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SYMMETRIES AND RECURSION OPERATORS OF VARIABLE COEFFICIENT KORTEWEG-de VRIES EQUATIONS

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Olver's method of finding the existence of infinitely many symmetries for an evolution equation is found to be true for the nonisospectral case. The recursion operators are developed from the Lax-pairs and this method is extended for nonisospectral problems. It is found that the minimum number of different infinite set of symmetries is the same as the number of independent similarity transformation groups associated with the given evolution equation. The relation between travelling wave solution and similarity transformation is also discussed. The results are applied to the variable coefficient Korteweg-de Vries equations.

1. Introduction

Associated with a given exactly solvable equation, there are two important aspects of fundamental studies, »methods of finding soliton-like particular solution« and »algebraic properties«. We have different powerful methods of finding exact particular solutions, like Auto-Bäcklund transformation (ABT), Inverse Scattering Technic (IST) etc.[1] Regarding the algebraic properties, one looks for a Lie group acting on the manifold in such a way that solutions of the dynamical system are mapped into solutions and finding a set of infinitely many conserved quantities of the equation which are in involution. It is found that a hierarchy of exactly solvable equation can be generated recursively with the help of an integro-differential operator, given various names in literature[2–6], such as recursion operator, strong symmetries etc. The recursion operator is understood as a special Lie algebra deformation with linear interpolation properties and for a recursion operator it is shown that the eigenvector decompositions are time invariant[7].

In this paper we will show that, Lax-pair, recursion operator, symmetries, etc., like algebraic properties are also associated with nonisospectral eigenvalue problems of nonlinear differential equations. The method of constructing recursion operator from isospectral Lax-pair is extended for nonisospectral problem. Olver's[2] condition for the existence of infinite set of symmetries is found to be applicable for nonisospectral problems. The Lie point group transformation method is used to find the number of different infinite symmetries associated with a differential equation. Our results are applied to a variable coefficient Korteweg-de Vries (VKdV) equation.

2. *Recursion operator from Lax-pair*

First, we will introduce a VKdV equation, a model for isospectral problem,

$$u_t = -\alpha t^n u u_x - \beta t^{2n+1} u_{xxx} \tag{2.1}$$

where n is any real number and α and β are arbitrary constants. The VKdV equation is a standard model for variable depth shallow water waves[8,9]. The isospectral Lax-pair of this equation is reported earlier[9],

$$\frac{\beta}{\alpha} t^{2(n+1)} \psi_{xx} + \frac{1}{6} t^{n+1} \psi - \frac{(n+1)}{6\alpha} x \psi = \lambda \psi \tag{2.2}$$

$$4\beta\, t^{2n+1} \psi_{xxx} + \alpha\, t^{n+1} u \psi_x + \frac{\alpha}{2} t^n u_x \psi = -\psi_t \tag{2.3}$$

where λ is the eigenvalue corresponding to the eigenfunction. It can be easily shown that $\frac{\partial \lambda}{\partial t} = 0$.

The VKdV equation can be transformed to a KdV-type equation[10],

$$v_t = -\beta\, t^{2n+1} (u_{xxx} - 6 v v_x) - \frac{(n+1)}{t} (x\, v_x + 2v) \tag{2.4}$$

using the transformation

$$u = -\frac{6\beta}{\alpha} t^{n+1} v + \frac{n+1}{\alpha} t_x^{-(n+1)}. \tag{2.5}$$

