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SUPERSYMMETRIC ALGEBRA OF THE HYDROGEN ATOM AND THE TENSOR FORCE

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Using the factorization property of a three-dimensional hydrogen-atom-like Hamiltonian $H^{(0)}$, we generate a series of related Hamiltonians $H^{(n)}$ $(n = 1, 2, 3)$ corresponding to a definite number of fermions, which are block diagonal components of an 8×8 matrix supersymmetric Hamiltonian. We find that $H^{(0)}(\alpha) = H^{(3)}(-\alpha)$ and $H^{(1)}(\alpha) = H^{(2)}(-\alpha)$, with α being the fine-structure constant. By introducing the »spin« operator in the space of fermions, we show that the total angular momentum operators, $J_k = L_k + S_k$ and $\vec{S}^2$, commute with the supersymmetric Hamiltonian and can be used to classify its eigenstates. We also show that the 3×3 matrix Hamiltonian $H^{(1)}$ (and similarly $H^{(2)}$) can be written in the form which explicitly exhibits the structure corresponding to the tensor and spin-spin interaction between two spin-1/2 particles. The relative strength between the Coulomb tensor and the spin-spin interaction is fixed by supersymmetry.

1. Introduction

The ideas of SUSY have been successfully applied not only in the relativistic field theory as an extension of Lorentz symmetry[1], but also as a symmetry in nuclear physics[2], relating paired and unpaired nucleons in nuclei, and in quantum-mechanical systems[3], relating the spectra and wave functions of different Hamiltonians.

The SUSY extension of quantum-mechanical systems in space dimensions ≥ 2 is most easily achieved by using the factorization property of the Schrödinger operator[4,5] whose spectrum is bounded from below, i. e. possesses a normalized groud-state wave function.

In this paper we apply the factorization method and its subsequent SUSY extension to the hydrogen-atom-like Hamiltonian.

The plan of the paper is as follows. In Sect. 2 we give a short presentation of the factorization method in quantum mechanics and its relation with supersymmetry. In Sect. 3 the factorization method is applied to the hydrogen-like Hamiltonian. In Sect. 4 we discuss the role of the spin operator and show that the appearance of the tensor-like interaction is the natural consequence of the supersymmetric extension of the Schrödinger equation of a particle in a central potential. The concluding remarks are given in Sect. 5.

2. *The factorization method and supersymmetry*

For any three-dimensional time-independent Hamiltonian $H_0 = \frac{1}{2} p_k^2 + V$ whose energy spectrum is bounded from below there exist three generators[6]

$$Q_k = \frac{1}{\sqrt{2}} (\mathrm{i} p_k + \partial_k \chi), \quad k = 1, 2, 3, \tag{2.1}$$

and their duals

$$\widetilde{Q}_{ij} = \varepsilon_{ijk} Q_k, \tag{2.1'}$$

in terms of which H_0 can be written in the factorized form[4,5]

$$H^{(0)} = H_0 - E_0 = Q_k^\dagger Q_k = \frac{1}{2} \mathrm{Tr} (\widetilde{Q}^\dagger \cdot \widetilde{Q}). \tag{2.2}$$

Here E_0 is the ground-state energy and $\chi = -\ln \psi_0$, with ψ_0 being the normalized ground-state wave function of H_0 satisfying

$$Q_k \psi_0 = 0, \tag{2.3}$$

and

$$\widetilde{Q}_{ij} \psi_0 = 0.$$

By introducing a set of three fermion creation and annihilation operators f_k ($k = 1, 2, 3$), with the usual anticommutation relations

$$\{f_k^\dagger, f_l\} = \delta_{kl}, \quad \{f_k^\dagger, f_l^\dagger\} = 0 = \{f_k, f_l\}, \tag{2.4}$$

we can construct a supersymmetric Hamiltonian $\hat{H}$

$$\hat{H} = \{\hat{Q}^\dagger, \hat{Q}\} = \hat{Q}^\dagger \hat{Q} + \hat{Q} \hat{Q}^\dagger \tag{2.5}$$

whose supercharges $\hat{Q}$ and $\hat{Q}^\dagger$ are defined by

$$\hat{Q} = Q_l f_l^\dagger, \qquad \hat{Q}^\dagger = Q_l^\dagger f_l \tag{2.6}$$

and satisfy

$$\hat{Q}^2 = 0, \qquad \hat{Q}^{+2} = 0,$$
$$[\hat{H}, \hat{Q}] = 0, \qquad [\hat{H}, \hat{Q}^\dagger] = 0. \tag{2.7}$$

Since the Hamiltonian (2.5) also commutes with the fermion-number operator $N_F = f_l^\dagger f_l$, it can be written as a direct sum of Hamiltonians $H^{(n)}$ corresponding to the fixed number of fermions, $n = 0, 1, 2$ and 3:

$$\hat{H} = \bigoplus_{n=0}^{3} H^{(n)}. \tag{2.8}$$

Note that $\hat{H}$ also has a more convenient form

$$\hat{H} = H^{(0)} + [Q_r, Q_s^\dagger] f_r^\dagger f_s \tag{2.9}$$

which shows how the supersymmetric extension of $H^{(0)}$ is achieved.

