varphi(r))^2 - \frac{A_1}{A_2}\right] r^2 = 0. \tag{2.14}$$

For the equation without the cubic term we know that a simple power series solution exists about $r = 0$, we thus try

$$\varphi(r) = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4 \tag{2.15}$$

where we will retain only up to the quartic term in the power series solution. Substituting Eq. (2.15) into Eq. (2.14), we have

$$r(a_1) + r^2\left(6a_2 + \frac{\omega^2}{c^2} a_0 - a_0^3 A_2 + A_1 a_0\right) +$$

$$+ r^3\left(12a_3 + \frac{\omega^2}{c^2} a_1 - 2a_0^2 a_1 A_2 - a_1 a_0^2 A_2 + A_1 a_1\right) = 0 \tag{2.16}$$

$$a_0 = \text{arb.}$$

$$a_1 = 0$$

$$a_2 = \frac{a_0^3 A_2 - a_0\left(\frac{\omega^2}{c^2} + A_1\right)}{6} \tag{2.17}$$

$$a_3 = 0$$

$$a_4 = \text{to be determined from B. C.}$$

For region I we have $\varphi(r) = 0$ at $r = 0$, $\varphi_{,r}(r = R_1) = 0$ at $r = R_1$; this gives $a_0 = 0$, $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, where we find the a_4 coefficient from the B. C., thus

$$\varphi(r) = 0 \text{ for } 0 \leq r \leq R_1. \tag{2.18}$$

For region II we have $\varphi,_r(r = R_1) = 0$, $\varphi(r = R_2) = 0$; giving upon using the solution Eq. (2.17) with a_4 to be determined by the B. C., we have

$$a_0 + \frac{\left[a_0^3 A_2 - a_0\left(\frac{\omega^2}{c^2} + A_1\right)\right]R_2^2}{6} + a_4 R_2^4 = 0$$

$$\frac{2\left[a_0^3 A_2 - a_0\left(\frac{\omega^2}{c^2} + A_1\right)\right]R_1}{6} + 4a_4 R_1^3 = 0 \qquad (2.19)$$

eliminating a_4 we find from Eq. (2.19)

$$a_0^2 = \frac{\frac{\left(\frac{\omega^2}{c^2} + A_1\right)R_2^2}{6} - 1 - \frac{R_2^4}{12R_1^2}\left(\frac{\omega^2}{c^2} + A_1\right)}{\frac{A_2 R_2^2}{6} - \frac{A_2 R_2^4}{12 R_1^2}} \qquad (2.20)$$

with

$$a_2 = \frac{a_0^3 A_2 - a_0\left(\frac{\omega^2}{c^2} + A_1\right)}{6}$$

$$a_4 = -\frac{1}{12R_1^2}\left[a_0^3 A_2 - a_0\left(\frac{\omega^2}{c^2} + A_1\right)\right].$$

To calculate the relation of ω to Q in region II we have from Eq. (2.4) using $e^{\lambda} \cong \cong e^{\nu} \cong 1$ and keeping only up to the quadratic terms in r

$$Q = \frac{4\pi\omega}{c}\int_{R_1}^{R_2} r^2\left[a_0 + \left(\frac{a_0^3 A_2 - a_0 A_1}{6}\right) r^2\right]^2 \mathrm{d}r \qquad (2.21)$$

where we have not included the ω^2 term in a_2 assuming it smaller than the other terms. Thus

$$\omega = \frac{cQ}{4\pi \int_{R_1}^{R_2} r^2 \left(a_0 + \frac{a_0^3 A_2 - a_0 A_1}{6} r^2\right)^2 \mathrm{d}r} \qquad (2.22)$$

for region II.