Corresponding to the KdV-type equation (2.4) we have a nonisospectral Lax-pair

$$\Phi_{xx} = (v + \mu)\, \Phi \tag{2.6}$$

$$\Phi_t = -\beta\, t^{(2n+1)} v_x \Phi + \left[-\frac{(n+1)}{t} v_x + 2\beta\, t^{(2n+1)} (v + 2\mu) \right] \Phi_x \qquad (2.7)$$

where μ is the time dependent eigenvalue and Φ is the respective eigenfunction. It can be easily verify that

$$\frac{\partial \mu}{\partial t} = -\frac{2(n+1)}{t} \mu. \qquad (2.8)$$

Now we will construct the recursion operators for both equations using a well known procedure[11,12] For an evolution equation of the type

$$u_t = K(u, x, t) \qquad (2.9)$$

where $u(x, t)$ is an element of the space S of functions on the real line, vanishing rapidly for $|x| \to \infty$. Let $K(u, x, t)$ be some differential map on the space S, depending explicitly on the variables x and t The directional derivative of a function $\Phi(u)$ in S is defined by

$$\Phi'(u)[w] = \frac{\partial}{\partial \varepsilon} \Phi(u + \varepsilon w)|_{\varepsilon=0}. \qquad (2.10)$$

Let

$$A(K(u, x, t))\, w = \frac{\partial}{\partial \varepsilon} K(u + \varepsilon w)|_{\varepsilon=0} \qquad (2.11)$$

then

$$w_t = A(K(u, x, t))\, w \qquad (2.12)$$

is called the linearization[2] of Eq. (2.9) about $u(x, t)$. Let the Lax-pair of Eq. (2.9) be

$$L(u, x, t, \Phi, \lambda) = 0 \qquad (2.13)$$

$$M(u, x, t, \Phi, \lambda) = \Phi_t \qquad (2.14)$$

where (2.13) is the eigenvalue problem with λ being the eigenvalue. Then there exists a transformation $T(\Phi, \lambda)$ mapping the eigenfunction $\Phi(x, t)$ of (2.13) into the solution $w(x, t)$ satisfying the linearized equation (2.12). Then using the inverse transformation $T^{-1}(\Phi, \lambda)$, we can bring the eigenvalue problem (2.13) into

$$R(u, x, t)\, w = f(\lambda)\, w \qquad (2.15)$$

where $f(\lambda)$ is a function of λ alone when we consider isospectral eigenvalue problem. And $f(\lambda)$ is time dependent when we have nonisospectral problem (Eq. (2.13)). Then $R(u, x, t)$ will be the required recursion operator.

Corresponding to the VKdV equation (2.1),

$$K(u, x, t) = -\alpha\, t^n\, u u_x - \beta\, t^{2n+1}\, u_{xxx} \tag{2.16}$$

the linearized equation is given by

$$w_t = (-\beta\, t^{2n+1}\, D_x^3 - \alpha\, t^n\, u\, D_x - \alpha\, t^n\, u_x)\, w \tag{2.17}$$

then

$$A\,(K(u, x, t)) = -\beta\, t^{2n+1}\, D_x^3 - \alpha\, t^n\, u D_x - \alpha\, t^n\, u_x \tag{2.18}$$

where $D_x = \frac{\partial}{\partial x}$.

The transformation

$$w = D_x(\psi^2) \tag{2.19}$$

takes the solution of (2.2) and (2.3) into the solution of (2.17). The inverse transformation

$$\psi^2 = D_x^{-1}(w) \tag{2.20}$$

when substitutes in Eq. (2.2) yields

$$\left[t^{2(n+1)}\, D_x^2 + \frac{2\alpha}{3\beta}\, t^{n+1}\, u - 2\frac{(n+1)}{3\beta}\, x + \left(\frac{\alpha}{3\beta}\, t^{n+1}\, u_x - \frac{(n+1)}{3\beta}\right) D_x^{-1}\right] w = \\ = \frac{4\alpha\,\lambda}{\beta}\, w. \tag{2.21}$$

This implies that the recursion operator $R(u, x, t)$ of the VKdV Eq. (2.1) is given by

$$R(u, x, t) = t^{2(n+1)}\, D_x^2 + \frac{2\alpha}{3\beta}\, t^{n+1}\, u - 2\frac{(n+1)}{3\beta}\, x + \left(\frac{\alpha}{3\beta}\, t^{n+1}\, u_x - \right. \\ \left. - \frac{(n+1)}{3\beta}\, D_x^{-1}\right). \tag{2.22}$$

In the above case $f(\lambda) = 4\alpha\lambda/\beta$ is independent of the variable t. Now let us consider the nonisospectral case (Eqs. (2.6) and (2.7)).