The fact that the action of the supercharge $\hat{Q}$ changes the fermion number of the state by one,

$$N_F \hat{Q} = \hat{Q}(N_F + 1), \tag{2.10}$$

can be used to find relations which connect Hamiltonians differing in the number of fermions by one. They are

$$Q_l H^{(0)} = H^{(1)}_{lk} Q_k,$$
$$\tilde{Q} \cdot H^{(1)} = H^{(2)} \cdot \tilde{Q}, \tag{2.11}$$

(matrix multiplication is implied)

$$Q_l H^{(2)}_{lk} = H^{(3)} Q_k,$$

where

$$H^{(1)}_{lk} = Q_l Q_k^\dagger + (\tilde{Q}^\dagger \cdot \tilde{Q})_{lk},$$
$$H^{(2)}_{lk} = Q_l^\dagger Q_k + (\tilde{Q} \cdot \tilde{Q}^\dagger)_{lk}, \tag{2.12}$$
$$H^{(3)} = Q_k Q_k^\dagger = \frac{1}{2} \operatorname{Tr}(\tilde{Q} \cdot \tilde{Q}^\dagger).$$

Each energy level of an intermediate Hamiltonian $H^{(n)}$ in the chain (2.11) is doubly degenerate except the ground-state level of $H^{(0)}$, which is singly degenerate because of (2.3).

In the vector space of states the supersymmetric Hamiltonian $\hat{H}$ thus becomes an 8×8 matrix

$$\hat{H} = \begin{bmatrix} H^{(0)} & 0 & 0 & 0 \\ 0 & H^{(1)} & 0 & 0 \\ 0 & 0 & H^{(2)} & 0 \\ 0 & 0 & 0 & H^{(3)} \end{bmatrix}, \tag{2.13}$$

with the supercharge Q given by the following 8×8 matrix:

$$\hat{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ Q_l & 0 & 0 & 0 \\ 0 & \tilde{Q}_{lk} & 0 & 0 \\ 0 & 0 & Q_k & 0 \end{bmatrix}. \tag{2.14}$$

3. *The hydrogen-like atom*

In the case of the hydrogen-like atom with the Hamiltonian

$$H_0 = \frac{1}{2} p_k^2 - \frac{\alpha}{r}, \qquad (m = 1,\ \hbar = 1), \tag{3.1}$$

we find that

$$H^{(0)} = H_0 - E_0 = \frac{1}{2} p_k^2 - \frac{\alpha}{r} + \frac{\alpha^2}{2}, \tag{3.2}$$

where $E_0 = -\alpha^2/2$ is the ground-state energy of H_0.

For this Hamiltonian the ground-state wave function ψ_0 is explicitly known:

$$\psi_0 = C\,\mathrm{e}^{-\alpha r} \tag{3.3}$$

and

$$H^{(0)} \psi_0 = 0.$$

The function $\chi = -\ln \psi_0$ is known and of the form

$$\chi = -\ln C + \alpha r, \tag{3.4}$$

so that the operators Q_k become

$$Q_k = \frac{1}{\sqrt{2}} \left(\mathrm{i} p_k + \alpha \frac{x_k}{r} \right), \qquad k = 1, 2, 3, \tag{3.5}$$

in terms of which $H^{(0)} = Q_k^{\dagger} Q_k$.

The supersymmetric Hamiltonian again has the form (2.13), where now

$$H^{(0)} = \frac{1}{2} p_k^2 - \frac{\alpha}{r} + \frac{\alpha^2}{2},$$

$$H_{lk}^{(1)} = H^{(0)} \delta_{lk} - \frac{\alpha}{r^3} (x_l x_k - r^2 \delta_{lk}), \tag{3.6}$$

$$H_{lk}^{(2)} = H_{lk}^{(1)} (\alpha \to -\alpha),$$

$$H^{(3)} = H^{(0)} (\alpha \to -\alpha).$$

Note that the Hamiltonian $H^{(3)}$ describes the motion of a positron in the Coulomb field of a point nucleus (proton) and it therefore has no bound states; the energy levels are in the continuum.

However, the Hamiltonians $H^{(0)}$ and $H^{(1)}$ have bound states which are related through (2.11). The wave functions of $H^{(1)}$ may therefore be obtained from those of $H^{(0)}$, i. e. $\psi_k^{(1)} \sim Q_k \psi^{(0)}$ when $\psi^{(0)} \neq \psi_0$.