We now evaluate the metric in the two regions as well as the mass as a function of the Q charge. From the $^4_4$ component of the Einstein equation we have for the metric $(ds)^2 = e^{\nu}(dx^4) - e^{\lambda}(dr)^2 - r^2 (d\Theta)^2 - r^2 \sin^2 \Theta (d\overline{\varphi})^2$,

$$\frac{d}{dr}(r\, e^{-\lambda}) = 1 - \frac{8\pi G}{c^4} T^4_4 r^2. \tag{2.23}$$

From Eq. (2.8) we have for region I

$$e^{-\lambda} = 1 - \frac{8\pi G}{3c^4}\left(\frac{A_1^2}{4A_2}\right) r^2$$

where $\varphi(r) = 0$ for $0 \leq r \leq R_1$.

By integrating between R_1 and R_2 we have, using Eq. (2.8) for T^4_4,

$$(R_2 e^{-\lambda})_{R2} - (R_1 e^{-\lambda})_{R1} = (R_2 - R_1) - \frac{8\pi G}{c^4}\int_{R_1}^{R_2}\left[\frac{\omega^2}{2c^2}(\varphi(r))^2 + \frac{1}{2}(\varphi_{,r})^2 + \right.$$
$$\left. + \frac{A_2}{4}\left((\varphi(r))^2 - \frac{A_1}{A_2}\right)^2\right] r^2\, dr \tag{2.24}$$

or

$$(R_2 e^{-\lambda})_{R2} = R_1 - \frac{8\pi G}{3c^4}\left(\frac{A_1^2}{4A_2}\right) R_1^3 + (R_2 - R_1) - \frac{8\pi G}{c^4}\int_{R_1}^{R_2}\left[\frac{\omega^2}{2c^2}(\varphi(r))^2 + \right.$$
$$\left. + \frac{1}{2}(\varphi_{,r})^2 + \frac{A_2}{4}\left((\varphi(r)^2 - \frac{A_1}{A_2}\right)^2\right] r^2\, dr \tag{2.25}$$

or

$$(e^{-\lambda})_{R2} = 1 - \frac{8\pi G}{3c^4}\left(\frac{A_1^2}{4A_2}\right)\frac{R_1^3}{R_2} - \frac{8\pi G}{c^4 R_2}\int_{R_1}^{R_2}\left[\frac{\omega^2}{2c^2}(\varphi(r))^2 + \frac{1}{2}(\varphi_{,r})^2 + \right.$$
$$\left. + \frac{A_2}{4}\left((\varphi(r))^2 - \frac{A_1}{A_2}\right)^2\right] r^2\, dr \tag{2.26}$$

on the other hand for $r > R_2$ outside the Q matter we have where

$$V_{EEF} = \frac{A_1^2}{4A_2},\ r > R_2$$

$$e^{-\lambda} = 1 - \frac{2GM}{rc^2} - \frac{8\pi G}{3c^4}\left(\frac{A_1^2}{4A_2}\right) r^2. \tag{2.27}$$

To calculate the mass of the system we match Eq. (2.26) with Eq. (2.27) at $r = R_2$ to obtain

$$1 - \frac{2GM}{R_2c^2} - \frac{8\pi G}{3c^4}\left(\frac{A_1^2}{4A_2}\right)R_2^2 = 1 - \frac{8\pi G}{3c^4}\left(\frac{A_1^2}{4A_2}\right)\frac{R_1^3}{R_2} - \frac{8\pi G}{c^4R_2}\int\limits_{R_1}^{R_2}\left[\frac{\omega^2(\varphi(r))^2}{2c^2} + \frac{1}{2}(\varphi_{,r})^2 + \frac{A_2}{4}\left((\varphi(r))^2 - \frac{A_1}{A_2}\right)^2\right]r^2\,\mathrm{d}r$$

or (2.28)

$$M = \frac{R_2c^2}{2G}\left[\frac{8\pi G}{3c^4}\left(\frac{A_1^2}{4A_2}\right)\frac{R_1^3}{R_2} + \frac{8\pi G}{c^4R_2}\int\limits_{R_1}^{R_2}\left[\frac{\omega^2(\varphi(r))^2}{2c^2} + \frac{1}{2}(\varphi_{,r})^2 + \frac{A_2}{4}\left((\varphi(r))^2 - \frac{A_1}{A_2}\right)^2\right]r^2\,\mathrm{d}r\right] - \left[\frac{8\pi G}{3c^4}\left(\frac{A_1^2}{4A_2}\right)R_2^2\right]\frac{R_2c^2}{2G}$$