Corresponding to the KdV-type equation (2.4),

$$K(v, x, t) = -\beta\, t^{2n+1}\, (v_{xxx} - 6 v v_x) - \frac{(n+1)}{t}\, (x v_x + 2v). \tag{2.23}$$

The linearization gives

$$A(K(v, x, t)) = -\beta t^{2n+1}(D_x^3 - 6v D_x - 6v_x) - \frac{(n+1)}{t}(x D_x + 2). \tag{2.24}$$

The transformation (2.19) takes the solution of (2.6) and (2.7) into the solution of (2.24) then the inverse transformation (2.20) when inserts in (2.6), yields

$$t^{2(n+1)}(D_x^2 - 4v - 2v_x D_x^{-1}) w = f(\mu, t) w \tag{2.26}$$

where $f(\mu, t) = 2\mu t^{2(n+1)}$. Obviously, the recursion operator for the KdV-type equation (2.4),

$$R(v, x, t) = t^{2(n+1)}(D_x^2 - 4v - 2v_x D^{-1}). \tag{2.27}$$

Now using Olver's standard procedure[2)] we will show that the recursion operators $R(u, x, t)$ and $R(v, x, t)$ are going to give infinitely many symmetries or flows of the evolution Eqs. (2.1) and (2.4), respectively.

Let $M[u]$ denotes the algebra of polynomials in the variable $u(x, t)$ and its partial derivatives with respect to x. For $P \in M[u]$, let $[P]$ denotes differential ideal generated by P. Let V be the vector space of all formal polynomial partial differential operators acting on $M[u]$ and whose elements are of the form

$$R(u, x, t) = \sum_n P_n \partial^n \tag{2.28}$$

where $P_n \in M[u]$. We define the map

$$A: M[u] \to T \tag{2.29}$$

is the linearization of P. Let V_0 be the subspace of V generated by the operator $\partial^n = \frac{\partial^n}{\partial x^n}$ whose elements are of the form $\sum_{n=0}^{\infty} P_n \partial^n$ and $P_n \in M[u]$.

According to Olver, for a given differential equation of the type Eq. (2.9) has a recursion operator (2.28) belongs to V_0, satisfies

$$[A(K) - D_t, R] P \in [u_t - K] \tag{2.30}$$

for all $P[u]$, where $D_t = \frac{\partial}{\partial t}$, then $K(u, x, t)$ possesses an infinite series of commuting symmetries, where [.,.] denotes the commutator

$$[X, Y] = XY - YX \tag{2.31}$$

for all functions X and Y.

The recursion operator $R(u, x, t)$ and linearized equation $A(K(u, x, t))$ of the VKdV equation (2.1) are respectively (2.22) and (2.25). Hence

$$A(K(u, x, t))\cdot R(u, x, t) = -\beta\, t^{4n+3} D_x^5 - \frac{\alpha}{3} t^{3n+2} (5u D_x^3 + 10u_x D_x^2 + 5u_{xxx}) - \\ - \frac{\alpha^2}{3\beta} t^{2n+1} (u^2 D_x + 5uu_x) + \frac{t^{2n+1}}{3} (n+1)(2x D_x^3 + 7D_x^2) + \\ + 2\alpha \frac{(n+1)}{3\beta} t^n (x u D_x + x u_x + \frac{3}{2} u) - \left(\frac{t^{3n+2}}{3} u_{xxxx} + \frac{\alpha^2}{3\beta} t^{2n+1} uu_{xx} - \right. \\ \left. - \frac{\alpha^2}{3\beta} t^{2n+1} u_x^2 + \frac{\alpha}{3\beta}(n+1) t^n u_x\right) D_x^{-1} \tag{2.32}$$