4. *Supersymmetric algebra and spin*

It is easy to see that the supersymmetric Hamiltonian $\hat{H}$ does not commute with components of the orbital angular momenta L_k. In fact, we find that

$$[L_k, \hat{Q}] = \mathrm{i}\varepsilon_{klj} f_l^\dagger Q_j, \tag{4.1}$$

showing that the notion of spin is necessary. In the space of fermions we define components of a spin operator in the following way:

$$\mathrm{i}S_k = \varepsilon_{klj} f_l^\dagger f_j. \tag{4.2}$$

They satisfy the usual angular momentum commutation relations

$$[S_k, S_l] = \mathrm{i}\varepsilon_{klm} S_m. \tag{4.3}$$

Their commutation relations with the supercharge Q are exactly

$$[S_k, \hat{Q}] = - [L_k, \hat{Q}]. \tag{4.4}$$

Thus we conclude that the components of the total angular momenta $J_k = L_k + S_k$ should commute with the supersymmetric Hamiltonian $\hat{H}$, i. e.

$$[\hat{H}, J_k] = 0 \tag{4.5}$$

and are therefore conserved. The eigenstates of $\hat{H}$ can be classified not only according to the number of fermions, but also according to the value of the total angular momentum $\mathcal{J}$.

Since the square of the spin operator is expressed in terms of the fermion-number operator N_F,

$$\vec{S}^2 = S_k^2 = N_F(3 - N_F), \tag{4.6}$$

it also commutes with the super-Hamiltonian $\hat{H}$ and can be used to classify states.

For the Coulomb potential, it can be verified that the Hamiltonian $H^{(1,2)}$ can be written in the form

$$H^{(1)} = H^{(0)} + \frac{\alpha}{r^3}(\vec{S} \cdot \vec{r})^2, \qquad (S_i)_{jk} = -\mathrm{i}\varepsilon_{ijk}, \tag{4.7}$$

$$H^{(2)} = H^{(1)}(\alpha \to -\alpha),$$

showing the appearance of an additional repulsive tensor-like interaction.

Using the standard notation for the »tensor operator«

$$S_{12} = 2\left[3\frac{(\vec{S} \cdot \vec{r})^2}{r^2} - \vec{S}^2\right], \tag{4.8}$$

we rewrite $H^{(1)}$ in the familiar form

$$H^{(1)} = H^{(0)} + V_T S_{12} + V_S \vec{S}^2, \tag{4.9}$$

where

$$V_T = \frac{1}{2} V_S = -\frac{1}{6} V_{Coulomb}. \tag{4.10}$$

Supersymmetry fixed the relative strength between the Coulomb potential, the tensor potential and the spin-spin potential.

5. *Conclusion*

We have shown that the SUSY quantum mechanics of a hydrogen atom can be extended to complete supersymmetric algebra by introducing the »spin« operator. The complete set of commuting operators is $\hat{H}$, $\hat{Q}$ (or $\hat{Q}^\dagger$), $\vec{\mathcal{J}}^2$, $\mathcal{J}_3$ and $\vec{S}^2 = N_F(3 - N_F)$. The eigenfunctions of $\hat{H}$ may be sought among the common eigenfunctions of $\vec{\mathcal{J}}^2$, $\mathcal{J}_3$ and N_F.

The above approach can be applied to any centrally symmetric potential $V(r)$ having the energy spectrum bounded from below. The appearance of the tensor-like interaction is the natural consequence of the supersymmetric extension of a central potential problem Hamiltonian.

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Koristeći metodu faktorizacije Hamiltonijana za vodiku slični atom u tri dimenzije konstruirali smo niz povezanih hamiltonijana $H^{(n)}$ $(n = 1, 2, 3)$ s određenim brojem fermiona, koji čine dijagonalne blokove u 8×8 supersimetričnom matričnom hamiltonijanu. Nalazimo da je $H^{(0)}(\alpha) = H^{(3)}(-\alpha)$ i $H^{(1)}(\alpha) = H^{(2)}(-\alpha)$, gdje je α konstanta fine strukture. Uvodeći operator spina u prostor fermiona pokazano je da ukupni angularni moment $J_k = L_k + S_k$ i $\vec{S}^2$ komutiraju sa supersimetričnim hamiltonijanom te se mogu koristiti za klasifikaciju njegovih vlastitih stanja. Također je pokazano da se 3×3 matrični hamiltonijan $H^{(1)}$ (slično za $H^{(2)}$) može prikazati u obliku koji eksplicitno pokazuje strukturu koja odgovara tenzorskoj i spin-spin interakciji između dvije čestice spina 1/2. Relativna jakost između Coulombove, tenzorske i spin-spin interakcije određena je supersimetrijom.