where φ is given by $\varphi = a_0 + a_2r^2 + a_4r^4$ for $R_1 < r < R_2$ where

$$a_0^2 = \frac{\left(\frac{\omega^2}{c^2} + A_1\right)\frac{R_2^2}{6} - 1 - \frac{R_2^4}{12R_1^2}\left(\frac{\omega^2}{c^2} + A_1\right)}{\frac{A_2R_2^2}{6} - \frac{A_2R_2^4}{12R_1^2}} \tag{2.29}$$

$$a_2 = \frac{a_0^3A_2 - a_0\left(\frac{\omega^2}{c^2} + A_1\right)}{6}, \qquad a_4 = -\frac{1}{12R_1^2}\left[a_0^3A_2 - a_0\left(\frac{\omega^2}{c^2} + A_1\right)\right]$$

and from Eq. (2.22)

$$\omega = \frac{cQ}{4\pi\int\limits_{R_1}^{R_2} r^2\left(a_0 + \left(\frac{a_0^3A_2 - a_0A_1}{6}\right)r^2\right)^2\mathrm{d}r}.$$

To evaluate ω we approximate

$$a_0^2 \cong \frac{A_1\left(\frac{R_2^2}{6} - \frac{R_2^4}{12R_1^2}\right)}{A_2\left(\frac{R_2^2}{6} - \frac{R_2^4}{12R_1^2}\right)} = \frac{A_1}{A_2}.$$

3. Conclusion

We have clearly chosen a very simple example by first insisting that the power series $\varphi = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4$ obeys the differential equation up to the cubic term and determined the constant a_4 from the B. C. Actually if we determined a_4 from the equation it would disagree with the results we found by forcing it to obey the B. C. However, since we are only dealing with an approximate solution this model is at least suggestive. Also the calculation of ω in terms of Q we estimated by keeping only linear powers in ω and taking only up to quadratic terms in $\varphi(r)$; a more exact analysis would involve a solution of a cubic equation for ω in terms of Q. No attempt here has been made at rigour but only the suggestion that by solving Eq. (2.6) in each region and calculating each Q_l by Eq. (2.5) using the appropriate B. C. that either $\varphi(r) = 0$ or $\varphi(r_{,r}) = 0$ at each boundary, we may by successive integration of Eq. (2.9) determine the exterior metric in terms of the various Q charges in each layer. This is the point behind the calculation, namely that the exterior metric is sensitive to the global Q charges of each layer.

References

1) P. Langacker, Phys. Rep. **22** (1981) 185;
2) H. H. Thodberg, Phys. Rev. Lett. **56** (1986) 2423;
3) I. Bars and M. Visser, Phys. Rev. Lett. **57** (1986) 25;
4) G. B. Gelmini and M. Roncadelli, Phys. Lett. B **99** (1981) 411;
5) G. B. Gelmini and J. W. F. Valle, Phys. Lett. B **142** (1984) 181;
6) S. Coleman, *Lecture on Q Balls*, M. I. T. (November 21, 1986);
7) A. Cohen, S. Coleman, H. George and A. Monohar, Nucl. Phys. B **272** (1986) 301;
8) C. Wolf, Phys. Lett. A **117** (1986) 443;
9) C. Wolf, (submitted to Il Nuovo Cimento B).

DOMENI SLIČNA STRUKTURA Q KONFIGURACIJE U OPĆOJ TEORIJI RELATIVNOSTI

CARL WOLF

Department of Physics, North Adams State College, North Adams, MA 01247, U. S. A.

UDK 530.12

Originalni znanstveni rad

U općoj teoriji relativnosti proučavani su sukcesivni slojevi konzerviranih Q naboja. Vanjska metrika može se napisati kao kombinacija tih naboja da bi poslužila kao proba za takove konfiguracije. Za ilustraciju generalnog pristupa korišten je jednostavan model dva sloja.