and

$$R(u, x, t)\cdot A(K(u, x, t)) = -\beta\, t^{4n+3} D_x^5 - \frac{\alpha}{3} t^{3n+2} (5u D_x^3 + 10u_x D_x^2 + \\ + 9u_{xx} D_x + 3u_{xxx}) - \frac{\alpha^2}{\beta} t^{2n+1} \left(\frac{2}{3} u^2 D_x + uu_x\right) + \\ + \frac{(n+1)}{3} t^{2n+1} (2x D_x^3 + D_x^2) + \frac{\alpha}{3\beta}(n+1) t^n (2x u D_x + 2x u_x + u). \tag{2.33}$$

Therefore the commutators are given by

$$[A, R] = 2(n+1) t^{2n+1} D_x^2 - \frac{2\alpha}{3\beta}(\beta\, t^{3n+2} u_{xxx} + \alpha\, t^{2n+1} uu_x - \\ - (n+1) t^n u) + \frac{\alpha}{3\beta}(n+1) t^n u_x D_x^{-1} - \frac{\alpha}{3\beta}(\beta\, t^{3n+2} u_{xxxx} + \\ + \alpha\, t^{2n+1} uu_{xx} + \alpha\, t^{2n+1} u_x^2) D_x^{-1} \tag{2.34}$$

and

$$[D, R] = 2(n+1) t^{2n+1} D_x^2 + 2(n+1)\frac{\alpha}{3\beta} t^n u + \frac{2\alpha}{3\beta} t^{n+1} u_t + \\ + (n+1)\frac{\alpha}{3\beta} t^n u_x D_x^{-1} + \frac{\alpha}{3\beta} t^{n+1} u_{xt} D_x^{-1} \tag{2.35}$$

so that the condition (2.30) is satisfied. This implies that the recursion operator $R(u, x, t)$ will give an infinite series of symmetries of the VKdV equation (2.1).

For the KdV-type equation (2.4), the recursion operator $R(v, x, t)$ and linearized equation $A(K(v, x, t))$ are given by Eqs. (2.25) and (2.27), respectively. As earlier we can easily find the commutator

$$[A(K(v, x, t)), R(v, x, t)] = 2(n+1)\, t^{2n+1} D_x^2 + 4\beta\, t^{4n+3} (v_{xxx} - 6vv_x) + \\ + 4(n+1)\, t^{2n+1} xv_x + 2\beta\, t^{4n+3} (v_{xxxx} - 6vv_{xx} + 6v_x^2) D_x^{-1} + \\ + 2(n+1)\, t^{2n+1} xv_{xx} D_x^{-1} \tag{2.36}$$

and

$$[D_t, R(v, x, t)] = 2(n+1)\, t^{2n+1} (D_x^2 - 4v - 2v_x D_x^{-1}) - \\ - 4t^{2(n+1)} v_t - 2t^{2(n+1)} v_{xt} D_x^{-1} \tag{2.37}$$

so that the condition (2.30) is satisfied. In the next section we will show the commuting and noncommuting symmetries of both equations.

3. *Symmetries and similarity transformations*

It is well known[13] that Lie's point group transformation theory has been one of the powerful methods of integration of differential equations. When a partial differential equation is invariant under a Lie's point group transformation then it is possible to find similarity solutions of the equation and its number of independent variables can be reduced by one[14].

A partial differential equation with one dependent variable $u(x, t)$ and two independent variables x and t:

$$P(x, t, u, u_t, u_x, \dots) = 0 \tag{3.1}$$

is said to be invariant under the transformation

$$x^* = x + \varepsilon X(x, t, u) + O(\varepsilon^2) \tag{3.2}$$

$$t^* = t + \varepsilon T(x, t, u) + O(\varepsilon^2) \tag{3.3}$$

$$u^* = u + \varepsilon U(x, t, u) + O(\varepsilon^2) \tag{3.4}$$

when

$$P(x^*, t^*, u^*_{t^*}, u^*_{x^*}, u^*, \dots) = 0. \tag{3.5}$$

The infinitesimals X, T and U will determine the possible Lie point group transformation under which Eq. (3.1) is invariant. Obviously, they will determine the symmetries of the given differential equation. If we know the recursion operator and the infinitesimals X, T and U then we can determine different sets of infinite series of symmetries of a given partial differential equation. Moreover, one of these symmetries will generate an infinite set of commuting symmetries and that will give the conserved functionals and conserved covariants of the given equation.

The similarity transformation analysis of the VKdV equation (2.1) is reported earlier[9] and we obtained the infinitesimals,

$$X = \frac{a\,\alpha}{n+1}\, t^{n+1} + b \tag{3.6}$$

$$T = 0 \tag{3.7}$$

$$U = a \tag{3.8}$$

where a and b are parameters. The respective generators are

$$G_1 = \frac{\alpha}{n+1}\, t^{n+1} \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \tag{3.9}$$

$$G_2 = \frac{\partial}{\partial x} \tag{3.10}$$

and they will form an Abelian group

$$[G_1, G_2] = 0. \tag{3.11}$$

Eqs. (3.6), (3.7) and (3.8) indicate that there are two types of symmetries for the VKdV equation (2.1), and thay are η and τ

$$\eta = -\frac{\alpha}{3\beta}\, t^{n+1}\, u_x + \frac{(n+1)}{3\beta} \tag{3.12}$$

$$\tau = u_x. \tag{3.13}$$

Eqs. (3.12) and (3.13) gives two different family of infinite set of symmetries. The elements of the first set of symmetries be denoted by η_n and the that of the second set is τ_n, then

$$\eta_{n+1} = [R\,(u, x, t)]^n\, \eta \tag{3.14}$$

and

$$\tau_{n+1} = [R\,(u, x, t)]^n\, \tau, \qquad n = 0, 1, 2, \ldots \tag{3.15}$$

It can be found that the elements of η_n form a commuting Lie algebra whereas the elements of τ_n form a noncommuting Lie algebra. For example the first two elements of the noncommuting infinite set of symmetries are obtained

$$\tau_1 = u_x \tag{3.16}$$

$$\tau_2 = t^{2(n+1)}\, u_{xxx} + \frac{\alpha}{\beta}\, t^{n+1}\, u u_x - \frac{2\,(n+1)}{3\beta}\, x u_x - \frac{(n+1)}{3\beta}\, u \tag{3.17}$$

and

$$[\tau_1, \tau_2] = \frac{2(n+1)}{3\beta} \tau_1. \tag{3.18}$$

The two types of symmetries together will form an infinite dimensional noncommuting Lie algebra, for example

$$[\eta_2, \tau_1] = \frac{(n+1)}{\beta} \eta_1 \tag{3.19}$$

$$[\eta_3, \tau_2] = \frac{5(n+1)}{3\beta} \eta_2 \tag{3.20}$$

etc. $n = 0, 1, 2, 3, \ldots$. The Noether's operator[7] Θ is $\frac{\partial}{\partial x}$ and if the γ_n are the conserved covariants, then

$$\gamma_n = \Theta^{-1} \eta_n. \tag{3.21}$$

Obviously, the first conserved covariant,

$$\gamma_1 = -\frac{\alpha}{3\beta} t^{n+1} u + \frac{n+1}{3\beta} x. \tag{3.22}$$

If $I_n(u, x, t)$ are the conserved functionals of the given equation then[7]

$$\frac{\partial}{\partial t} I_n(u, x, t) = 0 \tag{3.23}$$

and

$$\gamma_n = \operatorname{grad} I_n(u, x, t). \tag{3.24}$$

Now we consider the similarity transformation of the KdV-type equation (2.4). The infinitesimal transformation under which the Eq. (2.4) remains invariant is given by

$$x^* = x + \varepsilon\, \xi(x, t, v) + O(\varepsilon^2) \tag{3.25}$$

$$t^* = t + \varepsilon\, \sigma(x, t, v) + O(\varepsilon^2) \tag{3.26}$$

$$u^* = u + \varepsilon\, \varkappa(x, t, v) + O(\varepsilon^2) \tag{3.27}$$

where ξ, σ and $\varkappa$ are the infinitesimals corresponding to the variables x, t and v. Then

$$\xi = a\, t^{n+1} + \frac{6\beta}{n+1} b, \tag{3.28}$$

$$\sigma = 0, \tag{3.29}$$

$$\varkappa = t^{-2(n+1)} b, \tag{3.30}$$

where a and b are the parameters and they give the generators

$$T_1 = t^{n+1} \frac{\partial}{\partial x} \tag{3.31}$$

$$T_2 = \frac{6\beta}{n+1} \frac{\partial}{\partial x} + tZ^{2(n+1)} \frac{\partial}{\partial u}. \tag{3.32}$$

These generators form an Abelian group,

$$[T_1, T_2] = 0. \tag{3.33}$$

This implies that the possible symmetries are δ and ν

$$\delta = t^{n+1} v_x, \tag{3.34}$$

$$\nu = - \frac{6\beta}{n+1} v_x + t^{-2(n+1)}. \tag{3.35}$$

It can be easily verified that we can have two infinite sets of symmetries. They are the following,

$$\delta_{n+1} = [R(v, x, t)]^n \delta \tag{3.36}$$

$$\nu_{n+1} \; [R(v, x, t)]^n \nu, \qquad n = 0, 1, 2, \dots. \tag{3.37}$$

Eqs. (3.36) will form a commuting infinite dimensional Lie algebra and they will give the conserved covariances ϱ_n,

$$\varrho_n = \Theta^{-1} \delta_n \qquad n = 0, 1, 2, 3, \dots \tag{3.38}$$

where Θ^{-1} is the inverse Noether's operator which is D^{-1}.

For example, the first and second elements of the commuting symmetries,

$$\delta_1 = t^{n+1} v_x \tag{3.39}$$

$$\delta_2 = t^{3(n+1)} (v_{xxx} - 6v \, v_x). \tag{3.40}$$

The first and second elements of the noncommuting symmetries,

$$\nu_1 = - \frac{6\beta}{n+1} v_x + t^{-2(n+1)} \tag{3.41}$$

$$\nu_2 = -\frac{6\beta}{n+1} t^{2(n+1)} (v_{xxx} - 6vv_x) - 4v - 2xv_x \tag{3.42}$$

and the associated non-Abelian Lie algebra is given by,

$$[\nu_m, \nu_n] = 2(m-n)\, \nu_{m+n-1} \tag{3.43}$$

$$[\delta_m, \nu_n] = (2m+1)\, \delta_{m+n-1}. \tag{3.44}$$

Other properties of these symmetries are discussed in a recent paper. Similar types of noncommuting symmetries have been reported earlier by several authors[11,15,16] without establishing their relation with similarity transformations. There is no conserved functional or covariance associated with these noncommuting symmetries.

4. *Discussions*

Symmetries of nonlinear evolution equations provide a major contribution to the better understanding of the exactly solvable dynamical systems. Recently, the relation between movable singularities and the algebraic integrability of nonlinear partial differential equations have been widely studied in different contexts. It is a well known conjecture that if a field equation has the Paileve′ property (PP), then it is completely integrable. Now we can extend this study to symmetries. If a nonlinear partial differential equation has PP then there exists at least one set of infinite series of symmetries generated by the recursion operator associated with the system. Then the first element of these set of symmetries can be obtained from the similarity transformations and the recursion operator can be developed from the Lax-pair as we discussed earlier.

The relation between soliton like solutions and the similarity groups associated with the evolution equation is interseting to note here. The travelling wave solution of a partial differential equation is a function of $(kx - \omega t)$ where k and ω are constants. Soliton is a travelling wave preserved under space and time translations and so for such evolution equations we can expect translational invariance under space and time variables. Eqs. (2.1) and (2.4) have space translational invariance but no time translational invariance. This implies that these two equations will not give any soliton type travelling wave type solutions.

The VKdV equation (2.1) can be transformed into the cylindrical KdV equation, which by rescaling can be written as

$$\omega_\sigma + \frac{1}{2\sigma}\omega + a\,\omega_{xxx} + b\,\omega\,\omega_x = 0, \tag{4.1}$$

where a and b are constant coefficients. Using the following transformation

$$u = \sqrt{\sigma}\,\omega \tag{4.2}$$

and

$$\sigma = t^{2(n+1)} \tag{4.3}$$

we get

$$u_\sigma + a\, u_{xxx} + b \frac{1}{\sqrt{\sigma}} u\, u_x = 0 \tag{4.4}$$

and

$$u_t + 2(n+1)\, t^{n+1} \left(a\, u_{xxx} + \frac{b}{t^{n+1}} u\, u_x \right) = 0. \tag{4.5}$$

So we can transform the well known results like exact solutions, Lax-pair etc. of the cylindrical KdV equation to the VKdV equation. The PP analysis of the cylindrical KdV equation is also well known[18].

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References

1) R. K. Bullough, P. J. Caudrey and H. M. Gibbs, *Solitons*, eds. R. K. Bullough and P. J. Caudrey (Springer, Berlin, 1980);
2) P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, Berlin, 1986);
3) B. Fuchssteiner, Non. Anal. Th. Meth. Appl. **3** (1979) 849;
4) F. Magri, J. Math. Phys. **19** (1978) 1156;
5) I. M. Gel'fand and I. Ya Dorfman, Funct. Anal. Appl. **13** (1979) 13;
6) P. D. Lax, Commun. Pure Appl. Math. **28** (1975) 141;
7) B. Fuchssteiner, Prog. Theor. Phys. **65** (1981) 861; **68** (1982) 1082; **70** (1983) 1508;
8) R. S. Johnson, J. Fluid Mech. **97** (1980) 701;
9) N. Nirmala, M. J. Vedan and B. V. Baby, J. Math. Phys. **27** (1986) 2640; **27** (1986) 2646;
10) Li-Yishen and B. V. Baby, ICTP, Trieste, Preprint, IC/86/398 (1987);
11) W. Oevel and A. S. Fokas, J. Math. Phys. **25** (1984) 918;
12) W. Stramp, J. Math. Phys. **25** (1984) 2095;
13) E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1944);
14) G. M. Bluman and J. D. Cole, *Similarity Methods for Differential Equations* (Springer, Berlin 1974);
15) B. Fuhssteiner, Prog. Theor. Phys. **70** (1983) 1508;
16) H. H. Chen, Y. C. Lee and J. E. Lin, Phys. Letts. **91A** (1982) 381;
17) A. C. Scott, F. Y. F. Chu and D. W. McLaughlin, J. IEEE Trans. **61** (1973) 2051;
18) W. H. Steeb, M. Kloke, B. M. Spieker and W. Oevel, J. Phys. **16** (1983) L447.

SIMETRIJE I OPERATORI REKURZIJE ZA KORTEWEG-de VRIESOVE JEDNADŽBE S VARIJABILNIM KOEFICIJENTOM

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Nađeno je da Olverova metoda nalaženja beskonačno mnogo simetrija jednadžbe gibanja vrijedi za neizospektralni slučaj. Operatori rekurzije izvedeni su iz Laxova para, a cijela metoda proširena je na neizospektralne probleme. Minimalni broj različitih beskonačnih skupova simetrije isti je kao i broj nezavisnih transformacija sličnosti pridruženih danoj jednadžbi gibanja. Veza između rješenja za putujući val i transformacije sličnosti je također diskutirana. Rezultati su primijenjeni na Korteweg-de Vriesove jednadžbe s promjenljivim koeficijentom.